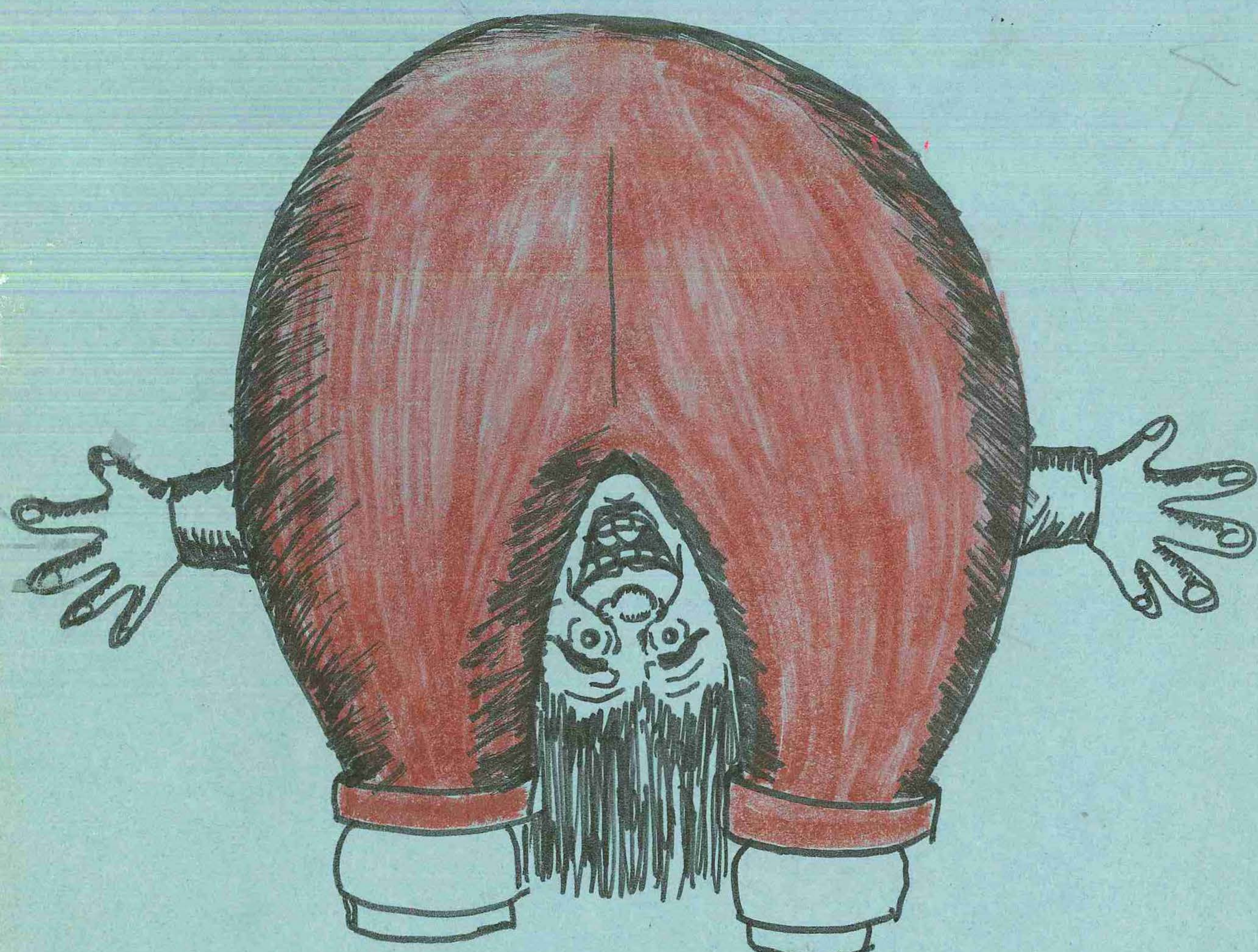
E&M and Energy Conversion

R.J. Marks II Class Notes
Rose-Hulman Institute of Technology (1971)
Texas Tech University (1976)

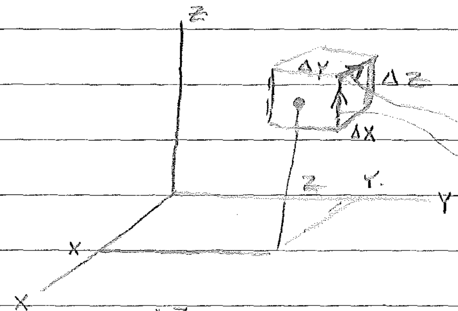
ELECTROMAGNETIC
WAVES



TUESDAY

VECTOR NOTATION: $\vec{A}$, $\bar{A}$

$\bar{a}_i$ = UNIT VECTOR ($\bar{a}_x = \bar{i}$)

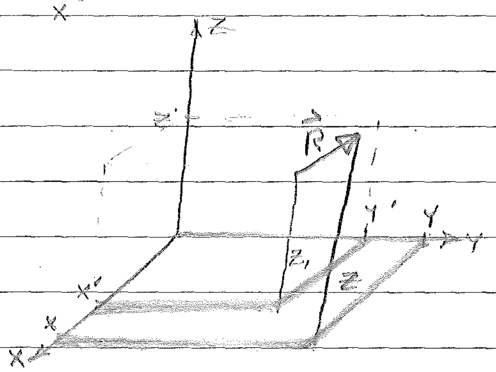


← RIGHT-HAND CO-ORDINATE SYSTEM

$\Delta V = \Delta X \Delta Y \Delta Z$

$\bar{dl} = dz\,\bar{a}_z$

$\bar{dl} = -dx\,\bar{a}_x$

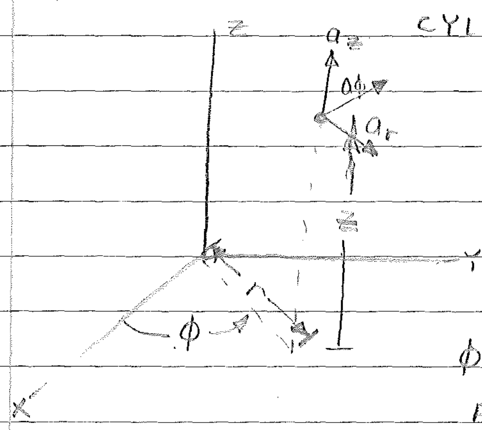


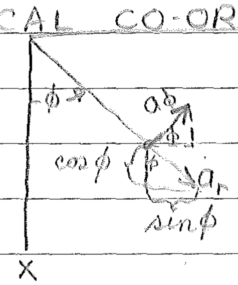
$\vec{R} = (x - x')\vec{a}_x + (Y - Y')\bar{a}_Y + (Z - Z')\bar{a}_z$

(PRIME POINT - SOURCE POINT)

(UNPRIMED - FIELD POINT)

CYLINDRICAL CO-ORDINATES





$\vec{a}_\phi \perp \vec{a}_r$

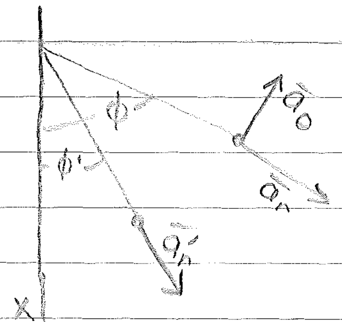
$$\begin{cases} \vec{a}_r = \cos\phi\,\bar{a}_x + \sin\phi\, \\ \vec{a}_\phi = -\sin\phi\,\bar{a}_x + \cos\phi\,\bar{a} \end{cases}$$

ϕ MEASURED COUNTER-CLOCKWISE FROM X AXIS

WORK 1, 2, AND 49 FOR TOMORROW

WED

QUIZ



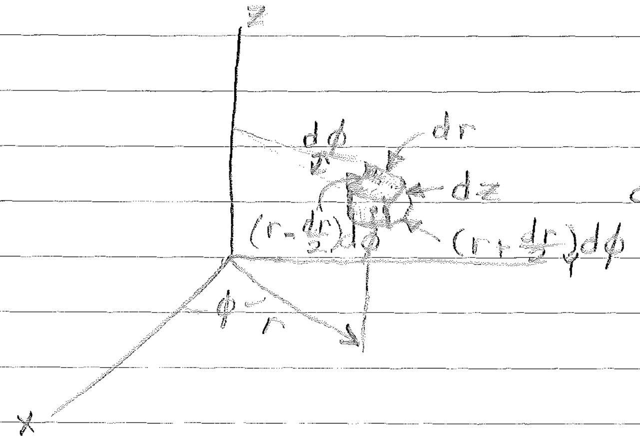
$\phi - \phi'$

$\cos(\phi-\phi')$

a_r $\sin(\phi-\phi')$

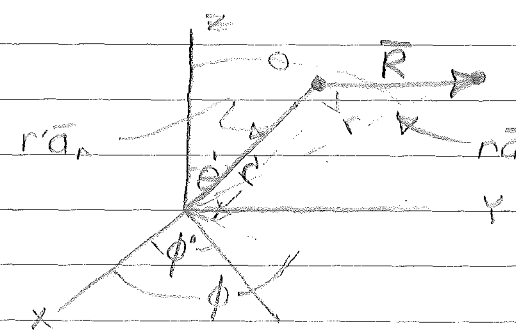
$\Rightarrow a_r = \cos(\phi-\phi')\vec{a}_r - \sin(\phi-\phi')\bar{a}_\phi$

MORE CYLYNDRICAL CO-ORDIANATES (DIFFERENTIAL VOLUME

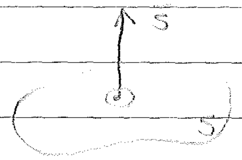


$dv = r\,dr\,d\phi\,dz$

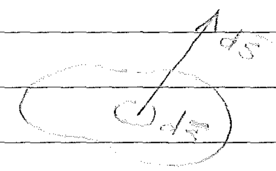
SPERICAL CO-ORDINATES



$\vec{R} = r\bar{a}_r - r'\bar{a}_{r'}$



SURFACE AREA = S

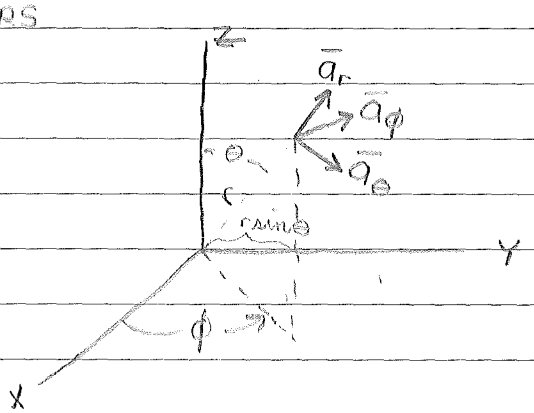


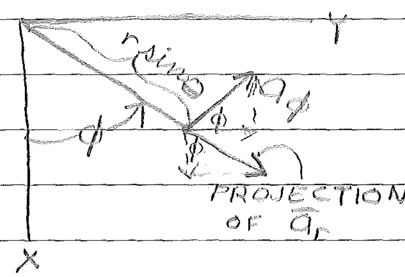
SURFACE VECTOR

MAGNITUDE = AREA

DIRECTION ⊥ TO AREA (POSITIVE OUTWARD FOR CLOSED SURFACE)

THURS



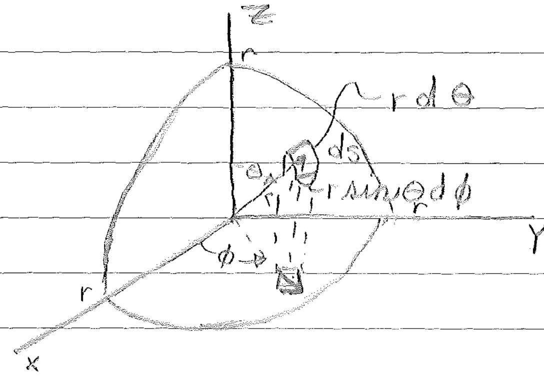


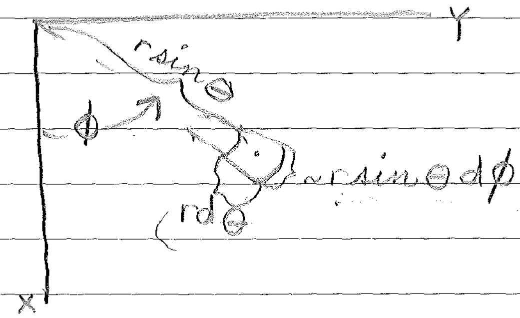
$\bar a_\phi = \cos\phi\, \bar a_y - \sin\phi\, \bar a_x$

1-5) $\bar A = 2\bar a_x + \bar a_y - 3\bar a_z$

$\bar A \cdot \bar a_\phi = -2\sin\phi + \cos\phi$

(IF $\bar A = \bar r$, THEN $\bar A \cdot a_\phi = 0$)





$\Rightarrow ds = r^2 \sin\theta \, d\theta \, d\phi$

FOR SURFACE VECTOR:

$d\bar{S} = r^2 \sin\theta \, d\theta \, d\phi \, \bar{a}_r$

PROJECTIONS ON AXIS

ON X $\sin\theta \cos\phi \bar{a}_x$

Y $\sin\theta \sin\theta \bar{a}_Y$

Z $\cos\phi \bar{a}_z$

$$\oint d\bar{S} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi \, \bar{a}_r$$

$$\Rightarrow \bar{a}_r = \sin\theta\cos\phi a_x + \sin\theta \sin\phi \bar{a}_Y + \cos\theta \bar{a}_z$$

$$\Rightarrow \oint d\bar{S} = \bar{a}_x r^2 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin^2\theta \cos\phi \, d\theta \, d\phi$$
$$+ \bar{a}_Y r^2 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin^2\phi \sin\phi \, d\theta \, d\phi$$
$$+ \bar{a}_z r^2 \int_0^{2\pi} \int_0^{\pi} \sin\theta \cos\theta \, d\theta \, d\phi$$

$$\oint d\bar{S} = 0$$

$\oint dS =$ AREA

$$= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin\theta \, d\theta \, d\phi = 4\pi r^2$$

$\Phi(x, y, z)$ = CONSTANT YIELDS SURFACE

$\frac{\delta \Phi}{\delta s} \ni s$ = SOME DIRECTION

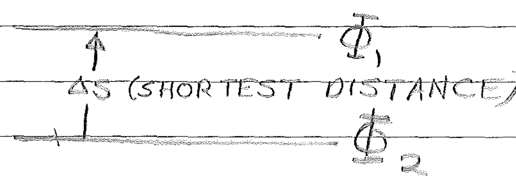
$$d\Phi = \frac{\delta \Phi}{\delta s} ds = \frac{\delta \Phi}{\delta x} dx + \frac{\delta \Phi}{\delta y} dy + \frac{\delta \Phi}{\delta z} dz$$

$$\Rightarrow s = \underbrace{\left(\frac{\delta \Phi}{\delta x} \bar{a}_x + \frac{\delta \Phi}{\delta y} \bar{a}_y + \frac{\delta \Phi}{\delta z} \bar{a}_z \right)}_{\nabla \Phi} \cdot \underbrace{\left(dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z \right)}_{d\bar{S}}$$

GRADIENT $\nabla = \bar{a}_x \frac{\delta}{\delta x} + \bar{a}_y \frac{\delta}{\delta y} + \bar{a}_z \frac{\delta}{\delta z}$

$\bar{A} \cdot \bar{B} = 0$ IF $\bar{A} \perp \bar{B}$ OR $|A| = 0$ OR $|B| = 0$

GRADIENT IS $\perp$ TO SURFACE Φ = CONST.



GRADIENT LIES IN GREATEST MAGNITUDE OF RATE OF CHANGE OF Φ

$\nabla\cdot\nabla\phi = \nabla^2\phi = \frac{\delta^2\phi}{\delta x^2} + \frac{\delta^2\phi}{\delta y^2} + \frac{\delta^2\phi}{\delta z^2}$

LAPLACIAN

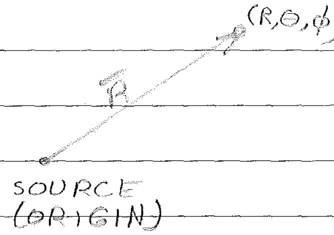
$\nabla^2\left(\frac{1}{R}\right)$

SOURCE @ ORIGIN OF SPH. CO-OR

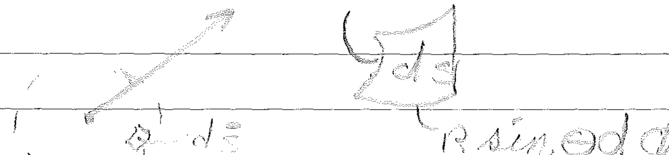
$\nabla \frac{1}{R} = \bar{a} \frac{\delta}{\delta R}\frac{1}{R} = -\bar{a}_r/R^2$

THURS.

$\nabla\cdot\nabla\left(\frac{1}{R}\right) = \nabla^2 \frac{1}{R}$



$\nabla^2\left(\frac{1}{R}\right) = \bar{a}_R \frac{\delta}{\delta R}\left(\frac{1}{R}\right) = -\bar{a}_r/R^2$

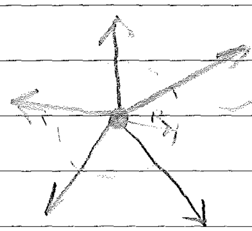


$\int \nabla^2 \frac{1}{R}\, dv = \int \nabla\cdot\nabla\left(\frac{1}{R}\right) dV = \oint \nabla\left(\frac{1}{R}\right)\cdot ds$

$\Rightarrow ds = R^2 \sin\theta\, d\theta\, d\phi\, \bar{a}_r$

$\Rightarrow \int \nabla^2 \frac{1}{R}\, dv = \oint \cancel{\frac{-\bar{a}_r}{R^2}}\, \cancel{R^2} \sin\theta\, d\theta\, d\phi\, \bar{a}_r$

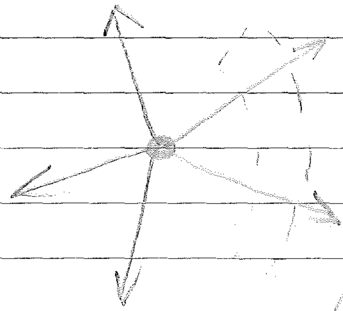
$= \int_0^{\pi}\int_0^{2\pi} -\sin\theta\, d\theta\, d\phi = -4\pi$



FIELD INTENSITY $= \frac{lines}{AREA} = \frac{5}{4\pi R^2}$

@ R=0, FIELD INTENSITY = 4π

$\oint \frac{q}{R^2} \cdot dS$ = # OF LINES OUT OF SURFACE



$$\oint \nabla^2 \left(\frac{1}{R}\right) dv = \begin{cases} -4\pi & \text{IF } R=0 \text{ IS IN VOLUME} \\ 0 & \text{IF } R=0 \text{ IS NOT IN VOLUME} \end{cases}$$

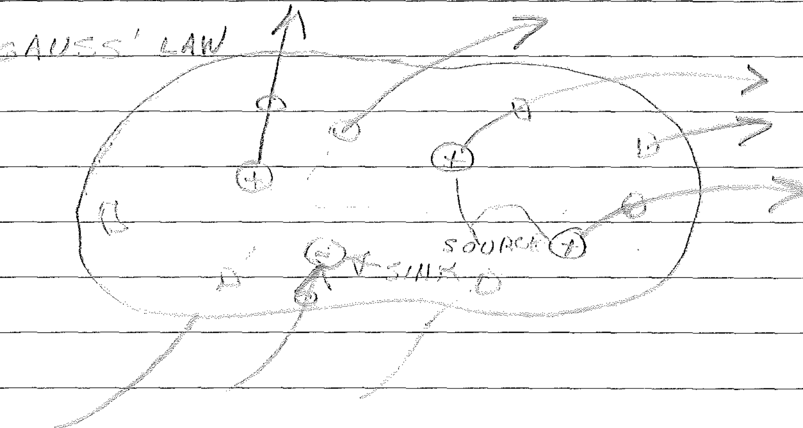
(ANALAGOUS TO DEL FUNCTION $\int_{t_1}^{t_2} \delta(t-t_0)\,dt = \begin{cases} 1 \\ 0 \end{cases}$)

$\Rightarrow \nabla^2\left(\frac{1}{R}\right) = -4\pi\,\delta(R)$ ∋ $\delta(R)$ IS A SPACIAL DEL FUNCTION

CHARGE DENSITY

$\int \rho\, dV$: FOR POINT SOURCE $\rho = q\,\delta(R)$ OR $q\,\delta(R-R_0)$

GAUSS' LAW



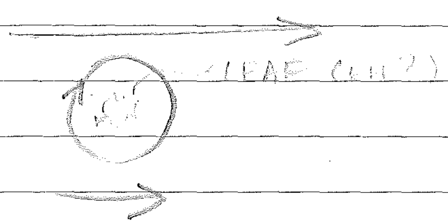
$\oint F \cdot d\bar{S} = \int \nabla \cdot F\, dV$

5 OUT − 3 IN = 3 SOURSES − 1 SINK

2 = 2

THE CURL

$(\nabla \times \bar{F}) \cdot \bar{a}_n = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint \bar{F} \cdot d\bar{l}$ = curl F = rot F



$\int \nabla \times F \cdot d\vec{S} = \oint F \cdot dl$

MON

TEST NEXT TUESDAY

REVIEW

$\nabla \phi$

$\nabla \cdot \vec{F}$

$\nabla \times \vec{F}$

$\int \nabla \cdot \vec{F}\, dv = \oint \vec{F} \cdot d\vec{s}$ (GAUSS' LAW)

$\int \nabla \times \vec{F} \cdot d\vec{s} = \oint \vec{F} \cdot d\vec{l}$

$\nabla \times \nabla \phi = 0 \qquad \nabla \cdot \nabla \times \vec{F} = 0$

$\nabla \times \vec{F} = \vec{C} \ni \vec{C}$ = CURRENT DENSITY (VORTEX)

$\nabla \cdot \vec{F} = S \ni S$ = CHARGE DENSITY

FOR A GIVEN S AND $\vec{C}$ $\exists$ A UNIQUE $\vec{F}$

IRROTATIONAL FIELD — ROTATIONAL (SOLENOIDAL) FIELD

$\hookrightarrow \nabla \times \vec{F} = 0 \qquad \hookrightarrow \nabla \cdot \vec{F} = 0$

$\Rightarrow \vec{F} = \nabla \phi \qquad \Rightarrow \vec{F} = \nabla \times \vec{A}$

$$F = F_1 + F_2$$

$\nabla \times F_1 = 0 \quad \nabla \cdot F_1 = S \qquad \nabla \cdot F_2 = 0 \quad \nabla \times F_2 = C$

$F_1 = -\nabla \phi \qquad \vec{F}_2 = \nabla \times \vec{A}$

$$\vec{F} = -\nabla \phi + \nabla \times \vec{A}$$

← VECTOR POTENTIAL

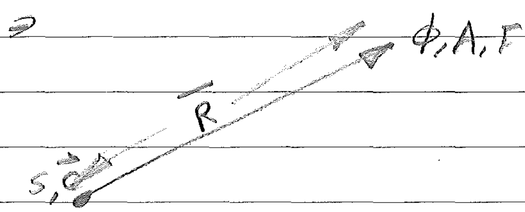
SCALAR POTENTIAL

$\nabla \cdot \vec{F} = -\nabla \cdot (\nabla \phi) + \cancel{\nabla \cdot \nabla \times \vec{A}} = -\nabla^2 \phi = S$

$\nabla \times \vec{F} = \cancel{\nabla \times (-\nabla \phi)} + \nabla \times \nabla \times \vec{A} = \underbrace{\nabla (\nabla \cdot \vec{A})}_{\text{CHOOSE } \nabla \cdot \vec{A} = 0} - \nabla^2 \vec{A} = \vec{C}$

$\therefore \nabla^2 \phi = -S \qquad \nabla^2 \vec{A} = -\vec{C}$

$\phi = \frac{1}{4\pi} \int \frac{S}{R}\, dV; \quad \vec{A} = \frac{1}{4\pi} \int \frac{\vec{C}}{R}\, dV$



9. Equivalently, (8-1) is a requirement for the linear independence of the the three vectors which means the following:

$$k_1\vec{a} + k_2\vec{b} + k_3\vec{c} = \vec{0}$$

means that $k_1 = k_2 = k_3 = 0$ for $\bar{a}$, $\bar{b}$, and $\bar{c}$ to be linearly independent.

10. In tensor or indicial notation the scalar triple product can be denoted as follows:

$$\left.\begin{aligned}\bar{a}\cdot\bar{b}\times\vec{c} &= a_i \epsilon_{ijk} b_j c_k \\ &= \epsilon_{ijk} a_i b_j c_k \\ &= \epsilon_{kij} a_i b_j c_k \\ &= \bar{a}\times\bar{b}\cdot\bar{c}\end{aligned}\right\}$$

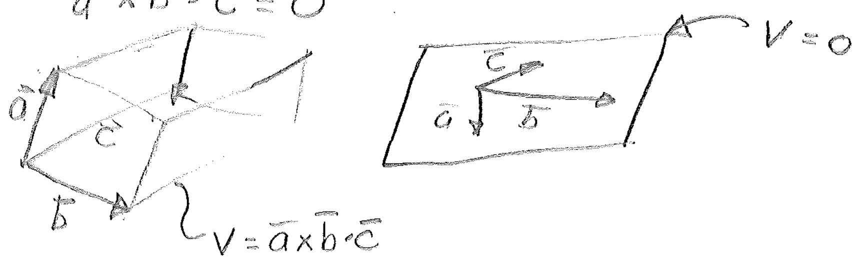
Using the properties of the permutation operator we deduce the following:

$$\vec{a}\cdot\vec{b}\times\vec{c} = \vec{b}\cdot\vec{c}\times\vec{a} = \vec{c}\cdot\vec{a}\times\vec{b}$$

It can also be shown that

11. Physically, in absolute value the scalar triple product represents the volume of the parallelpiped formed by $\vec{a}$, $\vec{b}$, and $\vec{c}$. Hence, if $\bar{a}$, $\bar{b}$, and $\bar{c}$ are coplanar it is evident that

$$\bar{a}\times\bar{b}\cdot\bar{c} = 0$$



$\vec{a}\times\vec{b}$ YIELDS AREA OF SIDE

ELEMENTS OF VECTOR CALCULUS

1. Vector calculus applies to vector functions which in turn define vector fields. Hence, the calculus of vectors is applicable to vector fields.

2. Essentially, vector calculus is merely an extension of ordinary calculus and involves "scalar" calculus applied to the scalar components of a vector function. Hence, all the ideas of ordinary or scalar calculus apply to vector calculus.

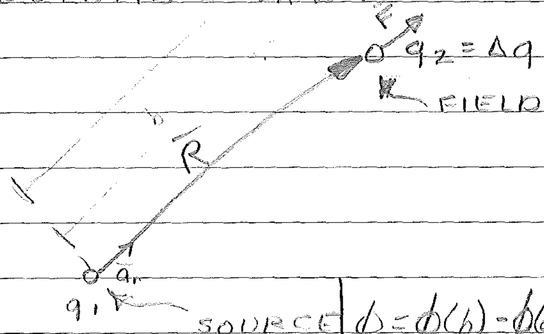
3. Let $v(t) = (u_1, u_2, u_3)$ be a vector function. Then the derivative of $v(t)$ is defined by the following: $\bar{U}'(t) = \frac{d\bar{u}}{dt}$

$$= \lim_{\Delta t \to 0} \frac{\bar{U}(t+\Delta t) - \bar{U}(t)}{\Delta t}$$

TUESDAY

$\vec{E} = \lim_{\Delta q \to 0} \frac{\vec{F}}{\Delta q}$

COULOMB'S LAW: $\vec{F} = \frac{q_1 q_2}{4\pi\epsilon_0 R^2}\bar{a}_r$



$\Rightarrow \vec{E} = \lim_{\Delta q \to 0} \frac{q_1 \Delta q}{4\pi\epsilon_0 R^2}\vec{a}_r \frac{1}{\Delta q}$
$= \frac{q_1}{4\pi\epsilon_0 R^2}\vec{a}_r$

$\phi = \phi(b) - \phi(a) = -\int_a^b \vec{E}\cdot d\vec{l}$ DEFINITION OF (POTENTIAL FIELD)

IF $\vec{E} = -\nabla\phi$

$= -\int_a^b (\nabla\phi)\cdot d\bar{l} = \int_a^b d\phi = \phi(b) - \phi(a)$

(WORK) $= \int \vec{F}\cdot d\bar{l} = -q\int \vec{E}\cdot d\vec{l} \Rightarrow \phi = \text{WORK}/q$

$\phi(b) - \phi(a) = \phi = -\int_a^b \left(\frac{q}{4\pi\epsilon_0 r^2}\right)\vec{a}_r \bullet (dr\,\bar{a}_r)$

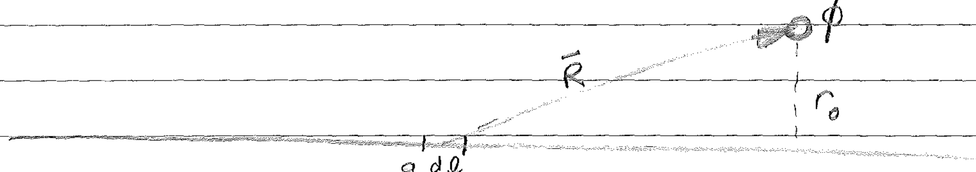
$= \frac{q}{4\pi\epsilon_0} \left.\frac{-1}{r}\right|_a^b$

$= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{b} - \frac{1}{a}\right)$

SOMETIMES LET $a = \infty$

$\Rightarrow \phi = \frac{q}{4\pi\epsilon_0 R}$

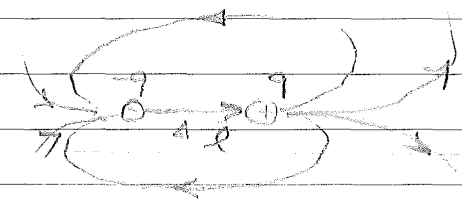
$dq = \rho\, dv \Rightarrow \vec{E} = \int \frac{\rho\, dv}{4\pi\epsilon_0 R^2}\bar{a}_r$



THURS

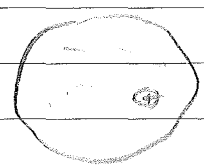
$\bar{D} = \epsilon_0 \vec{E} + \vec{P}$

$\phi = \dfrac{-\vec{m} \cdot \bar{a}_R}{4\pi\epsilon_0 R^2}$



$\vec{m} = q\,\Delta \vec{l}$

ATOM

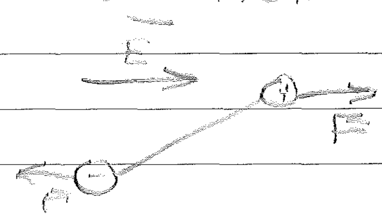


$\bar{P} = \alpha_e E_i$

ELECTRONIC POLARIZABILITY

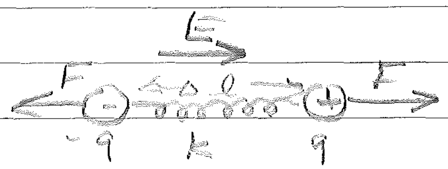
$\alpha_e = 4\pi\epsilon_0 R^3$

IONIZED ATOM



$\vec{P} = N\alpha_o E_i$

ORIENTATION POLARIZABILI



$\bar{P} = N\alpha_i E_i$

IONINIC POLARIZABILITY

$qE_i = k\Delta l \Rightarrow \Delta l = \dfrac{qE_i}{k} \Rightarrow \vec{m} = q\Delta l = \dfrac{q^2 E_i}{k}$

ATOMS

$\therefore \vec{P} = N\vec{m} = Nq^2 \dfrac{E_i}{k}$

$\Rightarrow \alpha_i = q^2/k$

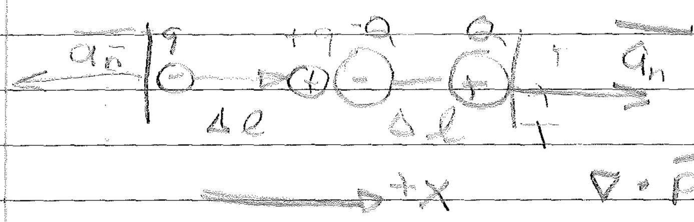
INTERNAL FIELD CONSTANT

$\bar{E}_i = \bar{E} + \dfrac{\gamma}{\epsilon_0} \vec{P}$

$\bar{P} = N\alpha E_L = N\alpha(\vec{E} + \frac{\gamma}{\epsilon_0}\bar{P})$

$\Rightarrow \bar{P} = \frac{N\alpha}{1 - N\alpha\gamma/\epsilon_0}\vec{E}$ → ANALAGOUS TO $\frac{e_o}{e_i} = \frac{A}{1 - A\beta}$

$\nabla\cdot\vec{D} = \mathrm{DIV}\,\vec{D} = \epsilon_0\nabla\cdot\vec{E} + \nabla\cdot\vec{P}$



$\nabla\cdot\bar{P} > 0$

$\rho_P = -\nabla\cdot\bar{P}$
$\rho_{PS} = \bar{P}\cdot a_n$

DO 18, 19, 23, 24

MON

~~H 3~~

$f_0(t) = M\frac{d^2x}{dt^2} + kx + b\frac{dx}{dt}$

MON

(CLOSED BOOK) IDENTITIES GIVEN

KNOW LAWS AND DEFINITIONS

CHAPTS 1, UP TO 95, (CEPT Pgs 60-78)

$z = \sqrt{K'} \int_1^R \frac{dR}{\sqrt{\ln R}}$

23) $E_r \uparrow q_e$

$I = \frac{dq}{dt} = q_e V$

$q_e = I/V$ coul/m

$E_r = \frac{q_e}{2\pi\epsilon_0 r}$

$F_r = q_e E_r = \frac{q_e q_e}{2\pi \epsilon_0 r} = m \frac{d^2 r}{dt^2}$

$\frac{q_e q_e}{2\pi\epsilon_0 m} = r \frac{d^2 r}{dt^2}$

IF E_r IS ASSUMED CONSTANT, $r = .34$ mm

BUT IF NOT!

$C_P\ q_e = \rho_0 \pi r_0^2 \Rightarrow \frac{q_e \pi \rho_0}{2\pi\epsilon_0 m} = C = R \frac{d^2 R}{dt^2} \Rightarrow R = \frac{r}{r_0}$

$V = dz/dt$

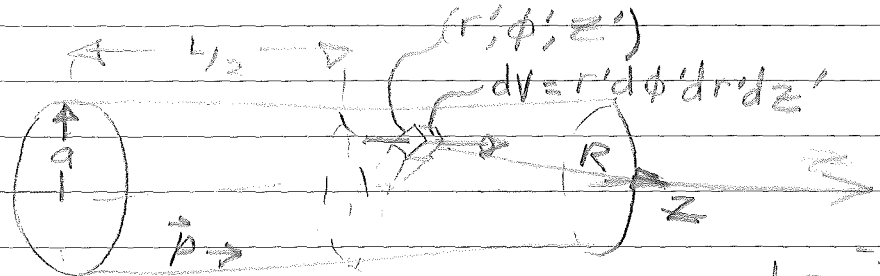
$\Rightarrow z = \sqrt{K'} \int_1^R \frac{dR}{\sqrt{\ln R}} \Rightarrow K = \frac{V^2}{2} = \frac{q_e \rho_0}{2\epsilon_0 m}$

$q_e V = \frac{1}{2} m v^2$

1.5 kV

---

25)



$\phi = \frac{-m \cdot \bar{a}_r}{4\pi\epsilon_0 R^3}$

$F = \nabla\phi = \frac{m}{4\pi\epsilon_0 r^3} (\cos\theta \bar{a}_r + \sin\theta \bar{a}_\theta)$

$m = \rho\, dV$

FOR $z > 0$, ~~ρθ~~ $\bar{a}_r = \bar{a}_z$ ON Z AXIS

FOR $z < 0$, $\theta = \pi$; $a_r = -a_z$

$dE_{AXIS} = \frac{\rho\, dV\, \bar{a}_z}{2\pi\epsilon_0 R^3}$

$E_{AXIS} = \int_{z=-L/2}^{L/2} \int_{\phi'=0}^{2\pi} \int_{r'=0}^{a} \frac{r' d\phi' dr' dz'}{[r'^2 + (z - z')^2]^{3/2}}$

(22)

$\vec{\nabla} \cdot \vec{D} = \rho$

$\int \nabla \cdot \vec{D}\, dv = \oint \vec{D} \cdot d\vec{s} = \int \rho\, dv = q$

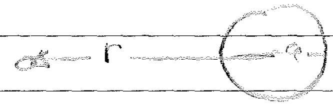
$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

$\vec{\nabla} \cdot \vec{D} = \rho = \epsilon_0 \vec{\nabla} \cdot \vec{E} + \nabla \cdot \vec{P} = \epsilon_0 \vec{\nabla} \cdot \vec{E} - \rho_P$

$\vec{\nabla} \cdot \vec{E} = \rho + \rho_P / \epsilon_0$

$E_{max} = 30\, kV/cm$

$\phi = 5 \times 10^6\, V$



$\oint \vec{D} \cdot d\vec{s} = D_r\, 4\pi r^2 = Q$

$E_r = Q / 4\pi\epsilon_0 r^2$

$\phi = -\int E\, dl = -\int_\infty^r \frac{Q}{4\pi\epsilon_0 r^2}\, dr = Q / 4\pi\epsilon_0 r$

$Q = 4\pi\epsilon_0 r \phi$

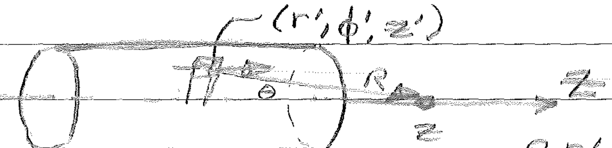
$E_r = \phi / R \quad \frac{E}{} \quad 4\pi\epsilon_0 r \phi / 4\pi\epsilon_0 r^2$

$30 \times 10^3\, cm = \frac{5 \times 10^6}{a} \Rightarrow a = 167\, cm$

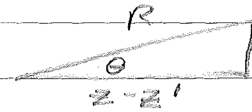
25

THURS.

$$\phi = \frac{\vec{m}\cdot\vec{a}_r}{4\pi\epsilon_0 R^2} = \frac{-\vec{m}\cdot\nabla\left(\frac{1}{R}\right)}{4\pi\epsilon_0}$$



$$E_{AXIS} = \frac{a_z}{2\pi\epsilon_0}\int_{-\frac{l}{2}}^{l/2}\int_0^a\int_0^{2\pi}\frac{\rho\, r'\, d\phi'\, dz'\, dr'\cos\theta}{[r'^2+(z-z')^2]^{3/2}}$$



$$\Rightarrow \cos\theta = \frac{z-z'}{R}$$

$$\vec{E}_{AXIS} = \frac{\rho\bar{a}_z}{2\epsilon_0}\ln\left(\frac{1+\left(\frac{a}{z-l/2}\right)^2}{1+\left(\frac{a}{z+l/2}\right)^2}\right)^{\frac{1}{2}}$$

$$W = \frac{1}{2}\sum_i q_i\phi_i = \frac{1}{2}\int\rho\phi\, dv = \frac{1}{2}\int(\nabla\cdot\vec{D})\phi\, dV$$

DUE TO ALL OTHER CHARGES

NOW $\nabla\cdot(\phi\vec{D}) = \phi\nabla\cdot\vec{D} + \vec{D}\cdot\nabla\phi$

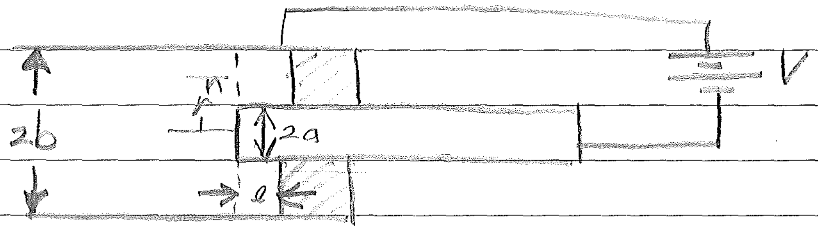
$$\Rightarrow W = \frac{1}{2}\int\left[(-\vec{D}\cdot\nabla\phi) + \nabla\cdot(\phi\vec{D})\right]dv$$

$$= \frac{1}{2}\int\vec{D}\cdot\vec{E}\, dv + \frac{1}{2}\oint\phi\vec{D}\cdot d\vec{s} \to 0$$

$$\therefore W = \frac{1}{2}\sum q_i\phi_i = \frac{1}{2}\int\rho\phi\, dV = \frac{1}{2}\int\vec{D}\cdot\vec{E}\, dV$$

(31)

PROB. 3-16



$\oint \bar{D} \cdot d\bar{S} = q = D_r 2\pi r l$

$E_r = \frac{q}{2\pi \epsilon_0 r l}$

$V = \phi = -\int E \cdot dl$

$= -\int_a^b \frac{q}{2\pi \epsilon_0 r l} dr = \frac{-q}{2\pi \epsilon_0 l} \ln \frac{b}{a}$

$W = \frac{\epsilon_0}{2} \int E^2 dv$

$q = \frac{-2\pi \epsilon_0 l V}{\ln b/a}$

$E_r = \frac{-V}{\ln \frac{b}{a} r} \Rightarrow W = \frac{\epsilon_0}{2} \int \frac{V^2}{R^2 \ln b/a} r \, d\phi \, dr \, dz$

$\vec{W} = \frac{\epsilon_0}{2} \int_0^{2\pi} \int_0^l \int_a^b \frac{V^2 r}{R^2 \ln b/a} d\phi \, dr \, dz$

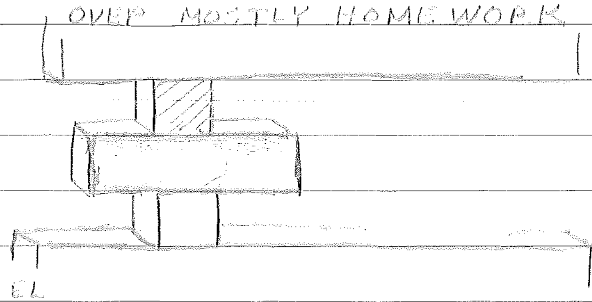
$\Longrightarrow \quad \vec{W} = V^2 \epsilon_0 \pi l / \ln b/a \quad (= \frac{1}{2} C V^2)$

$\vec{F} = \frac{-dw}{dl} = \frac{-\epsilon_0 \pi V^2}{\ln b/a}$

MON

NEXT TUESDAY - OPEN BOOK MULTIPLE GUESS

OVER MOSTLY HOMEWORK



EL

$$d(\vec{D}\cdot\vec{E}) = \vec{D}\cdot d\vec{E} + \vec{E}\cdot d\vec{D}$$

MAG

$$d(\vec{B}\cdot\vec{H}) = \vec{B}\cdot d\vec{H} + \vec{H}\cdot d\vec{B}$$

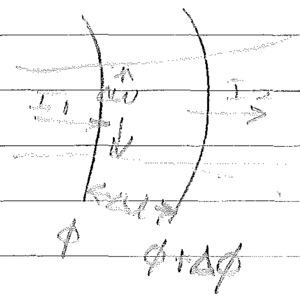
$$W = -\int \vec{F}\cdot dl = -\int q\vec{E}\cdot dl$$

$$\int f_v\, dV = \oint \vec{T}\, ds \Rightarrow T = \epsilon_0\left[(a_n\cdot E)E - \frac{E^2}{2}a_n\right]$$

MON

TEST TOMORROW

CURVILINEAR SQUARES



$E = -\frac{\Delta \phi_1}{\Delta l_1}$

$J = \sigma E = -\sigma\, \Delta\phi_1 / \Delta l_1$

$I_1 = J\, \Delta w_1 (1) = -\sigma\, \Delta\phi_1 \frac{\Delta w_1}{\Delta l_1}$

$I_2 = -\sigma\, \Delta\phi_2 \frac{\Delta w_2}{\Delta l_2}$

FLOW OUT = FLOW IN $\Rightarrow I_2 = I_1$

$\nabla \times E = 0$	$\nabla \times E = 0$
$J = \sigma \bar{E}$	$\bar{D} = \epsilon \bar{E}$
$\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0$	$\Delta \cdot D - \rho = 0$
$\nabla \cdot J = 0$	$\nabla \cdot D = 0$

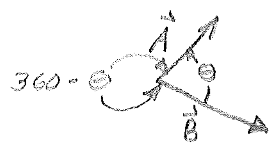
4. Multiplication of a vector $\vec{A}$ by a scalar k is defined to be a vector $\vec{P}$ given by $\vec{P} = k\vec{A} = (kA_x, kA_y, kA_z)$

Physically, $\vec{P}$ is a vector that has a magnitude that is $|k|$ times that of $\vec{A}$ with the same line of action and same sense if k > 0 and opposite sense if k < 0.

$k < 0$
$|k| > 1 \Rightarrow k < -1$

5. The scalar or dot product of two vectors and is a scalar and is defined by

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta = \vec{B} \cdot \vec{A}$$



Equivalently,

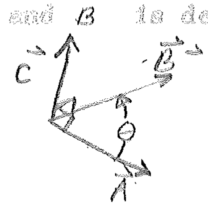
$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

Note the following

a. $\vec{A} \cdot \vec{B}$ can be +ve, -ve, or even zero (when $\theta = 90^\circ$)

b. $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

6. The vector or cross product of two vectors $\vec{A}$ and $\vec{B}$ is defined to be a vector $\vec{C}$ as follows:



a. $|\vec{C}| = |\vec{A}||\vec{B}| \sin\theta$

b. $\vec{C}$ is perpendicular to the plane defined by $\vec{A}$ and $\vec{B}$.

c. $\vec{C}$ obeys the "right hand rule".

Equivalently,

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ \hat{i} & \hat{j} & \hat{k} \end{vmatrix}$$

$$= (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}$$

or in triplet notation

$$\vec{A} \times \vec{B} = (A_x, A_y, A_z) \times (B_x, B_y, B_z)$$
$$= (A_y B_z - A_z B_y,\ A_z B_x - A_x B_z,\ A_x B_y - A_y B_z)$$

7. We shall illustrate the use of triplet notation and vector algebra from selected problems in your textbook:

Prob 3 (246) GIVEN $\vec{a} = (1,3,2)$ $\vec{c} = (3,{}^{-}4,{}^{-}1)$
$\vec{b} = (0,{}^{-}1,4)$

a) $(\vec{a}+\vec{b})\cdot\vec{c} = (1,2,6)\cdot(3,{}^{-}4,{}^{-}1)$
$= (3-8-6) = {}^{-}11$

b) $\vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c} = (3-12-2)+(0+4-4)$
$= {}^{-}11$

Prob 9 (252) GIVEN: $\vec{a} = (1,2,{}^{-}3)$
$\vec{b} = (1,2,0)$
$\vec{c} = ({}^{-}1,1,0)$

$\vec{a}\times(\vec{b}-c) = (1,2,{}^{-}3)$
$\times\,(2,1,0)$
$= (2\times 0-(-3)\cdot 1,\ {}^{-}3\times 2-1\times 0,\ 1\times 1-2\times 2)$
$= (3,{}^{-}6,{}^{-}3)$

8. Some useful properties of vector algebra are the following:

a. $\vec{A}\cdot(\vec{B}\pm\vec{C}) = \bar{A}\cdot\bar{B} \pm \bar{A}\cdot C$

b. $(k\pm m)\bar{A} = k\bar{A} \pm m\bar{A}$

c. $(\bar{A}+\bar{B})+\bar{C} = \bar{A}+(\bar{B}+\bar{C})$

d. $\bar{A}+\bar{B} = \bar{B}+\bar{A}$

e. $k\vec{A} = \vec{A}k$

f. $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

g. $\vec{A}\times\vec{B} = -\vec{B}\times A$

h. $\vec{A}\cdot\vec{B} = 0 \Rightarrow \vec{A} \perp \vec{B}$

i. $\vec{A}\times\vec{B} = \vec{0} \Rightarrow \vec{A} \parallel \vec{B}$

j. $\vec{A}\times(\vec{B}\pm\vec{C}) = \vec{A}\times\vec{B} \pm \vec{A}\times\vec{C}$

k. $(\vec{A}\pm\vec{B})\times\vec{C} = \vec{A}\times\vec{C} \pm \vec{B}\times C$

#. l. $\vec{A}\cdot\vec{A} = |\vec{A}|^2 \equiv A^2$

m. $\vec{A}\times\vec{A} = 0$

TENSOR MANIPULATION OF VECTORS

1. By means of indicial notation it is possible to manipulate vectors in a highly efficient and convenient manner. For this purpose we need only to consider the following items:
   a. Tensor notation of vectors and scalars
   b. Summation convention
   c. Definition of Kronecker delta
   d. Definition of permutation symbol
   e. The $\epsilon\delta$ -identity

   With these five items it will be possible to represent as well as derive numerous identities involving vectors and scalars.

2. We have previously defined a vector in tensor form or in [illegible] "indicial notation" as follows:

   (2-1) $\vec{A} = A_i$

   where it is important to ntoe that "i" is a free index and can have values 1, 2, or 3 in the "real world". Hence, A_i represents a typical vector component and in essence the entire vector $\vec{A}$.

3. A scalar will lack a free index so familiar representations like A x, k, T, etc. are scalars as well some others involving the summation convention.

4. The summation convention introduced by Albert Einstein merely eliminates the summation symbol $\sum$ for the case of two dummy subscripts. The convention can be generally represented by the following

   (4-1) $(\quad)_i(\quad)_i = \sum_{i=1}^{n}(\quad)_i(\quad)_i$

   where "n" is usually 3 in the "real world".

5. Examples of the summation convention are the following:

a) $\vec{A}\cdot\vec{B} = A_1B_1 + A_2B_2 + A_3B_3 = A_iB_i$

b) $a_{ii} = a_{11} + a_{22} + a_{33}$

c) $a_{ij}b_{ik} = a_{1j}b_{1k} + a_{2j}b_{2k} + a_{3j}b_{3k}$ $\qquad (a_{1j} \equiv (a_{11}, a_{12}, a_{13}))$

(DIAD) $\vec{A}\vec{B} = A_xB_x\hat{i}\hat{i} + A_xB_y\hat{i}\hat{j} + A_xB_z\hat{i}\hat{k}$
$+ A_yB_x\hat{j}\hat{i} + A_yB_y\hat{j}\hat{j} + A_yB_z\hat{j}\hat{k}$
$+ A_zB_x\hat{k}\hat{i} + A_zB_y\hat{k}\hat{j} + A_zB_z\hat{k}\hat{k} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$

6. The Kronecker delta is probably familiar and will be defined below:

(6-1) $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

In other words, $\delta_{11} = \delta_{22} = 1$ but $\delta_{13} = \delta_{20,1} = 0$

7. To illustrate the use of the Kronecker delta consider the following examples:

a) $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$

b) $\delta_{ij}A_j = ?$

$= \delta_{i1}A_1 + \delta_{i2}A_2 + \delta_{i3}A_3$

FOR $i = 1,\ \delta_{ij}A_j = A_1$

$i = 2,\ \delta_{ij}A_j = A_2$

$i = 3;\ \delta_{ij}A_j = A_3$

$\therefore \delta_{ij}A_j = A_i$

8. The permutation symbol offers a compact means for representing a cross product. It is defined byt the following

(8-1) $\epsilon_{ijk} = \begin{cases} 1 & \text{IF } ijk = 123,\ 231,\ \text{OR } 312 \\ -1 & \text{IF } ijk = 321,\ 213,\ \text{OR } 132 \\ 0 & \text{FOR ALL OTHER CASES} \end{cases}$ $\qquad \overset{\xrightarrow{+1}}{\underset{\xleftarrow[-1]{}}{123}}$

9. With the permutation symbol the cross product can be represented as follows:

(9-1) $\vec{A}\times\vec{B}\Big|_{i^{TH}\text{COMP}} = \epsilon_{ijk}A_jB_k$

Let us verify this below:

In the usual manner we have in triplet notation,

$$\vec{A}\times\vec{B} = (A_1, A_2, A_3) \times (B_1, B_2\ B_3) = (A_2B_3 - A_3B_2,\ A_3B_1 - A_1B_3,\ A_1B_2 - A_2B_1)$$

For i = 1 from (9-1) we have

$$\vec{A}\times\vec{B}\Big|_{1ST\ COMP} = \epsilon_{123}A_2B_3 + \epsilon_{132}A_3B_2$$

$$= A_2B_3 - A_3B_2$$

For i = 2 there results

$$\vec{A}\times\vec{B}\Big|_{2nd\ COMP} = \epsilon_{213}A_1B_3 + \epsilon_{231}A_3B_1$$

$$= A_3B_1 - A_1B_3$$

For i = 3 we get

$$\vec{A}\times\vec{B}\Big|_{3RD} = \epsilon_{312}A_1B_2 + \epsilon_{321}A_2B_1$$

$$= A_1B_2 - A_2B_1$$

Therefore, $\vec{A}\times\vec{B} = \epsilon_{ijk}A_jB_k$

10. Note two inherent properties of the permutation symbol as follows:

(10-1) $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{\substack{kji \\ kij}}$

and

(10-2) $\epsilon_{ijk} = -\epsilon_{kji}$

11. The so-called $\epsilon\delta$-identity can be proved as a theorem with considerable effort and is given by the following:

(11-1) $\epsilon_{ijk}\epsilon_{irs} = \delta_{jr}\delta_{ks} - \delta_{js}\delta_{kr}$

The proof follows from the basic definitions of the Kronecker delta and permutation symbol and will not be done here. Note that there must be one common subscript only while the others are different.

12. As an example of the application of the $\epsilon\delta$-identity let us prove (1) on page 258 of your textbook as follows:

(1) $\vec{b}\times(\vec{c}\times\vec{d}) = (\vec{b}\cdot\vec{d})\vec{c} - (\vec{b}\cdot\vec{c})\vec{d}$

We proceed as follows, beginning with the left hand side of (1)

$$\vec{b}\times(\vec{c}\times\vec{d}) = \epsilon_{ijk}\, b_j\, \epsilon_{k\ell m}\, c_\ell\, d_m$$

$$= \epsilon_{kij}\,\epsilon_{k\ell m}\, b_j\, c_\ell\, d_m$$

$$= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})\, b_j\, c_\ell\, d_m$$

$$= \delta_{i\ell}\, c_\ell\, \delta_{jm}\, d_m\, b_j - \delta_{im}\, d_m\, \delta_{j\ell}\, c_\ell\, b_j$$

$$\delta_{ij} A_j = A_i$$

$$\therefore\ \vec{b}\times(\vec{c}\times\vec{d}) = c_i d_j b_j - d_i c_i b_i$$

$$= (\vec{b}\cdot\vec{d})\vec{c} - (\vec{b}\cdot\vec{c})\vec{d}$$

where we have used freely the commutative law for scalar components as well as relationships in items 2, 5, and 7.

SCALAR AND VECTOR FIELDS

1. Given a space (3-dimensional in most engineering situations) if at each point in the spade there is defined a scalar quantity or a vector quantity, the situation is called a scalar or a vector field, respectively.

2. In notational form if P is a point in space and f(P) or $\vec{V}(P)$ are the unique associated scalar or vector values, respectively, then f(P) and $\vec{V}(P)$ denote scalar and vector fields.

3. In 3-dimensional space $f(P) = f(x,y,z)$ and $\vec{V}(P) = \vec{V}(x,y,z)$.

4. Fields also exist for "time space" as follows: $f(t)$ and $\vec{V}(t)$ We also have fields in spaces that are both geometric and time in nature so that we may have $f(x,y,z,t)$ and $\vec{V}(x,y,z,t)$.

5. A geometric space is said to be spanned by three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ if any vector $\vec{V}$ in the space is given by

$$\vec{V} = k_1\vec{a} + k_2\vec{b} + k_3\vec{c}$$

6. The particular vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ will span a 3-dimensional vector space since any vector in this space can be represented in terms of these particular three.

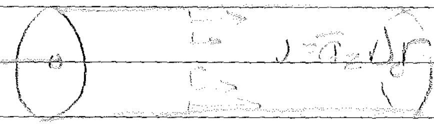
7. There are an infinite variety of vectors that will span a space, but the essential ingredient is that the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ be non-parallel and non-coplanar.

8. A simple test (see page 256, Theorem 1, in your textbook) for 3 vectors to be non-parallel and non-coplanar is the following:

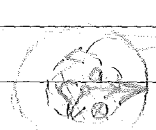
(8-1) $\vec{a}\cdot\vec{b}\times\vec{c} \neq 0$

This is called the scalar triple product.

6-13)



$(\vec{B} = B_\phi \vec{a}_\phi) \quad (d\vec{l} = r\,d\phi\,\vec{a}_\phi)$



$\oint \vec{B}\cdot d\vec{l} = \mu_0 I = \mu_0 \int \vec{J}\cdot d\vec{s}$

$B\,2\pi r = \mu_0 \int_0^{2\pi}\int_0^{r} (J_0 r \vec{a}_z)\cdot(r\,d\phi\,dr\,\vec{a}_z)$

⊕ INTO $\quad = \mu_0 \int_0^{2\pi}\int_0^{r} J_0 r^2\,dr\,d\phi$

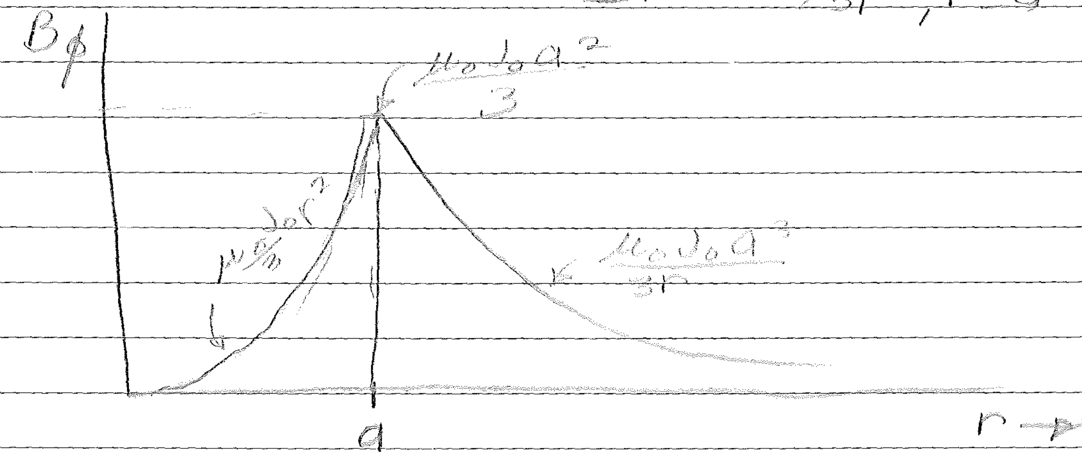
⊙ OUT OF $\quad = \frac{2\pi J_0 r^3}{3}\mu_0$

$\Rightarrow B = \frac{\mu_0 J_0 r^2}{3}$ FOR $r < a$

FOR $r > a$; $B_\phi(2\pi r) = \int_{r=0}^{a}\int_0^{2\pi} \vec{J}\cdot d\vec{s}$

$= \frac{2\pi J a^3 \mu_0}{3}$

$= \mu_0 J_0 a^2/3r$; $r > a$



MONDAY

CLASS TO BE RESECTIONED - GOT AFTERNOON SECTION

5-12) $W = \frac{1}{2}\int \vec{D}\cdot\vec{E}\,dV = \frac{\epsilon}{2}\int E^2\,dV$

DEF $\vec{D} = \epsilon_0\vec{E} + \vec{P}$ $\qquad \vec{P} = \chi_e \epsilon_0 \vec{E}$ (SPECIAL CASE)

$\Rightarrow \vec{D} = \epsilon_0(1+\chi_e)\vec{E}$

$= K_e \epsilon_0 \vec{E} = \epsilon\vec{E}$

DEF $\vec{B} = \mu_0(\vec{H} + \vec{M})$ $\qquad \vec{M} = \chi_m \vec{H}$ (SPECIAL CASE)

$\Rightarrow \vec{B} = \mu_0(1+\chi_m)\vec{H}$

$= K_m \mu_0 \vec{H}$

$= \mu \vec{H}$

$\nabla\times\vec{B} = \mu_0(\nabla\times\vec{H} + \nabla\times\vec{M}) = \mu_0 \vec{J}$

(DO 3-16) (3-2, 3-4 → HALL EFFECT)

TUES

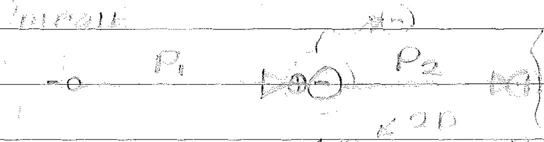
REVIEW

$\vec{B} = \mu_0(\vec{H} + \vec{M})$ | $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

$\nabla \cdot \vec{B} = \mu_0(\nabla \cdot \vec{H} + \nabla \cdot \vec{M}) = 0$ | $\nabla \cdot \vec{D} = \epsilon_0 \nabla \cdot \vec{E} + \nabla \vec{P} = \rho$ (FREE CHARGE DENSITY)

$\mu_0 \nabla \cdot \vec{H} = -\mu_0 \nabla \cdot \vec{M} = \rho_m$ (MAGNETIC CHARGE DENSITY) | $\epsilon_0 \nabla \cdot \vec{E} = \rho + \rho_p$, $\rho_p = -\nabla \cdot \vec{P}$

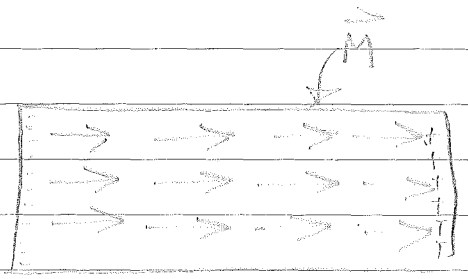
DIPOLE



$\rho_p = -\nabla \cdot \vec{P} = -\frac{\delta P_x}{\delta x} \approx -\frac{P_x(x+\Delta x) - P(x)}{\Delta x}$

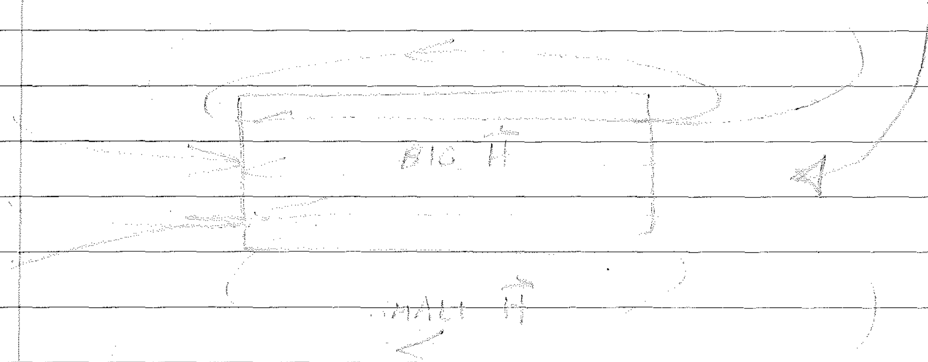
$= -\frac{P_2 - P_1}{\Delta x} < 0$

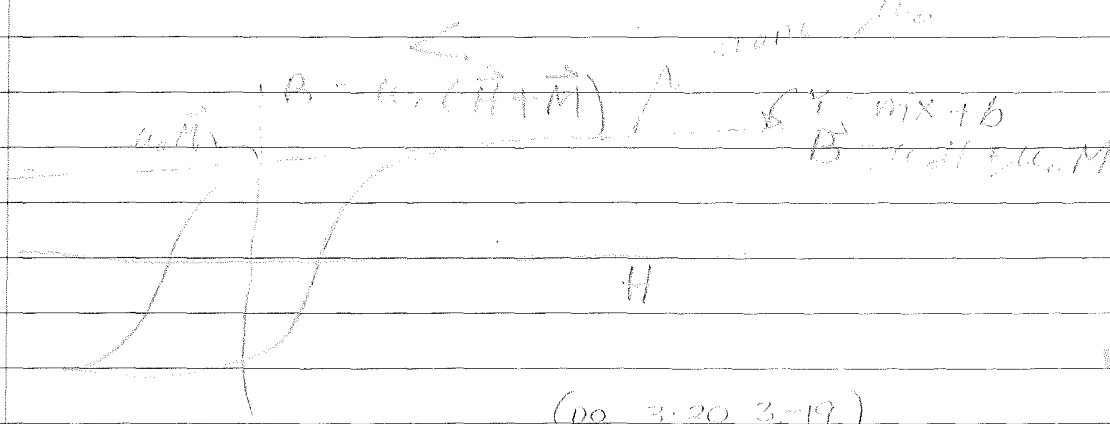
$\vec{B} = \mu_0(\vec{H} + \vec{M})$ $\rightarrow M$



$\rho_m = \mu_0 \vec{H}$

$B \propto \vec{M} + \vec{H}$



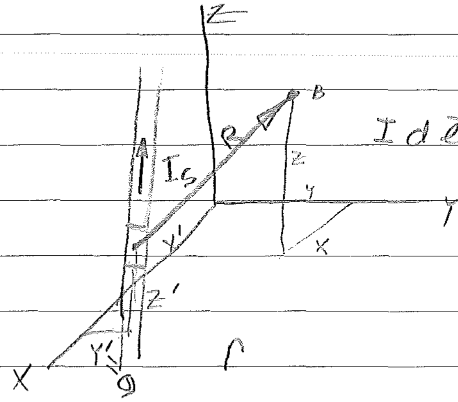


(DO 3-20, 3-19)

FU

WED

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{I\, d\ell \times \bar{a}_r}{R^2}$$

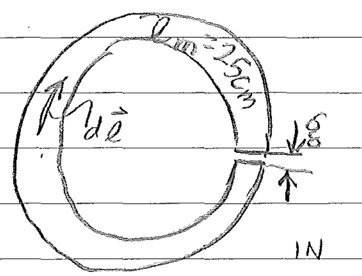


$$\vec{R} = (x - x')\vec{a}_x + Y\vec{a}_Y + (z - z')\vec{a}_z = R\vec{a}_r$$

$$I\, d\vec{\ell} = I_s\, dx'\, dz'\, \vec{a}_z$$

$$\Rightarrow \vec{B} = \frac{\mu_0}{4\pi}\int_{z'=-\infty}^{z'=\infty}\int_{x'=-d/2}^{d/2} \frac{I\, dx'\, \vec{a}_z\, dz' \times [(x-x')\vec{a}_x + Y\vec{a}_Y + (z-z')\vec{a}_z]}{[(x-x')^2 + Y^2 + (z-z')^2]^{3/2}}$$

PROB 3-19



$$\oint \vec{H} \cdot d\vec{\ell} = I = 0$$

$$H_m l_m + H_g l_g = 0$$

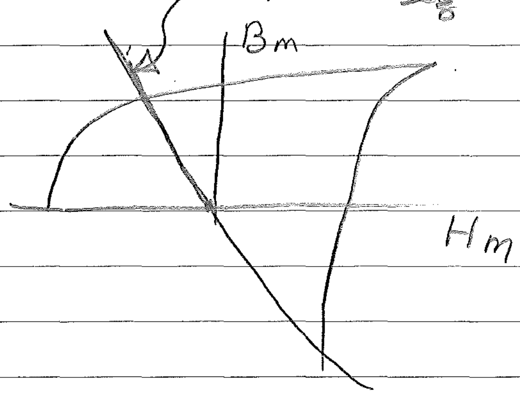
$$B_g = B_m$$

($B \neq \mu H$ MUST GET FROM B-H CURVE)

IN THE GAP $B_g = \mu_0 H_g$ $(= B_m)$

$$\Rightarrow H_m l_m + H_g l_g = 0 = H_m l_m + \frac{B_m}{\mu_0} l_g$$

$$\Rightarrow B_m = \frac{-\mu_0 l_m}{l_g} H_m = \frac{-4\pi 10^{-7}(0.25)}{(0.1 \times 10^{-2})} H_m = -\pi 10^{-4} H_m$$



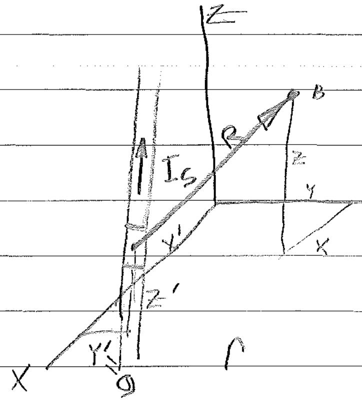
TWO ANSWERS FOR EACH PART, DEPENDING ON CURRENT DIRECTION

$H_m = -40 \Rightarrow B_m = 4\pi \times 10^{-3}\ \frac{Wb}{m^2}$

FU

WED

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{I\,d\ell \times \bar{a}_r}{R^2}$$

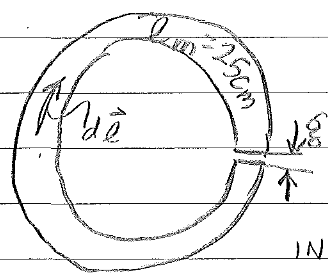


$$\vec{R} = (x - x')\,\vec{a}_x + y\,\vec{a}_y + (z - z')\,\vec{a}_z = R\bar{a}$$

$$I\,d\vec{\ell} = I_s\,dx'\,dz'\,\bar{a}_z$$

$$\Rightarrow \vec{B} = \frac{\mu_0}{4\pi}\int_{z'=-\infty}^{z'=\infty}\int_{x'=-d/2}^{d/2} \frac{I\,dx'\,\vec{a}_z \times [(x-x')\bar{a}_x + y\bar{a}_y + (z-z')\bar{a}_z]\,dz'}{[(x-x')^2 + y^2 + (z-z')^2]^{3/2}}$$

PROB 3-19



$$\oint \vec{H}\cdot d\vec{\ell} = I = 0$$

$$H_m \ell_m + H_g \ell_g = 0$$

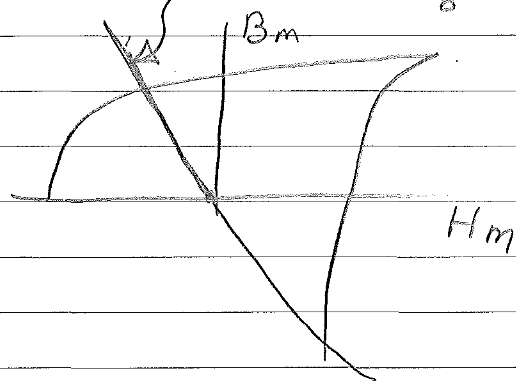
$$B_g = B_m$$

($B \neq \mu H$ MUST GET FROM B-H CURVE)

IN THE GAP $B_g = \mu_0 H_g$ $(= B_m)$

$$\Rightarrow H_m \ell_m + H_g \ell_g = 0 = H_m \ell_m + \frac{B_m}{\mu_0}\ell_g$$

$$\Rightarrow B_m = \frac{-\mu_0 \ell_m}{\ell_g} H_m = \frac{-4\pi 10^{-7}(0.25)}{(0.1 \times 10^{-2})} H_m = -\pi 10^{-4} H_m$$

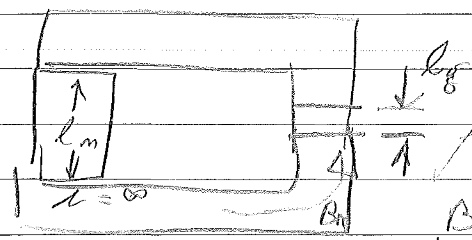


TWO ANSWERS FOR EACH PART, DEPENDING ON CURRENT DIRECTION

$H_m = -40 \Rightarrow B_m = 4\pi \times 10^{-3}\ \frac{WEB}{m^2}$

3-20) IN MAGNETIC FIELD

$W = \int \vec{B} \cdot H \, dV)$

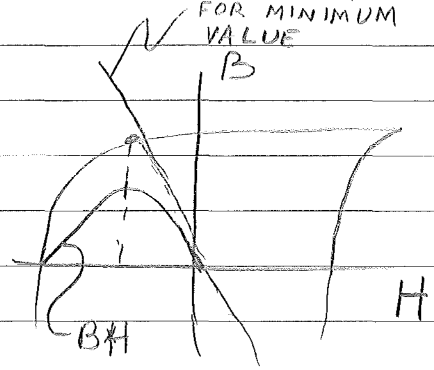


$H_m l_m + H_g l_g = 0$

$B_m A_m = B_g A_g \Rightarrow A_m = B_g A_g / B_m$

$A_m l_m = -A_m l_g H_g / H_m$

$A_m l_m = -\dfrac{l_g H_g B_g A_g}{H_m B_m} = \dfrac{-B_g^2}{\mu_0 H_m B_m} A_g l_g$ $\qquad B_g = \mu_0 H_g$



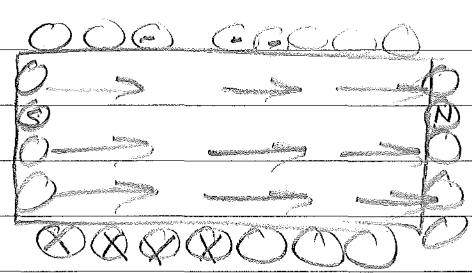
$\vec{J}_p = \dfrac{\delta \vec{P}}{\delta t}$

$\nabla \cdot \vec{J}_p = \dfrac{\delta}{\delta t}(\nabla \cdot \vec{P}) = -\dfrac{\delta \rho_p}{\delta t}$

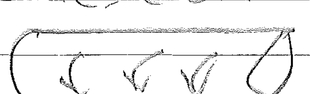
$\Rightarrow \nabla \cdot \vec{J}_p + \dfrac{\delta \rho_p}{\delta t} = 0$

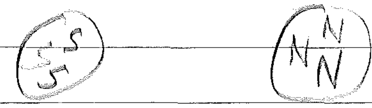
CHARGES AND CURRENTS (OR NOT AND) DIPOLE MOMENTS

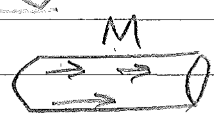
(7-1 7-2)



ELEC CUR. MAGNETIC POLES (OR CHARGES





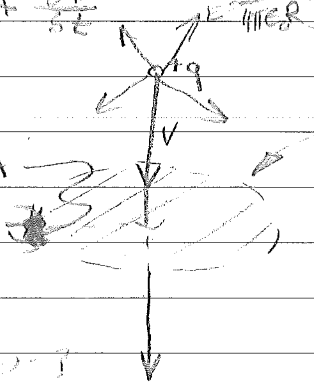


ALL 3 ARE EQUIVALENT

TUES.

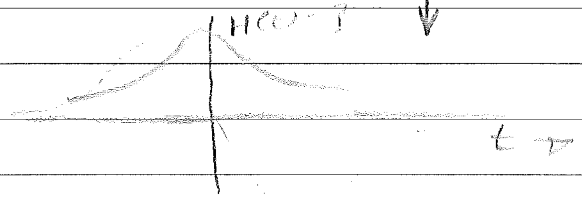
TEST ON 5, 6, 7, 8

$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$ $\qquad \oint \vec{H} \cdot d\vec{l} = \frac{\partial}{\partial t} \int \vec{D} \cdot d\vec{S}$

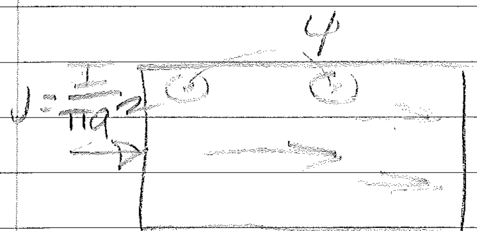


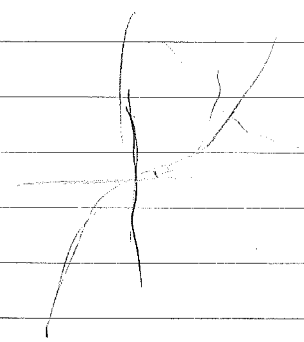
COMPUTE $\vec{H}(t)$ @ A

$D = \epsilon_0 E$



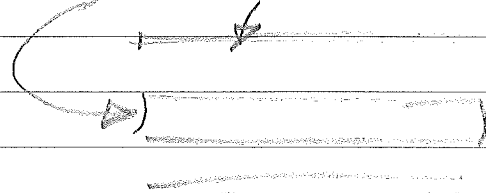
$L = \Psi / I = N\phi / I$





$W = \frac{1}{2} \int \vec{B} \cdot \vec{H} \, dv = \frac{1}{2} L I^2$

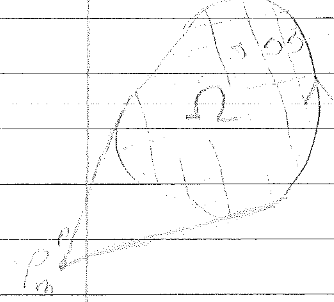
$\left(L = \frac{\mu_0}{8\pi} + \frac{\mu_0}{2\pi} \ln b/a \ \ H/m\right)$



THURS

7-8) $\varphi_m = \frac{1}{4\pi} \int \frac{\bar{r} \cdot d\bar{s}}{r^3}$

$$\varphi_m = \frac{-1}{4\pi} \int \mu \cdot \nabla \left(\frac{1}{r}\right) d\bar{V}$$



$I \Delta \bar{s}$

$\frac{\bar{a}_r}{R^2}$

$$= \frac{-1}{4\pi} \int \mu \cdot \bar{a}_r \, dV' / R^2$$

$$= \frac{1}{4\pi} \int \frac{I d\bar{s} \cdot \bar{a}_r}{R^2}$$

ON 5, 6, 7, 8 (ALLOWED 1 $8\frac{1}{2}$ BY 11 SHEET)

CHAPT 9.

$\nabla \times \bar{E} = \frac{-\delta \bar{B}}{\delta t}$

$\nabla \times \bar{H} = \bar{J} + \frac{\delta \bar{D}}{\delta t}$

$\nabla \cdot \bar{D} = \rho$

$\nabla \cdot \bar{B} = 0$

$\nabla \cdot \nabla \times \bar{H} = \nabla \cdot J + \frac{\delta}{\delta t} \nabla \cdot \bar{D} = 0$

$\Rightarrow$ CONS. OF CHARGE FROM MAXWELL

$\nabla \cdot \bar{J} + \frac{\delta \rho}{\delta t} = 0$

$\bar{B} = \nabla \times \bar{A}$

$\nabla \times \bar{E} = -\frac{\delta}{\delta t}(\nabla \times \vec{A})$

$\nabla \times (\vec{E} - \frac{\delta \bar{A}}{\delta t}) = 0$

$\bar{E} - \frac{\delta \bar{A}}{\delta t} = -\nabla \phi$

$\Rightarrow \vec{E} = -\nabla \phi + \frac{\delta \bar{A}}{\delta t}$

FOR NON-TIME VARYING:

$$\phi = \frac{1}{4\pi\epsilon} \int \frac{\rho}{R} dV \quad \text{AND} \quad \vec{A} = \frac{1}{4\pi} \int \frac{\bar{J}}{R} dV$$

$\nabla \times \nabla \times E = \frac{-\delta}{\delta t}(\nabla \times \bar{B}) = \frac{-\delta}{\delta t}\left(\mu J + \mu \frac{\delta \bar{D}}{\delta t}\right)$

$\bar{B} = \mu H \qquad \bar{D} = \epsilon \bar{E}$

$\Rightarrow \underbrace{\nabla \times \nabla \times \vec{E}}_{\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}} + \mu \epsilon \frac{\delta^2 \bar{E}}{\delta t^2} = -\mu \frac{\delta J}{\delta t}$

$\nabla \cdot \vec{E} = 0$ IF $\rho = 0$

$\Rightarrow \nabla^2 E - \mu \epsilon \frac{\delta^2 \bar{E}}{\delta t^2} = \mu \frac{\delta \bar{J}}{\delta t}$ (WAVE EQ)

$\bar{J} = \sigma \bar{E}$

$\nabla^2 E - \mu \sigma \frac{\delta \bar{E}}{\delta t} - \mu \frac{\delta^2 \bar{E}}{\delta t^2} = 0$

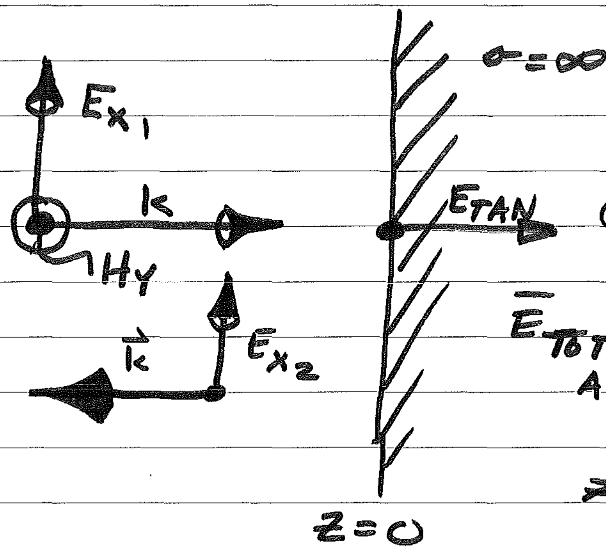
$\frac{d^2 v}{dx^2} - (LR + GC) \frac{\delta v}{\delta t} - LC \frac{\delta^2 v}{\delta t^2} = 0$

THURS.

9-7) $\vec{E} = jE_0 \sin k_0 z \,\vec{a}_z$

$\vec{H} = \frac{1}{\eta} E_0 \cos k_0 z \,\vec{a}_Y$

$\eta = \sqrt{\mu_0/\epsilon_0}$



LET $E_{x_1} = E_1 e^{j(\omega t - \beta z)}$
$= E_1 e^{j(\omega t - kz)}$

(OR) $E_{x_1} = E_1 e^{-jkz}$

$E_{x_2} = E_1 e^{jkz}$

$\bar{E}_{TOT} = E_1 e^{-jkz} + E_2 e^{jkz}$

ALSO $E_{TOT} = E_{TAN}\big|_{z=0}$

$= E_1 + E_2$

$\cancel{\Rightarrow E_1 = E_2} \Rightarrow E_1 = -E_2$

$E_{TOT} = E_1 (e^{-jkz} - e^{jkz})$

$= -j\underbrace{2E_1}_{\substack{\| \\ E_0 \text{ IN } 9\text{-}7}} \sin kz$

BACK TO 9-7

$\vec{S} = \vec{E} \times \vec{H}$ = PWR FLOW / M^2 = $\frac{\text{WATTS}}{M^2}$

$\vec{E}(x, y, z, t) = Re(e^{j\omega t} E)$

$= -E_0 \sin kz \sin \omega t \,\vec{a}_x$

$\vec{H}(x, y, z, t) = \frac{E_0}{\eta} \cos kz \cos \omega t \,\vec{a}_Y$

$\Rightarrow \vec{S} = \frac{-E_0^2}{2} \sin kz_0 \cos k_0 z \sin \omega t \cos \omega t \,\bar{a}_z$

$\vec{S}_{AVE} = \frac{1}{T} \int_0^T \frac{-E_0^2}{2} \sin kz \cos kz \sin \omega t \sin \omega t \,\bar{a}_z \, dt$

$\ni T = 2\pi/\omega$

$P_{AV} = \frac{1}{T}\int_0^T v\,i\,dt = \frac{1}{2}Re(VI^*)$

SIMILARLY; $\bar{S}_{AVE} = \frac{1}{2}Re(\bar{E}\times\bar{H}^*)$

$= \frac{1}{2}Re[(jE_0 \sin k a_x)\times(\frac{E_0}{\eta}\cos k_0 z \bar{a}_y)]$

$= 0$

MON

TUES

WORK 10.5, 10.6

SET WAVE EQ - RESTRICT IT, ~FOR SINUSOIDAL

-MOST GENERAL SOLUTION~

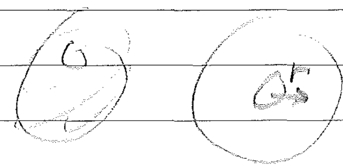
WED.

FOR TEM MODE

$$\nabla^2 \bar{E} + K^2 \bar{E} = \nabla^2_{xy} E + \nabla^2_z E_z + k^2 E_z = 0$$

ALSO $\nabla^2_{xy} \bar{E} = 0$

$(E(x,y)e^{j(\omega t + \beta z)}$



$\int D \cdot ds = q$

$E_r = \frac{q}{2\pi\epsilon_0 r}$

$V = \frac{q}{2\pi\epsilon_0} \ln b/a$

$E_r = \frac{V}{r \ln b/a}$

$\bar{E} = \frac{V}{r \ln b/a} e^{j(\omega t - \beta z)} \bar{a}_r$

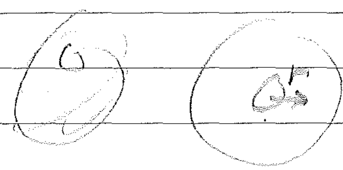
WED.

FOR TEM MODE

$$\nabla^2 \bar{E} + k^2 \bar{E} = \nabla^2_{xy} E + \nabla^2_z E_z + k^2 E_z = 0$$

ALSO $\nabla^2_{xy} \bar{E} = 0$

$(E(x,y)\, e^{j(\omega t \ldots)}$



$\int D \cdot ds = q$

$E_r = \frac{q}{2\pi\epsilon_0 r}$

$V = \frac{q}{2\pi\epsilon_0} \ln b/a$

$E_r = \frac{V}{r \ln b/a}$

$$\bar{E} = \frac{V}{r \ln b/a}\, e^{j(\omega t - \beta z)}\, \bar{a}_r$$

SOLUTIONS

$$D = \frac{Q}{A} = \epsilon E = \epsilon_0 \frac{2d}{x+d} E \qquad E = \frac{Q(x+d)}{2\epsilon_0 d A}$$

1 $$V = \int E\,dx = \int_0^d \frac{Q(x+d)}{2\epsilon_0 d A}\,dx = \frac{3dQ}{4\epsilon_0 A} \qquad \boxed{C = \frac{Q}{V} = \frac{4\epsilon_0 A}{3d}}$$

$$\vec{P} = (\epsilon_r - 1)\epsilon_0 \vec{E} = \left(\frac{2d}{x+d} - 1\right)\epsilon_0 \frac{Q(x+d)}{2\epsilon_0 d A}(-\bar{a}_x) = \frac{-Q}{2dA}(d-x)\bar{a}_x$$

$$\boxed{\rho_p = -\nabla \cdot \vec{P} = -\frac{\partial P_x}{\partial x} = \frac{-Q}{2dA}} \qquad \boxed{\rho_{ps} = \vec{P} \cdot \bar{a}_n \Big|_{x=d} = 0}$$

2. $$\oint \bar{H} \cdot d\bar{l} = H 2\pi r = NI$$

$$\phi = \int \bar{B} \cdot d\bar{S} = \int_a^b \frac{\mu NI}{2\pi r}(t\,dr) = \frac{\mu NIt}{2\pi} \ln \frac{b}{a}$$

$$L = \frac{N\phi}{I} = \frac{\mu N^2 t}{2\pi} \ln \frac{b}{a}$$

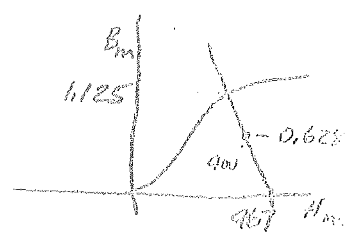
3 $$\oint \bar{H} \cdot d\bar{l} = I \qquad H_m l_m + H_g l_g = NI \qquad B_g = \mu_0 H_g = B_m$$

$$H_m l_m + \frac{B_m}{\mu_0} l_g = NI$$

$$B_m = \mu_0\left(-\frac{l_m}{l_g} H_m + \frac{NI}{l_g}\right) = \mu_0(-7500 H_m + 3.5 \times 10^6)$$

At $B_m = 0$

$$H_m = \frac{3.5 \times 10^6}{7.5 \times 10^3} = 467 \frac{a\text{-}t}{m} \quad \text{(HORIZONTAL INTERCEPT)}$$



At $H_m = 400$

$$B_m = \mu_0(-7.5(0.4) + 3.5)10^6 = 0.628 \text{ wb/m}^2$$

FROM PLOT

$$B_m = 1.125 \frac{\text{wb}}{\text{m}^2} = B_g$$

Also

$$B_m = \mu_0\left(\frac{-l_m}{l_g} H_m + \frac{NI}{l_g}\right) = \mu_0(H_m + M)$$

$$M = \frac{NI}{l_g} - \left(1 + \frac{l_m}{l_g}\right) H_m = 3.5 \times 10^6 - 7501 H_m = 3.5 \times 10^6 (1 - 0.75)$$

$$= 875 \times 10^3 \text{ a-t/m}$$

4. $\vec{E} = \vec{v} \times \vec{B} = \omega r B_0 \vec{a}_r$

$$\int \vec{E} \cdot d\vec{l} = \int_0^a \omega r B_0 \, dr = \omega B_0 \frac{a^2}{2} \qquad V = -\omega B_0 \frac{a^2}{2}$$

5. $\rho_m = -\nabla \cdot \vec{M} = -\frac{\partial M_z}{\partial z} = -2z M_0 (a-r)$

$\rho_{sm} = \vec{M} \cdot \vec{a}_n = M_0 (a-r) \frac{L^2}{4}$ at $z = \pm \frac{L}{2}$

$\vec{J}_m = \nabla \times \vec{M} = -\bar{a}_\phi \frac{\partial M_z}{\partial r} = \bar{a}_\phi M_0 a z^2$

$\vec{J}_{ms} = \vec{M} \times \vec{a}_n = 0$ at $r = a$.

6. $V = 0$ ($\vec{E}$ inside of a perfect conductor $= 0$ always!)

In magnetic circuit problems where the length l and the cross-sectional area A of the flux paths are well defined, it is convenient to define the reluctance of the magnetic path as

$$\mathcal{R} = \frac{l}{\mu A} \qquad \text{amp-turns/weber} \tag{3.61}$$

The reluctance $\mathcal{R}$ of a flux path, the mmf Ni which causes the flux, and the flux ϕ are related by

$$\phi = \frac{Ni}{\mathcal{R}} \qquad \text{webers} \tag{3.62}$$

The induced emf caused by a changing magnetic flux is given by

$$\oint \mathcal{E} \cdot ds = -\frac{d\phi}{dt} \qquad \text{volts} \tag{3.63}$$

If a conductor of length l is moving with a velocity $\mathbf{u}$ perpendicularly to a magnetic field of flux density B, the voltage induced between the ends is

$$v = Blu \qquad \text{volts} \tag{3.64}$$

The energy density of a magnetic field is

$$U = \tfrac{1}{2}\mathbf{H} \cdot \mathbf{B} \qquad \text{joules/meter}^3 \tag{3.65}$$

For an electromagnetic wave, the velocity of propagation is given by

$$u = 1/\sqrt{\mu\epsilon} \qquad \text{meters/sec} \tag{3.66}$$

The ratio of the electric field intensity to the magnetic field intensity in a plane electromagnetic wave is

$$\frac{\mathcal{E}}{H} = \sqrt{\frac{\mu}{\epsilon}} \qquad \text{ohms} \tag{3.67}$$

PROBLEMS

3.1. An electron is traveling parallel to a straight, current-carrying conductor. If the speed of the electron is 10^6 meters/sec and the current in the conductor is 50 amp, what force is exerted on the electron if it is 10 cm from the wire? (The charge of an electron is $q_e = -1.602 \times 10^{-19}$ coulomb.)

● 3.2. Consider an xyz coordinate system. The electric field is known to be zero in this test. An electron is projected through the origin at a speed of 2.00×10^6 meters/sec. If it is projected in the direction of the x axis, the sidewise acceleration is found to be zero. If it is projected in the direction of the positive y axis, the acceleration is found to be 1.78×10^{17} meters/sec² in the positive z direction. What are the direction and magnitude of the magnetic flux density B at the origin at this moment?

● 3.3. Refer to Fig. P3.3. A nonmagnetic insulating disk is inclined with its axis at an angle θ to a magnetic field B and is rotating at an angular speed of ω rad/sec. A point charge q is located on the periphery of the disk. Find the instantaneous torque when the charge is in positions 1, 2, 3, and 4. The radius of the disk is r. Note that the torque tends to turn the axis of the disk into line with B.

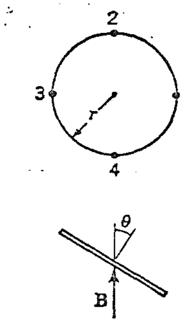


FIG. P3.3. Charge rotating in magnetic field.

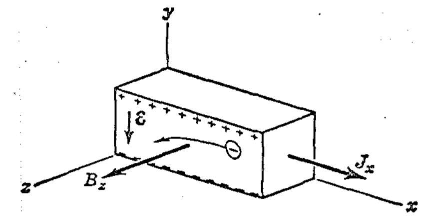


FIG. P3.4. Hall effect.

● 3.4. This problem concerns the *Hall effect*. Consider a solid body such as that shown in Fig. P3.4. The substance carries a current density J_x and is subjected to a magnetic field B_z. The sketch shows the effect if the current is one of negative charges (i.e., electrons). The magnetic field deflects the moving charges downward, thus charging the lower face negatively and leaving the upper face with a positive charge. This produces an electric field in the negative y direction that, in equilibrium, annuls the deflecting effect of the magnetic field on the charge carriers.

(*a*) Show that, if the current is that of positive charge carriers (i.e., holes), the electric field is opposite, i.e., in the positive y direction.

(*b*) The Hall coefficient is defined by $R_H = \mathcal{E}_y/J_xB_z$. Assume that all the charge carriers are alike and that each has a charge q (positive for holes, negative for electrons). The number of charge carriers per unit volume is N. Express R_H in terms of q and N.

(*c*) For copper at room temperature, $R_H = -5.5 \times 10^{-11}$ volt-meter³/weber-amp. The electronic charge is -1.602×10^{-19} coulomb. Compute the number of free electrons per unit volume. Compare this with the number of atoms per unit volume as computed by Avogadro's number, which is 6.02×10^{23} atoms per gram atom; i.e., the number of atoms per gram of a substance is $6.02 \times 10^{23}/A$, where A is the atomic weight of the substance. The atomic weight of copper is 63.6, and its density is 8.89 grams/cm³.

(*d*) The Hall coefficient for zinc is positive. What does this fact suggest concerning the charge carriers in zinc?

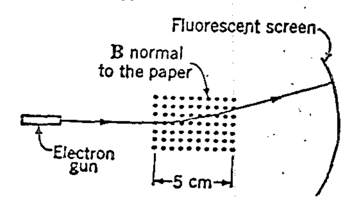


FIG. P3.5. Magnetic deflection of electron beam.

3.5. In some cathode-ray tubes the deflection of the beam of electrons is accomplished with an electric field; in others, by a magnetic field. Figure P3.5 shows the magnetic field scheme. If the electrons enter the region of the magnetic field with a velocity of 22.9×10^6 meters/sec, determine the magnetic flux density B that is required to make the beam emerge at an angle of 10°.

3.6. Consider two parallel wires of diameter much smaller than d, the distance between their centers. Suppose that the current carried by the wires is 5000 amp in opposite directions. If the spacing between them is 0.5 meter, calculate the force in pounds between two 10-meter-length sections of the wires. (One pound is equivalent to 4.448 newtons.) Is the force one of repulsion or attraction?

3.7. In the d'Arsonval meter of Fig. P3.7, the poles of the permanent magnet are shaped around the cylindrical iron core so that there is a uniform, radial magnetic

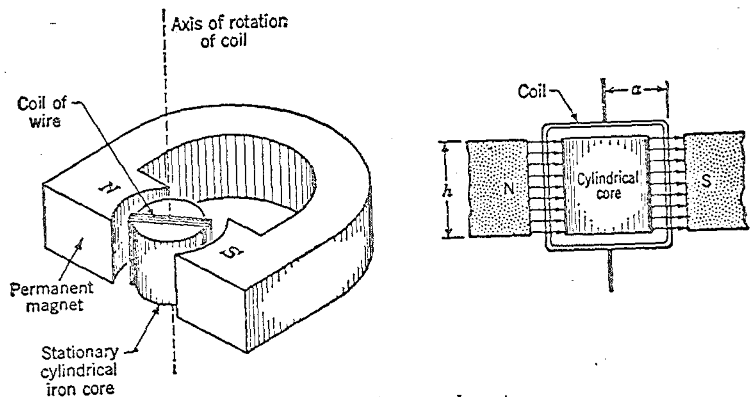


FIG. P3.7. D'Arsonval meter.

field B in the air gap. The rectangular coil has N turns and a radius a. The magnetic field acts over a height h. The coil carries a current of i amp.

(a) Determine an expression for the torque tending to turn the coil.

(b) The coil turns against a spring that resists rotation with a torque $\mathfrak{T} = k\theta$, where k is the torsional spring constant in newton-meters/radian and θ is the angular deflection in radians. Write the expression for the equilibrium value of θ.

3.8. A uniform magnetic field with a density of B webers/meter² emerges through a small plane area of A meters². The angle between B and the normal to the area is θ. Write the expression for the magnetic flux ϕ that emerges from the surface.

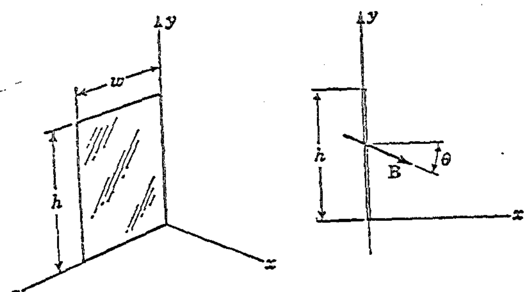


FIG. P3.9. For the computation of magnetic flux.

3.9. Figure P3.9 shows two views of a rectangular area of width w and height h. Through this surface emerges a magnetic flux that has a density $B = 1/\sqrt{1 + y^2}$.

The angle between the vector B and the normal to the yz plane is $\theta = \cot^{-1} y$. Determine the expression for the magnetic flux ϕ that emerges through the surface.

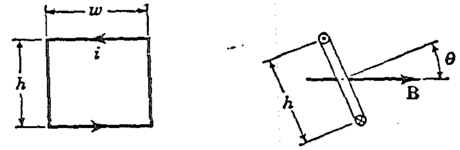


FIG. P3.10. For the computation of torque.

3.10. A rectangular turn of wire has dimensions h and w, as shown in Fig. P3.10, and carries a current i. The coil is placed in a uniform magnetic field so that the vector B makes an angle θ with the normal to the plane of the rectangle.

(a) Determine the expression for the torque that tends to turn the rectangle at right angles to B.

(b) Express the result of part a in terms of the magnetic moment of the current-carrying rectangle.

(c) A small bit of ferromagnetic material of volume V has a uniform magnetic polarization M. Use the results of part b, and discuss the torque that the ferromagnetic body will experience when placed in a magnetic field B.

3.11. A typical sample of iron is fairly well saturated magnetically at $B = 1.6$ webers/meter². Take $H = 8000$ amp/meter² at this value of B. Compute the magnetic moment per atom. See Prob. 3.4 for Avogadro's number. The atomic weight of iron is 55.85, and its density is 7.87 grams/cm³.

3.12. The flux density in a certain substance is 1.0 weber/meter². Calculate the magnetic field intensity H and the magnetic polarization M if the material has a relative permeability of (a) 1.00002; (b) 4.0; (c) 10,000.

3.13. Find the exact expression for the total magnetic flux threading the toroidal core shown in Fig. P3.13.

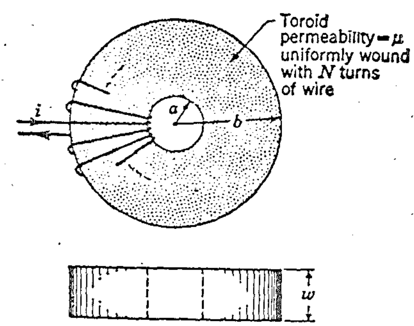


FIG. P3.13. A thick toroidal core.

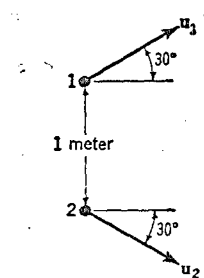


FIG. P3.14. For the calculation of forces between two moving charges.

3.14. At a particular instant of time, two isolated electrons are travelling in the plane of the paper with velocities whose magnitudes are 3×10^7 meters/sec for each one and whose directions are shown in Fig. P3.14. Compute all of the forces exerted on the two particles at that instant, including gravitational effects between each other and between each one and the earth. The electrons are near the earth's surface.

3.15. Using Gauss's law and the mmf law, show that, if there are no electrical forces tending to defocus a uniform cylindrical beam of charged particles moving with a common velocity, then this velocity must be the velocity of light.

3.16. Figure P3.16 shows a d-c electromagnet in which we wish to obtain a flux density of 1 weber/meter² in the air gap. Let $g = 0.25$ in. Assume that the steel core has infinite permeability.

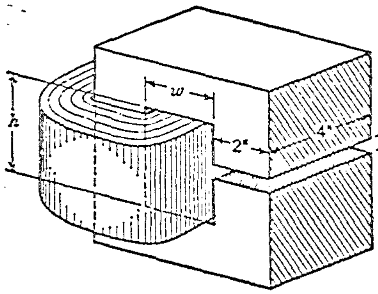


FIG. P3.16. For the design of a d-c electromagnet.

(a) How many ampere-turns are required on the coil?

(b) The "window" has dimensions w and h. Assume that half the area of the window is filled with copper, the remaining area being insulation and air spaces. A conservative value of current density in the copper is 1000 amp/in.². What is the required area of the window?

(c) From the results of part b select values for the dimensions w and h, and estimate the "mean length of turn" for the coil, i.e., the length of one turn of a conductor lying at the middle of the coil. Compute the electric field intensity in the copper and the voltage required for one turn. ($\rho = 2.08 \times 10^{-8}$ ohm-meter at the reasonable operating temperature of 70°C.) The coil is to use 80 volts d-c. How many turns are required? What is the cross-sectional area of the copper of one wire? What will be the current in the coil?

(d) Make a detailed check of the coil design as follows: From the table of wire gages in Appendix C, select the standard wire nearest to the one computed in part c. Add 0.003 in. to the diameter for enamel insulation. Assume that the width of the winding is the dimension h minus 0.5 in., and compute the turns per layer and the number of layers required. Compute the "build" of the coil in the direction of the dimension w, assuming 0.060 in. pressboard under the coil, 0.005 in. paper between each layer, and a wrapper 0.020 in. thick. The calculated build should lie between $0.75w$ and $0.9w$. The latter limit allows for 10 per cent "bulge" in winding. If the first wire size that you choose does not work out properly, choose another and try again.

3.17. Refer to Fig. P3.16. Let $w = 3$ in., $h = 6$ in., $g = 0.25$ in., and assume that the core has a constant permeability of $400\mu_0$.

(a) Compute the (approximate) mean length of magnetic path in the core.

(b) Compute the reluctance of each of the two parts of the magnetic circuit, neglecting leakage flux and fringing at the gap.

(c) The coil has 2000 turns of wire. What current is required to produce a magnetic flux of 5×10^{-3} weber across the gap?

3.18. Refer to Fig. P3.16. Let $w = 3$ in., $h = 6$ in., and $g = 0.10$ in. The coil has 2000 turns of wire. The B-H relationship for the steel of the core is given in the table below:

B, webers/meter²	0	0.5	1.0	1.2	1.4	1.6	1.7
H, amp/meter	0	76	219	414	1280	5580	[illegible]

(a) Compute the (approximate) mean length of the magnetic path in the core.

(b) Neglect leakage flux and fringing at the gap. For each of the values of B in the table, compute the magnetic flux ϕ, the total mmf required by the magnetic circuit, and the current required in the coil.

(c) Plot flux ϕ versus current i to scale. Note the linear portion of the curve at the beginning, often termed the "air-gap line."

3.19. A permanent magnet, such as the one shown in Fig. 3.21, has a magnetic path 25 cm long in the magnetic material. The B-H curve of the material is given in

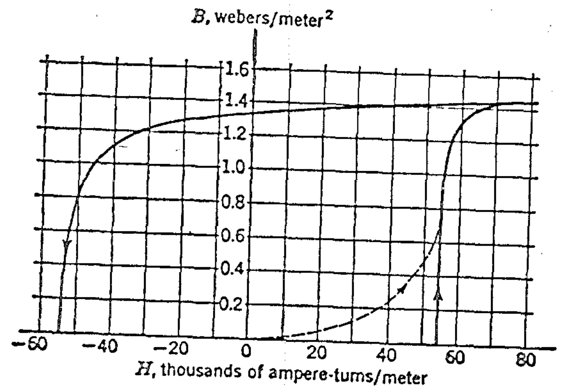


FIG. P3.19. B-H curve for an Alnico steel.

Fig. P3.19. Neglect leakage flux and fringing at the gap, and compute the magnetic flux density in the air gap for gap lengths of:

(a) 0.1 cm (b) 1 cm (c) 2 cm

3.20. With the permanent magnet of Fig. P3.20 we wish to produce a flux density of B_g webers/meter² in the air gap of area A_g and length l_g, and we wish to do this with a permanent magnet of minimum volume. The permanent magnet has a cross-sectional area A_m and a length l_m that are to be selected so that the volume $A_m l_m$ is a

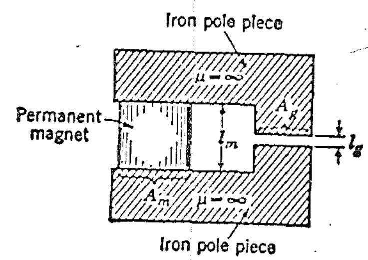


FIG. P3.20. Design of permanent magnet.

minimum. The iron pole pieces may be considered to have infinite permeability. Neglect leakage flux and fringing at the gap.

(a) Show that $A_m l_m = -B_g^2 l_g A_g / \mu_0 B_m H_m$, where B_m is the flux density in the permanent magnet and H_m is the (negative) field intensity in the permanent magnet. This relation shows that the criterion for minimum $A_m l_m$ is that the product $-B_m H_m$ be a maximum.

(b) Refer to the B_m-H_m curve of Fig. P3.19. Plot the product $-B_m H_m$ versus B_m for the portion of the curve in the second quadrant. For this material, what values of B_m and H_m satisfy the criterion of part a?

(c) Given $A_g = 10$ cm², $l_g = 0.25$ cm, and $B_g = 1.3$ webers/meter². Determine A_m and l_m, assuming the material of Fig. P3.19.

(d) If the area determined in part c should be increased by 50 per cent to take care of leakage flux, determine the proper diameter for the cylindrical magnet.

3.21. A sectional view of a loudspeaker magnet is shown in Fig. P3.21. The diameter of the circular pole is 5 cm. The radial length of the gap is 0.20 cm, and the thickness of the rectangular steel structure is 1.0 cm. We wish to have a flux density of 1.3 webers/meter² in the gap. The permanent magnet is to operate at 1.0 weber/meter² (see the B-H curve of Fig. P3.19). The steel pole piece and the rectangular structure may be assumed to have infinite permeability.

(a) Neglect leakage flux and fringing at the gap, and determine the required length and cross-sectional area of the permanent magnet.

(b) If the area determined in part a should be increased by 50 per cent to take care of leakage flux, determine the proper diameter for the cylindrical permanent magnet.

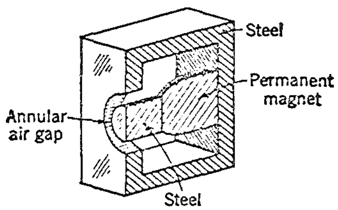


FIG. P3.21. Sectioned view: half of magnetic structure of loudspeaker magnet.

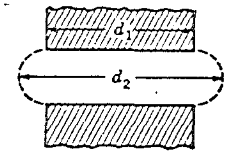


FIG. P3.22. Effective area to account for fringing.

3.22. When the length of an air gap is appreciable compared with its breadth and depth, there will be an appreciable fringing flux at the edges of the gap that should be included in the total flux. The additional flux can be taken into account approximately by imagining that the breadth of the air gap is increased by the amount indicated in Fig. P3.22, where the dotted lines are semicircles. Given a gap of 1 cm length, compute the ratio of effective area of the air gap to the area of one pole face if:

(a) The pole pieces are circular with a diameter of 5 cm.

(b) The pole pieces are rectangular with dimensions 5 by 100 cm.

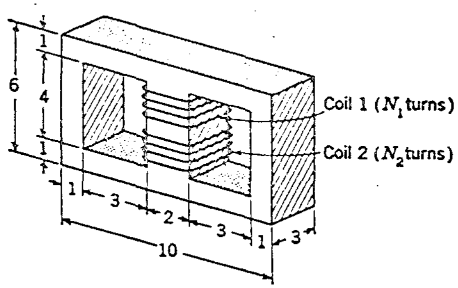


FIG. P3.23. A transformer. Dimensions are in centimeters.

3.23. (a) For the transformer shown in Fig. P3.23, calculate the reluctance to flux set up by current flowing in the N_1 turns. Assume a constant permeability $\mu = 2000\mu_0$.

(b) If the current through the primary winding increases linearly with time according to the law $i = 0.1t$ amp, where t is measured in seconds, calculate how large the product N_1N_2 must be for a constant voltage of 1 volt to be induced in the secondary winding.

3.24. The ring shown in Fig. P3.24 has a constant permeability μ. The mean length of the magnetic path is l, and the cross-sectional area of the magnetic path is A. The wire that goes through the opening of the ring carries a current i that is a function of time.

(a) Determine the expression for the magnetic flux ϕ in the ring as a function of i.

(b) Determine the expression for the voltage v induced in the N-turn coil.

(c) Let $i = I_m \sin \omega t$, where t is time in seconds. Determine the expression for v as a function of time. Sketch both v and i versus t.

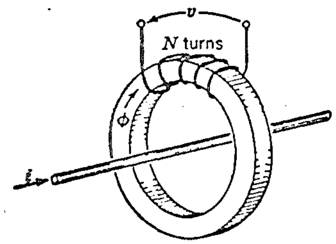


FIG. P3.24. Induced emf.

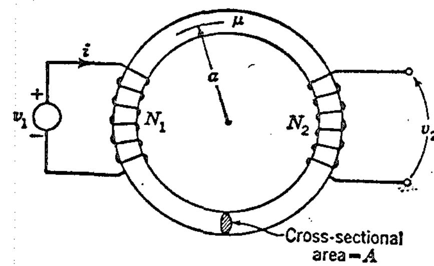


FIG. P3.25. Transformer with toroidal core.

3.25. Refer to Fig. P3.25. A voltage $v_1 = V_m \sin \omega t$, where t is time, is applied to a coil of N_1 turns which encircles a high permeability toroid. Assuming that μ is constant, and neglecting leakage flux, calculate:

(a) The expression for the flux threading the coil as a function of time

(b) The expression for current i as a function of time

(c) The expression for v_2 as a function of time

Sketch v_1, v_2, ϕ, and i versus time.

3.26. The sketches of Fig. P3.26 show an instrument used in measuring the vibration of machines. The magnetic flux density in the gap is $B = 1.0$ weber/meter².

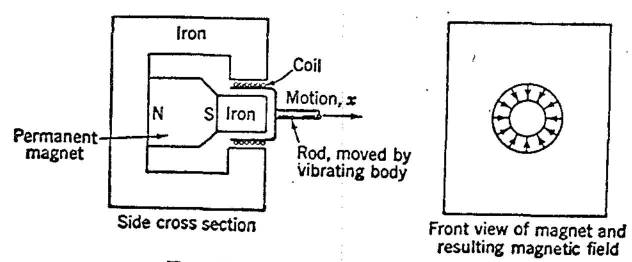


FIG. P3.26. Vibration pickup.

The coil has 20 turns and a diameter of 3 cm. If the motion of the coil is $x = 10^{-3}$ sin 377t meter, what is the voltage induced in the coil?

3.27. The phonograph pickup of Fig. P3.27 has an N-turn coil of area A. The permanent magnet supplies a uniform magnetic field of density B. The sidewise motion of the stylus rotates the coil.

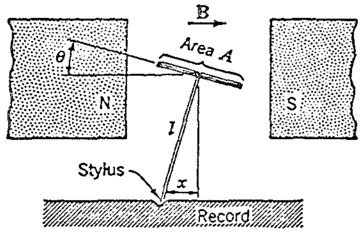


FIG. P3.27. Moving-coil phonograph pickup.

(a) Determine the expression for the voltage induced in the coil in terms of θ. What does this expression reduce to for small θ? Write this approximate expression in terms of x ($x \ll l$).

(b) Let $N = 10$, $A = 0.10$ cm², $B = 0.3$ weber/meter², $l = 0.7$ cm, and $x = 10^{-2} \sin 2\pi ft$ cm, where f is the frequency of the sinusoidal signal in cycles per second and t is the time in seconds. Determine the induced voltage as a function of time for $f = 100$ cycles per second (cps), 1000 cps, 10,000 cps.

(c) For the data in part b, what is the induced emf in millivolts for an instantaneous stylus velocity of 10 cm/sec?

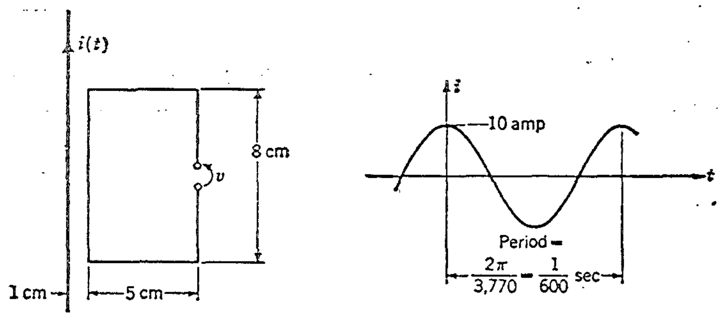


FIG. P3.28. Voltage induced by changing current.

3.28. A current of 10 cos 3770t amp flows through a thin wire as shown in Fig. P3.28. Calculate as a function of time the voltage which is induced between the terminals of the rectangular coil. Sketch v versus t.

3.29. The sketch in Fig. P3.29 shows a parallel-plate air-dielectric capacitor. Neglect fringing effects at the edges.

(a) Assuming that a charge $q = Q \sin \omega t$ is uniformly distributed on the top plate with an equal and opposite charge on the bottom plate, determine the displacement current through a circle of radius r and the resulting B around the circumference of this circle.

(b) Considering only the electric flux (neglecting the effect of the magnetic flux), find the voltage across the capacitor. Evaluate this numerically for $a = 5$ cm, $d = 0.2$ cm, $Q = 4.5 \times 10^{-9}$ coulomb.

(c) Find the induced emf around the path 12341 by $\oint \mathcal{E} \cdot ds$, as caused by the magnetic flux. Evaluate this, using the data in (b), for frequencies of 10^6 and 10^9 cps. What conclusions can be drawn?

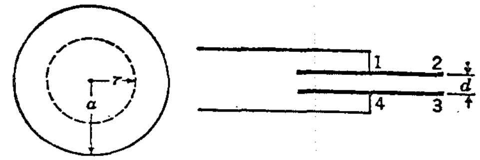


FIG. P3.29. Effect of displacement current.

3.30. The betatron sketched in Fig. P3.30 has cylindrical pole pieces. A magnetic flux density of B webers/meter² is changing at the rate dB/dt. Neglect fringing flux. It is observed that a charge placed in the air gap experiences a force even when it is not moving. What are the magnitude and direction of the induced electric field within the gap at a radius r?

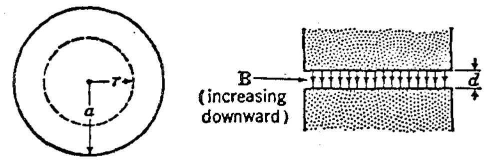


FIG. P3.30. Betatron.

3.31. The top sketch in Fig. P3.31 shows two resistors connected in a simple circuit. A magnetic flux within the loop is changing at the rate of 1 weber/sec. A high-impedance d-c voltmeter is connected successively in the three positions shown. State the reading of the voltmeter in each of the three positions, and give the reasons for your answers.

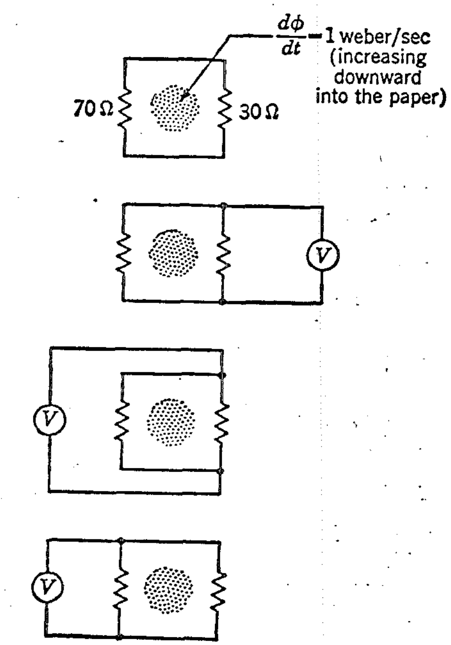


FIG. P3.31. Induced emf.

3.32. This problem is similar to Prob. 3.31, except that the lower conductor is looped around the flux as shown in Fig. P3.32. Four high-impedance d-c voltmeters are arranged as shown. State the readings of the meters, and give the reasons for your answers.

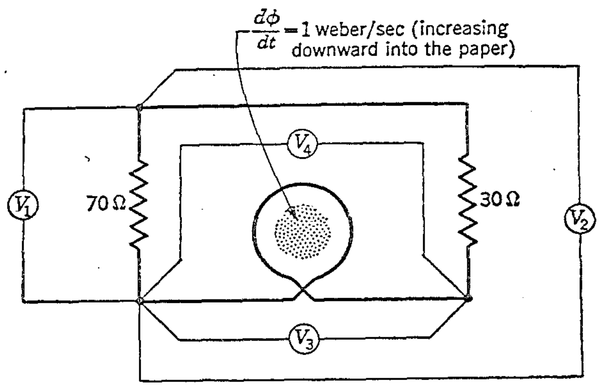


FIG. P3.32. What do the voltmeters read?

3.33. (*a*) A circular thin sheet of nichrome of resistivity ρ is suspended perpendicular to a magnetic field which is changing at the rate of dB/dt webers/meter²-sec. Sketch the direction of the induced electric field at various points in the sheet and the paths of the resulting currents (often termed "eddy currents"). Calculate the magnitude of the total induced current. Assume that the resistivity of the sheet is large, so that the induced currents are correspondingly small and do not affect the value of B appreciably. (See Fig. P3.33.)

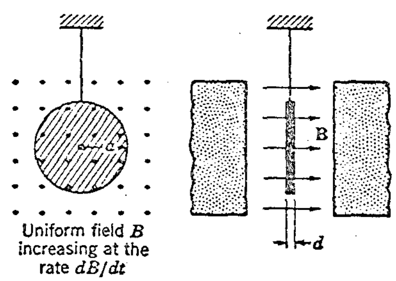


FIG. P3.33. For the calculation of eddy currents.

(*b*) If the sheet is free to turn, what will be its stable equilibrium position?

(*c*) What will be the stable equilibrium if B is in the direction shown but is decreasing with respect to time?

3.34. Given a toroid with a mean length of magnetic path l, a cross-sectional area A normal to the magnetic flux, and a permeability μ. The toroid is wound with N turns of wire that carry a current of i amp. Obtain an expression for the energy in the magnetic field of the toroid.

3.35. An electromagnet is shown in Fig. P3.16. The length of the air gap is one one-hundredth the length of the flux path in the iron. The relative permeability of the iron is 5000. Calculate the ratio of the energy stored in the air gap compared with that in the iron.

3.36. Consider two parallel pole faces, each of area 1 meter² and located 1 cm apart in air. In the gap between the pole faces, the magnetic flux density is $B = 1.6$ webers/meter². Compute the energy stored in the gap. How high would this energy lift a 150-lb man?

3.37. (*a*) Consider two plane parallel pole faces of area A, located a distance x apart in air. The magnetic flux density is B. Neglecting fringing flux, write the expression for the energy stored in the gap.

(*b*) The distance between the pole faces, x, is increased by the small amount Δx. The flux density B stays the same as before. Write the expression for the increase in stored energy. From this, deduce the expression for the force of attraction that exists between the pole faces.

(*c*) Compute the magnetic force of attraction between pole faces 1 meter² in area if $B = 1.6$ webers/meter².

(*d*) Repeat the foregoing problem for the energy stored and force between two insulated plates, separated by air, that have an electric field of intensity $\mathcal{E}$ volts/meter between them. Evaluate the force for plates of 1-meter² area if $\mathcal{E} = 3 \times 10^6$ volts/meter, which is the approximate field intensity for the breakdown of air at standard pressure and temperature.

3.38. For the plane wave shown in Fig. 3.32, show that there is a rate of energy flow per square meter equal to

$$\mathbf{P} = \mathcal{E} \times \mathbf{H} \qquad \text{watts/meter}^2$$

(*Hint:* Begin with the expression for energy density, and note that **P** is the product of this density and the velocity of propagation.)

3.39. What is the time of travel of a radio wave in air, expressed in microseconds per mile?

3.40. A certain radio wave has an electric field, measured at a certain receiver, of $\mathcal{E} = 5 \times 10^{-3} \sin 2\pi ft$ volts/meter, where the frequency is $f = 10^6$ cps. The $\mathcal{E}$ and **H** vectors are normal to each other and to the direction of propagation, and the ratio of $\mathcal{E}$ to H is $\sqrt{\mu/\epsilon}$, just as in Eq. 3.52.

(*a*) Write the expression for the magnetic field intensity as a function of time.

(*b*) A rectangular loop antenna with an area of 0.02 meter² and 20 turns of wire is oriented with its plane normal to B. Write the expression for the emf induced in the antenna.

BOB MARKS

EE 342

Jan 72
CCR

Homework Test

"I shall try to correct errors where shown to be errors, and I shall adopt new views as fast as they shall appear to be true views."

...Abraham Lincoln

"The lady bearer of this says she has two sons who want to work. Set them at it if possible. Wanting to work is so rare a want that it should be encouraged."

... Abraham Lincoln

[illegible]

Problem number on ditto assignment sheets

Problem number from textbook

1 3. The expression for $\bar{a}_\phi$, the unit vector in spherical coordinates, in terms of the rectangular unit vectors is

(a) $-\sin \phi \, \bar{a}_x + \cos \phi \, \bar{a}_y$

(b) $\sin \theta \, (-\sin \theta \, \bar{a}_x + \cos \phi \, \bar{a}_y)$

(c) $\sin \phi \, \bar{a}_x - \cos \phi \, \bar{a}_y$

(d) $\cos \theta \, (\sin \phi \, \bar{a}_x - \cos \phi \, \bar{a}_y)$

2 4. The component of $\bar{R}$ in the direction of $\bar{a}_r$ (a_r in cylindrical coordinates) is

(a) $r - r' \sin(\phi - \phi')$

(b) $r' \sin(\phi - \phi')$

(c) $r \sin(\phi - \phi') - r' \cos(\phi - \phi')$

(d) $r - r' \cos(\phi - \phi')$

3 8. 1.12 The directional derivative in the direction of the unit vector given is

(a) $12\bar{a}_x + 4\bar{a}_y + 12\bar{a}_z$

(b) $112/\sqrt{50}$

(c) $28/\sqrt{50}$

(d) $(3\bar{a}_x + 4\bar{a}_y + 5\bar{a}_z)/\sqrt{50}$

4 5. 1.5 The component of $\bar{A}$ in the direction of $\bar{a}_r$ is:

(a) $2 \sin \theta \cos \phi$

(b) $2 \cos \theta - \sin \theta$

(c) $\sin \theta \, (2 \cos \theta + \sin \phi) - 3 \cos \theta$

(d) $-3 \cos \theta + 2 \sin \theta \cos \phi$

5 6. 1.8 $d\bar{S}$ over the side of the prism in the x-z plane is:

(a) $6(\bar{a}_x + \bar{a}_y)$

(b) $\frac{6}{2}(\bar{a}_x + \bar{a}_y)$

(c) $6 \, \bar{a}_y$

(d) $-6 \, \bar{a}_y$

6 10. 1.14 $\int \bar{A} \cdot d\bar{S}$ over the top of the cube is

(a) 1/24

(b) -1/48

(c) 2/3

(d) 1/96

7 11. 1.16 $\int \bar{F} \cdot d\bar{l}$ along the portion of the contour described by the line y=2 is

(a) -2

(b) 2

(c) -16/3

(d) 4

(b) $a^4/4$ (d) $a^3/3$

17. 1.25 The vectors which may be derived from the gradient of a scalar function are:

(a) $\vec{A}$, $\vec{B}$ (b) $\vec{A}$, $\vec{C}$ (c) $\vec{B}$, $\vec{C}$ (d) $\vec{C}$

17. 1.25 The vectors which may be derived from the curl of another vector are

(a) $\vec{A}$, $\vec{B}$ (b) $\vec{A}$, $\vec{C}$ (c) $\vec{B}$, $\vec{C}$ (d) $\vec{C}$

19. The answer to the problem is approximately ____ mile (1 mile = 1609 meters)

5.09 m

(a) 2 (b) 0.5 (c) 1.6 (d) 8

21. The electric field intensity in the p-n junction is:

(a) 4.25×10^{-3} v/m (c) 240 v/m

(b) 0.425 v/m (d) 240 Kv/m

22. The electric field varies with r as:

(a) $1/r^{\frac{1}{2}}$ (b) $1/r$ (c) $1/r^{3/2}$ (d) $1/r^2$

23. The radial force on the electron varies with r as

(a) $1/r^{\frac{1}{2}}$ (b) $1/r$ (c) $1/r^{3/2}$ (d) $1/r^2$

24. 3.1 The surface polarization charge density on the bottom is

(a) $-(\frac{A\ell^2}{4} + B)$ (c) $+(\frac{A\ell^2}{4} + B)$

(b) $-(A\ell^2 + B)$ (d) $+(A\ell^2 + B)$

25. 3.3 If one evaluates the electric field directly (i.e., not using the potential ϕ), it is necessary to evaluate an integral of the form

(a) $\int \frac{x\,dx}{(x^2+a^2)^{3/2}}$ (c) $\int \frac{dx}{(x^2+a^2)^{3/2}}$

(b) $\int \frac{x\,dx}{(x^2+a^2)}$ (d) $\int \frac{dx}{(x^2+a^2)}$

27. 3.5 The surface polarization charge on the right surface can be written as:

(a) $E_o \cos \theta_2/3\epsilon_o$ (c) $E_o \cos \theta_2/4$

(b) $E_o \cos \theta_1/3\epsilon_o$ (d) $E_o \cos \theta_1/4$

28. 3.5 In part (b) of the problem, it is necessary to differentiate the electric field with respect to ______ with ______ maintained constant.

(a) a, Q (c) a, ϕ

(b) (b/a), Q (d) (b/a), ϕ

42 30. 3.9 The polarization surface charge density on the surface between the dielectric and the oil (i.e. $r = r_o$) will be

(a) positive

(b) negative

(c) positive over half of the cylinder, negative over the other half

(d) zero

CHECK YOUR CARD!

Does your last mark appear in column #19?

Is your student number marked?

Is your name written on the <u>back</u> of the card?

EE 342 - Electromagnetic Fields 1971-72

Text: Plonsey & Collin, "Principles and Applications of Electromagnetic Fields," McGraw-Hill, 1961.

Week	Date	Subject	Chapter
1	Dec. 6 - 10	Coordinate Systems, & Vector Analysis	1
2	Dec. 13 - 17		
	CHRISTMAS VACATION		
3	Jan. 3 - 7	Electric Fields	3
4	Jan. 9 - 14	Magnetic Fields	5,6,7,8
5	Jan. 16 - 21		
6	Jan. 24 - 28		
7	Jan. 31 - Feb. 4	Electromagnetic Fields, Waves, & Antennas	9,10,11
8	Feb. 7 - 11		
9	Feb. 14 - 18		
10	Feb. 21 - 25		
11	Feb. 28 -	Final Exams	

I) VECTOR ANALYSIS

$$\nabla = \bar{a}_x \frac{\delta}{\delta x} + \bar{a}_y \frac{\delta}{\delta y} + \bar{a}_z \frac{\delta}{\delta z}$$

$\text{GRAD } \phi = \nabla \phi$

$\text{DIV } \vec{F} = \nabla \cdot \vec{F}$

$$\text{CURL } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\nabla^2 = \nabla \cdot \nabla = \frac{\delta^2}{\delta x} + \frac{\delta^2}{\delta y} + \frac{\delta^2}{\delta z}$$

IF $\nabla \cdot \vec{F} = 0$, $\vec{F}$ IS SOLENOIDAL

IF $\nabla \times \vec{F} = 0$, $\vec{F}$ IS IRROTATIONAL

GAUSS' LAW: $\int_V \nabla \cdot \vec{F}\, dV = \oint_S \vec{F} \cdot dS$

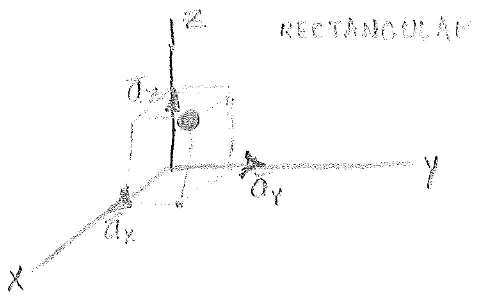
STOKES' THEOREM: $\oint_C \vec{F} \cdot d\vec{l} = \int_S \nabla \times \vec{F} \cdot d\vec{S}$

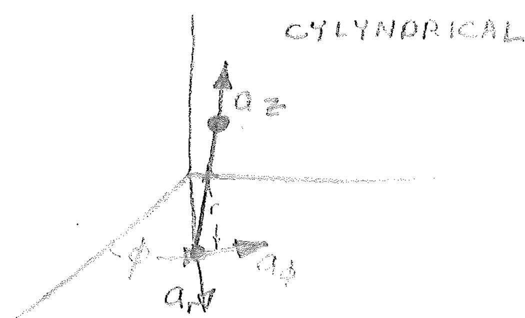
$R = r - r' = (x - x')\bar{a}_x + (y - y')\bar{a}_y + (z - z')\bar{a}_z$

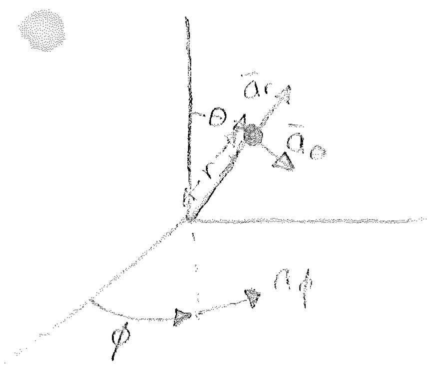
MOMENT OF A SOURCE SYSTEM: $m = \sum_{i=1}^{n} Q_i r_i'$

HELMHOLTZ'S THEOREM: $\forall \vec{F} \; \exists \; \vec{F}_1 + \vec{F}_2 = \vec{F} \ni \nabla \cdot F_1 = 0$ AND $\nabla \times F_2 = 0$

$\vec{F} = -\nabla \phi \; ; \; Q = \oint_S F \cdot dS$

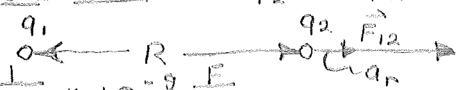






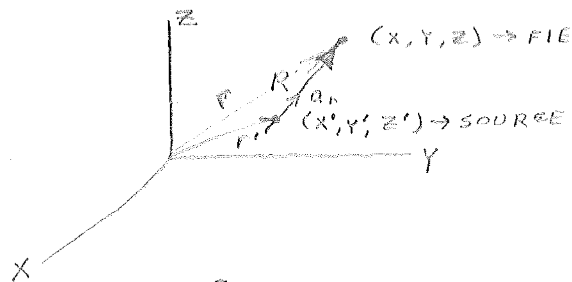
II) ELECTROSTATICS

A) <u>COULOMB'S LAW</u>: $\vec{F}_{12} = \dfrac{q_1 q_2}{4\pi \epsilon R^2}\vec{a}_r$



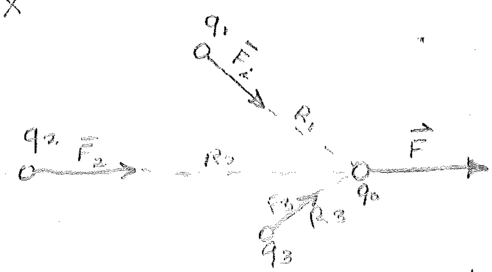
$\epsilon_0 = \dfrac{1}{36\pi} \times 10^{-9} \dfrac{F}{m}$

B) ELECTRIC FIELD



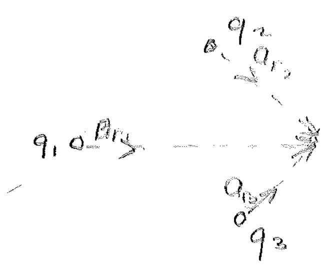
$\bar{a}_r = \dfrac{\bar{r} - \bar{r}'}{|\bar{r} - \bar{r}'|}$

$|\vec{R}| = |\bar{r} - \bar{r}'| = \left[(X - X')^2 + (Y - Y')^2 + (Z - Z')^2\right]^{\frac{1}{2}}$



$F = \dfrac{q_0}{4\pi\epsilon_0} \sum_{i=1}^{n} \dfrac{q_i \bar{a}_{R_i}}{R_i^2}$

DEFINITION: $\vec{E} = \lim_{\Delta q \to 0} \dfrac{\vec{F}}{\Delta q}$ (<u>ELECTRIC FIELD</u>)



$E = \dfrac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \dfrac{q_i}{R_i^2} \vec{a}_{R_i}$

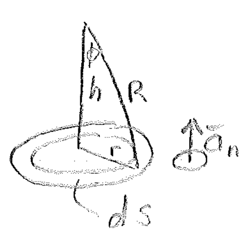
DEFINITION: $\rho = \lim_{\Delta V \to 0} \dfrac{\Delta q}{\Delta V}$ (<u>CHARGE DENSITY</u>)

$Q = \int_V \rho \, dV$

$\Rightarrow \vec{E} = \dfrac{1}{4\pi\epsilon_0} \int_V \dfrac{\rho(x', y', z')}{R^2} \bar{a}_R \, dV'$ (FOR VOLUME)

$= \dfrac{1}{4\pi\epsilon_0} \int_S \dfrac{\rho_s}{R^2} \bar{a}_R \, dS$

FOR AN INFINITLY CHARGED PLANE



$dE_z = \dfrac{\rho_s 2\pi r \, dr}{4\pi\epsilon_0 R^2} \cos\phi$

$E_z = \dfrac{\rho_s}{2\epsilon_0} \int_0^{\pi/2} \sin\phi \, d\phi = \dfrac{\rho_s}{2\epsilon_0} \vec{a}_n$

C) GAUSS' FLUX THEM

$$E \cdot ds = \frac{q}{4\pi\epsilon_0 r^2} \cdot ds$$

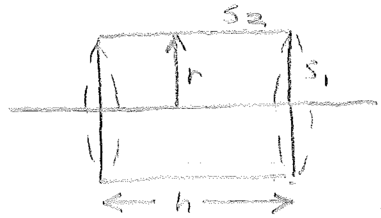
$$d\Omega = \frac{a_r \cdot ds}{r^2} \qquad 0 < \Omega < 4\pi$$

$$\oint E \cdot ds = \frac{q}{\epsilon_0} \quad \text{FOR } 1\ q$$

$$\oint_S E \cdot ds = \sum_{i=1}^{\#} \frac{q_i}{\epsilon_0} = Q/\epsilon_0$$

FIELD FROM INFINITE LINE CHARGE:

GAUSS' LAW



$$\oint_S E \cdot ds = \frac{Q}{\epsilon_0} = \int_{S_1} E \cdot ds + \int_{S_2} E \cdot ds$$

$$2\pi r h E_r = \frac{h}{\epsilon_0} \rho_\ell \Rightarrow E_r = \frac{\rho_\ell}{2\pi\epsilon_0 r}$$

FROM A CHARGED SPHERE

$$\oint E \cdot ds = \frac{Q}{\epsilon_0} \qquad r > a$$

$$= 0 \qquad r < a$$

$$E_r = \frac{Q}{4\pi\epsilon_0 r^2} \qquad r > a$$

$$= 0 \qquad r < a$$

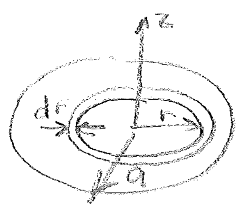
D) ELECTROSTATIC POTENTIAL

$$\text{WORK} = \phi(P_2) - \phi(P_2)$$

$$\vec{E} = -\nabla\phi \ni \phi = \frac{q}{4\pi\epsilon_0 R} + C \quad \text{OR} \quad \phi = \int_V \frac{\rho\, dV_1}{4\pi\epsilon_0 R} + C$$

$$\nabla \times \vec{E} = 0$$

POTENTIAL ON AXIS OF A CHARGED DISK



$$d\phi = \frac{\rho_s 2\pi r\, dr}{4\pi\epsilon_0 (z^2 + r^2)^{\frac{1}{2}}}$$

$$\phi_{AXIS} = \frac{\rho_s}{2\epsilon_0} \int_0^a \frac{r\, dr}{(z^2 + r^2)^{\frac{1}{2}}} + C$$

$$= \frac{\rho_s}{2\epsilon_0} \left[(a^2 + z^2)^{\frac{1}{2}} - |z| \right] + C$$

$$E_{AXIS} = -\frac{\partial\phi}{\partial z} = \frac{\rho_s}{2\epsilon_0} \left[1 - z(a^2 + z^2)^{-\frac{1}{2}} \right] \qquad z > 0$$

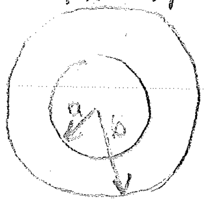
$$= \frac{\rho_s}{2\epsilon_0} \left[1 + z(a^2 + z^2)^{-\frac{1}{2}} \right] \qquad z < 0$$

POTENTIAL FROM A LINE CHARGE IS INFINITE

E) CONDUCTING BOUNDRIES

$E_n = \vec{n} \cdot \vec{E} = \rho_s / \epsilon_0$

FIELD 'TWIXT 2 CO-AXIAL CYLINDERS



$E_r = \frac{\rho_\ell}{2\pi\epsilon_0 r}$

$\phi(a) - \phi(b) = V = \int_a^b E_r\, dr = \left(\int_a^b \frac{dr}{r}\right) \frac{\rho_\ell}{2\pi\epsilon_0}$

$V = \frac{\rho_\ell}{2\pi\epsilon_0} \ln b/a$

$\rho_\ell = 2\pi\epsilon_0 V / \ln b/a$

$E_r = \frac{V}{r \ln (b/a)} \quad a < r < b$

F) POISSON'S EQUATION

$\oint_S E \cdot ds = \frac{1}{\epsilon_0} \int_V \rho\, dV$

$\nabla \cdot E = \rho / \epsilon_0$

$\nabla^2 \phi = -\rho / \epsilon_0$ FOR NO SOURCES, $\nabla^2 \phi = 0$

III) ELECTROSTATIC FIELDS IN MATERIAL BODIES, ENERGY, AND FORCES

A) POLARIZABILITY

FOR SHERICAL ATOM MODEL: $P = Np = N 4\pi \epsilon_0 R^3 E$

POLARIZABILITY: $\alpha_e = 4\pi R^3 \epsilon_0$ (ELECTRIC)

α_i (IONIC)

$\frac{p^2}{3kT}$ (ORIENTATIONAL)

$$P = N\left(\alpha_e + \alpha_i + \frac{p^2}{3kT}\right)E$$

$$= \epsilon_0 \chi_e E$$

ELECTRIC SUSCEPTIBILITY

B) FLUX DENSITY

$$D = \epsilon_0 E + P = \epsilon E$$

$$\nabla \cdot D = \rho$$

$$P = \epsilon_0 \chi_e E = (\epsilon - \epsilon_0)\vec{E}$$

$$\oint_S D \cdot ds = \int_V \rho \, dV = 0$$

$$n \cdot D_1 = n \cdot D_2 \qquad \frac{n \cdot E_1}{n \cdot E_2} = \frac{\epsilon_2}{\epsilon_1}$$

$$\frac{D_1}{D_2} = \frac{\epsilon_1}{\epsilon_2}$$

$$E_a = K E_d$$

DIELECTRIC

V) STATIONARY CURRENTS

OHM'S LAW: $\bar{J} = \sigma \bar{E}$

$I = \int_S \bar{J} \cdot d\bar{S}$; CURRENT THRU A SURFACE

FOR A CONSERVATIVE FIELD: $V = \Phi_1 - \Phi_2 = EL$

B) NONCONSERVATIVE FIELDS (EMF)

$\bar{J} = \sigma(\bar{E} + \bar{E}') \ni \bar{E}'$ IS A CONSERVATIVE FIELD

$R = \oint \frac{dl}{\sigma A}$ FOR A UNIFORM CYLINDRICAL CONDUCTOR

C) CONSERVATION OF CHARGE

$- \int_S \bar{J} \cdot d\bar{S} = - \int_V \nabla \cdot J \, dV$ (GAUSS' THEM)

NOW $\int_S \bar{J} \cdot d\bar{S} = \frac{\delta}{\delta t} \int_V \rho \, dV$

$\Rightarrow \int_V (\nabla \cdot J + \frac{\delta \rho}{\delta t}) dV = 0$ (CONSERVATION OF CHARGE)

OR $\nabla \cdot J + \frac{\delta \rho}{\delta t} = 0$ (DIFFERENTIAL FORM)

FOR STATIONARY FIELD, $\frac{\delta \rho}{\delta t} = 0 \Rightarrow \nabla \cdot J = 0$

WHEN CURRENTS FORM CLOSED LOOPS

D) RELAXATION TIME; $\tau = \epsilon / \sigma$

E) RESISTANCE OF ARBITRARY SHAPED CONDUCTORS

$\int_C \bar{E} \cdot d\bar{l} = \mathcal{E}$

$\int_A \bar{J} \cdot d\bar{S} = \sigma \int_A \bar{E} \cdot d\bar{S} = I$

$$\Rightarrow R = \frac{\Phi_1 - \Phi_2}{I} = \frac{\int_C \bar{E} \cdot d\bar{l}}{\int_A \sigma \bar{E} \cdot d\bar{S}}$$

$I^2 R = \int_V \bar{J} \cdot \bar{E} \, dV = \int_V \sigma \bar{E} \cdot \bar{E} \, dV = \frac{1}{\sigma} \int_V \bar{J} \cdot \bar{J} \, dV$

F) DUALITY TWIXT $\bar{J}$ AND $\bar{D}$

D = DISPLACEMENT FLUX DENSITY

FOR LINEAR ISOTROPIC MATERIALS

	CONDUCTOR	DIELECTRIC
	$\nabla \times \bar{E} = 0$	$\nabla \times \bar{E} = 0$
	$\bar{J} = \sigma \bar{E}$	$\bar{D} = \epsilon \bar{E}$
	$\nabla \cdot \bar{J} = 0$	$\nabla \cdot \bar{D} = 0$
IF ϵ & σ CONSTANT	$\nabla \times \bar{J} = 0$	$\nabla \times \bar{D} = 0$

$\bar{J} \Leftrightarrow \bar{D}$

$\sigma \Leftrightarrow \epsilon$

$C = \int_S \epsilon E \cdot ds \Big/ \int_L E \cdot dl$

$RC = \frac{\epsilon}{\sigma}$; IF FRINGING EFFECTS ARE NEGLECTED

G) JOULE'S LAW:

$P = \frac{dw}{dt} = I^2 R$

H) CONVECTION CURRENT

$\vec{J} = \rho \vec{V}$ ($\vec{V}$ IS VELOCITY) $\qquad \frac{1}{\sigma} = \rho$

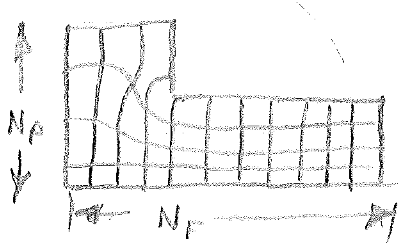
I) FLUX PLOTTING

1) TAKE ADVANTAGE OF SYMMETRY

2) DRAW IN BOUNDRIES SEPARATING CONDUCTING AND THOSE WITH 0 CONDUCTIVITY

3) STARTING WITH KNOWN POTENTIALS AND FLOW LINES, SCETCH FIELD, MAINTAINING ORTONALITY

4) REFINE TO CURVILINEAR SQUARES



$R = \frac{N_P}{\sigma N_F}$

$C = \epsilon N_F / N_P$

VI) STATIC MAGNETIC FIELD IN A VACUUM

A) AMPERE'S LAW OF FORCE

$1\mu_0 = $ FARADAYS/METER (PERMEABILITY)

B) MAGNETIC FIELD ($B = $ WEBER/M^2)

$\bar{F} = q\bar{V} \times \bar{B}$ (LORENTZ FORCE)

$\bar{B} = \nabla \times \bar{A} \ni \bar{A} = \frac{\mu_0}{4\pi} \int_V \frac{J'}{R} dV'$

C) MAGNETIC DIPOLE

FOR ARBITRARILY SHAPED LOOP:

$$\bar{M} = \frac{1}{2} \oint_c \bar{r} \times d\bar{r} = \frac{1}{2} \int_V \bar{r} \times \bar{J} \, dV$$

$\bar{T} = \bar{p} \times \bar{E}$; $\bar{T} = \bar{M} \times \bar{B}$ OR $\bar{T} = \int d\bar{M} \times \bar{B}$

D) MAGNETIC FLUX AND $\nabla \cdot \bar{B}$

$\nabla \cdot \bar{B} = 0$

$\psi = \int_S \bar{B} \cdot d\bar{S}$ = FLUX PASSING THRU A SURFACE

$= \int_S \nabla \times \bar{A} \cdot d\bar{S}$

$= \oint_C \bar{A} \cdot d\bar{\ell}$

E) AMPERE'S CIRCUITAL LAW

$\int_S \nabla \times \bar{B} \cdot d\bar{S} = \int_S \mu_0 \bar{J} \cdot dS = \oint_c \bar{B} \cdot d\bar{\ell}$

$\nabla \times \bar{B} = \mu_0 \bar{J}$

$\nabla \cdot \bar{E} = \rho / \epsilon_0$

$\nabla \cdot \bar{D} = \rho$

$\nabla \times \bar{D} = \nabla \times \bar{P}$

VII.) MAGNETIC FIELD IN MATERIAL BODIES

1) MAGNETIC DIPOLES TEND TO ALIGN IN ACCORDANCE WITH TORQUE

2) IF ATOM HAS NO MOMENT, EXTERNAL FIELD DISORTS, CREATING MAGNETIC DIPOLE

$\bar{M} = n\bar{m}$ $\left(\frac{AMP}{m}\right)$

B) EQUIVALENT VOLUME AND SURFACE POLARIZATION CURRENTS

$\bar{A} = \frac{\mu_0}{4\pi} \nabla \times \frac{\bar{m}}{R}$; $\left[A(x,y,z) = \int_V \nabla \times \frac{\bar{M}}{R} dV' \right]$

J_m = POLARIZATION CURRENT; J_{ms} = SURFACE POL. CORRENT

$\left(\begin{array}{l} J_m = \nabla' \times M \\ J_{ms} = \bar{M} \times n \end{array} \right)$

$m = \bar{M} dV = I d\bar{S}$

$\Rightarrow I_{x,y,z} = M_{xyx} dx$ (dy or dz) $\qquad J_{xyz} = -\frac{dM_{xyz}}{dx, dy, dz}$

C) MAGNETIC FIELD INTENSITY (H)

$\vec{H} = \frac{\bar{B}}{\mu_0} - \bar{M}$

$\Delta \times H = J$

$\bar{M} = X_m \bar{H}$

$\vec{B} = \mu_0 (1 + \chi_m) \bar{H} = \mu H$

D) $\bar{B} \cdot \bar{H}$ CURVE

$\int_s \nabla \times H \cdot ds = \oint_c \bar{H} \cdot d\bar{l} = \int_S J \cdot dS$

E) BOUNDRY CONDITIONS FOR $\bar{B}$ AND $\bar{H}$

F) SCALAR POTENTIAL FOR $\vec{H}$

($\bar{P}$ = DIPOLE MOMENT / VOLUME)

FOR A PERMANENT MAGNET: $\nabla \times \bar{H} = 0$; $\nabla \cdot \bar{H} = -\nabla \cdot \bar{M}$

$\bar{B} = \mu_0 (\bar{H} + \bar{M})$ (INSIDE BODY)

$\oint_c \bar{H} \cdot dl = \int_s J \cdot dS = NI$

VIII) QUASI-STATIONARY MAGNETIC FIELD

A) FARADAY'S LAW

$$\psi = \int_S \bar{B} \cdot d\bar{S}$$

$$\oint_c E \cdot dl = -\frac{\delta}{\delta t} \int_S \nabla \times \bar{A} \cdot d\bar{S} = -\frac{\delta}{\delta t} \bar{A} \cdot d\bar{l}$$

$$\nabla \times \bar{E} = -\frac{\delta B}{\delta t}$$

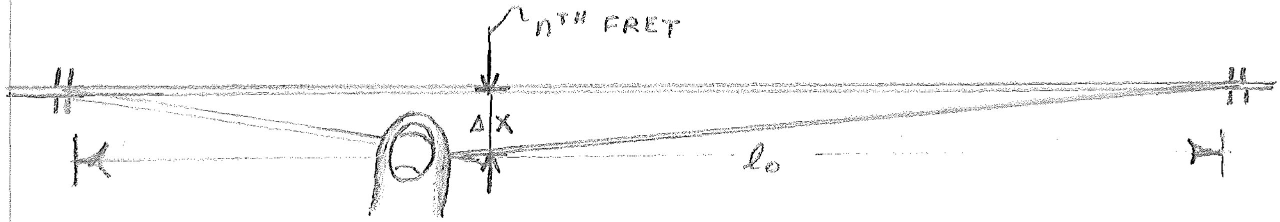
$$\bar{F} = q\bar{V} \times \bar{B} \Rightarrow \bar{E} = \frac{\bar{F}}{q} = \bar{V} \times \bar{B}$$

$$V_{INDUCED} = -\int_S \frac{\delta B}{\delta t} \cdot dS + \oint_c \bar{V} \times \bar{B} \cdot d\bar{l}$$

FOR MOTIONAL emf

$$\oint_c \bar{E} \cdot dl = -\frac{d}{dt} \int_S B \cdot dS$$

$$\psi_{21} =$$



WHAT DISTANCE ΔX MUST A STRING BY STRETCHED ON THE n^{TH} FRET TO RAISE IT ONE CHROMATIC STEP? THE STRING OBEYS HOOK'S LAW, AND IS INITIALLY UNDER TENSION T_0. ASSUME STRETCHED AND UNSTRETCHED LINEAR MASS DENSITIES EQUIVALENT.

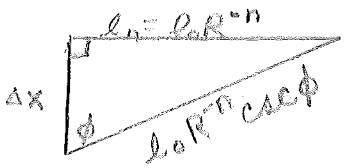
ANSWER:

UNSTRETCHED

$$f_n^2 = \frac{T_0}{\mu_n \ell_n^2} = \frac{T_0 \ell_0}{m \ell_n^2} \ni m = \text{MASS OF STRING}$$

$$= \frac{T_0}{m \ell_0 R^{-2n}} = \frac{T_0}{m \ell_0} R^{2n}$$

STRETCHED:



$$T_s = T_0 + k\Delta\ell$$

$$= T_0 + k(\ell_0 R^{-n}\csc\phi - \ell_0 R^{-n})$$

$$= T_0 + k\ell_0 R^{-n}(\csc\phi - 1)$$

$$\mu_s = \mu_n = \frac{m}{\ell_0}$$

$$\ell_s = \ell_0 R^{-n}\csc\phi$$

$$\Rightarrow f_s^2 = \frac{[T_0 + k\ell_0 R^{-n}(\csc\phi - 1)]\ell_0}{m \ell_0^2 R^{-2n}\csc^2\phi} = \frac{[T_0 + k\ell_0 R^{-n}(\csc\phi - 1)]}{m \ell_0 \csc^2\phi} R^{2n}$$

NOW: $$\frac{f_s^2}{f_n^2} = R^2 = \frac{[T_0 + k\ell_0 R^{-n}(\csc\phi - 1)]}{T_0 \csc^2\phi}$$

$$= \sin^2\phi + k\ell_0 \frac{(\sin\phi - \sin^2\phi)}{T_0}$$

$$\Rightarrow \sin^2\phi + \frac{k\ell_0}{T_0}(\sin\phi - \sin^2\phi) - R^2 = 0$$

$$\left(1 - \frac{k\ell_0}{T_0}\right)\sin^2\phi + \frac{k\ell_0}{T_0}\sin\phi - R^2 = 0$$

NOW $$\sin\phi = \ell_0 R^{-n} / (\Delta x_n^2 + \ell_0^2 R^{-2n})^{\frac{1}{2}}$$

$$\Rightarrow \left(1 - \frac{k\ell_0}{T_0}\right)\left(\frac{\ell_0^2 R^{-2n}}{\Delta x_n^2 + \ell_0^2 R^{-2n}}\right) + \frac{k\ell_0^2 R^{-n}}{T_0(\Delta x_n^2 + \ell_0^2 R^{-2n})^{\frac{1}{2}}} - R^2 = 0$$

$$(T_0 - k\ell_0)(\ell_0^2 R^{-2n}) + k\ell_0^2 R^{-n}(\Delta x_n^2 + \ell_0^2 R^{-2n})^{\frac{1}{2}} - T_0 R^{+2}(\Delta x_n^2 + \ell_0 R^{-2n}) = 0$$

LET $\psi_n = (\Delta X_n^2 + \ell_0^2 R^{-2n})^{\frac{1}{2}}$

$$\Rightarrow (T_0 - k\ell_0)(\ell_0^2 R^{-2n}) + k\ell_0^2 R^{-n}\psi_n - T_0 R^{+2}\psi^2 = 0$$

BY QUADRATIC THEM:

$$\psi_n = \frac{-k\ell_0^2 R^{-n} \pm \left[k^2\ell_0^4 R^{-2n} + 4T_0R^{+2}(T_0 - k\ell_0)(\ell_0^2R^{-2n})\right]^{\frac{1}{2}}}{-2T_0R^{+2}}$$

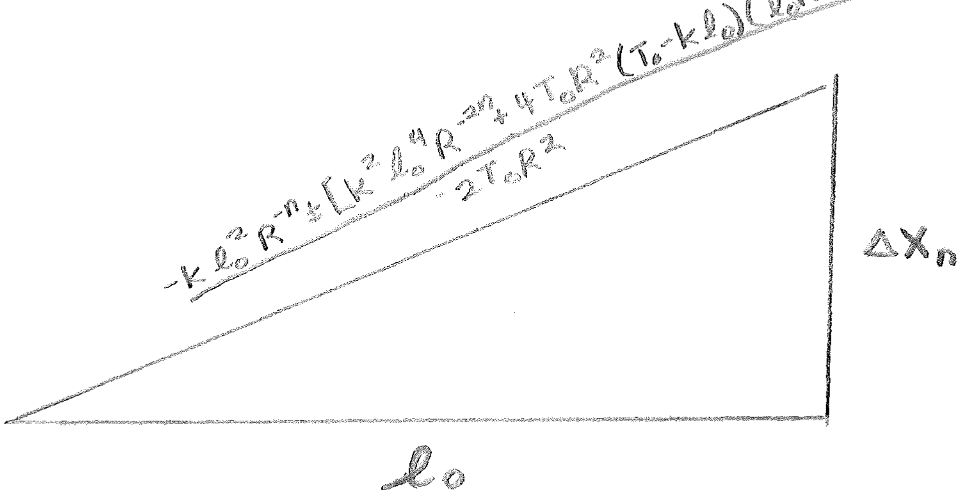
$$= (\Delta X_n^2 + \ell_0^2 R^{-2n})^{\frac{1}{2}}$$

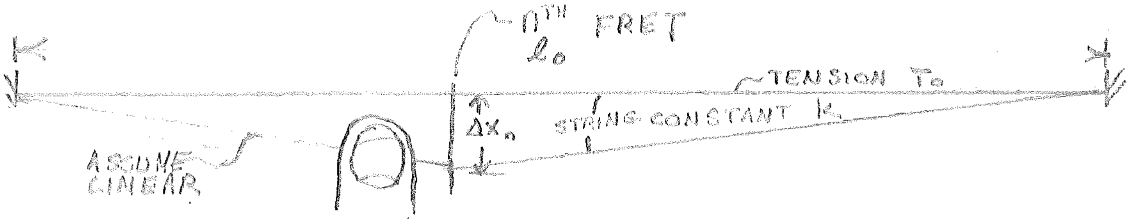
$$\Rightarrow \Delta X_n = \frac{\left(-k\ell_0^2R^{-n} \pm \left[k^2\ell_0^4R^{-2n} + 4T_0R^2(T_0 - k\ell_0)(\ell_0R^{-n})^2\right]^{\frac{1}{2}}\right)^2}{4T_0R^4}$$

$$- \ell_0^2 R^{-2n}$$

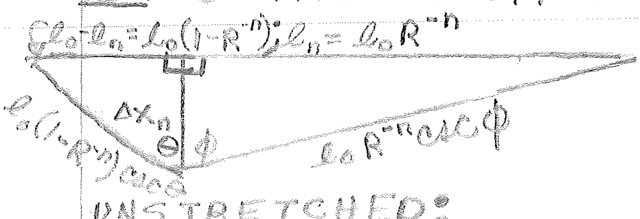
FOR $\ell_0 = \ell_n$ (ie $n = 0$)

$$\Delta X = \left[\frac{\left(-k\ell_0^2 \pm \left[k^2\ell_0^4 + 4T_0R^2(T_0 - k\ell_0)\,\ell_0^2\right]^{\frac{1}{2}}\right)^2}{4T_0R^4} - \ell_0^2\right]$$





FIND $\Delta x_n \ni$ PITCH OF STRING INCREASES ONE CHROMATIC STEP, ASSUME HOOK'S LAW.



LET $\phi = \operatorname{atan} \dfrac{l_0 R^{-n}}{\Delta x_n}$

m = STRING'S MASS

AND $\theta = \operatorname{atan} \dfrac{l_0(1-R^{-n})}{\Delta x_n}$

UNSTRETCHED:

$$\mu_n = \frac{m}{l_0}$$

$$l_n = l_0 R^{-n} \Rightarrow f_n^2 = \frac{T_0}{m l_0 R^{-2n}} = \frac{T_0}{m l_0} R^{2n}$$

STRETCHED:

$$\mu_s = \frac{m}{l_0[(1-R^{-n})\csc\theta + R^{-n}\csc\phi]}$$

$$T_s = T_0 + k l_0 R^{-n}(\csc\phi - 1)$$

$$l_s = l_0 R^{-n}\csc\phi$$

$$\Rightarrow f_s^2 = \frac{[T_0 + k l_0 R^{-n}(\csc\phi - 1)]\ [(1-R^{-n})\csc\theta + R^{-n}\csc\phi]}{m\ l_0\ R^{-2n}\csc\phi}$$

$$\frac{f_n^2}{f_s^2} = R^{-2} = \frac{[T_0 + k l_0 R^{-n}(\csc\phi - 1)][(1-R^{-n})\csc\theta - R^{-n}\csc\phi]}{\csc\phi\ T_0}$$

EE 342

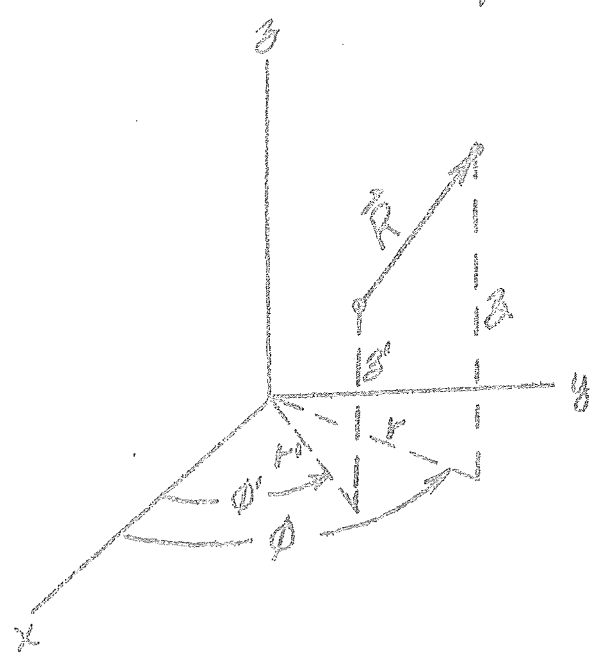
1. The following unit vectors in c

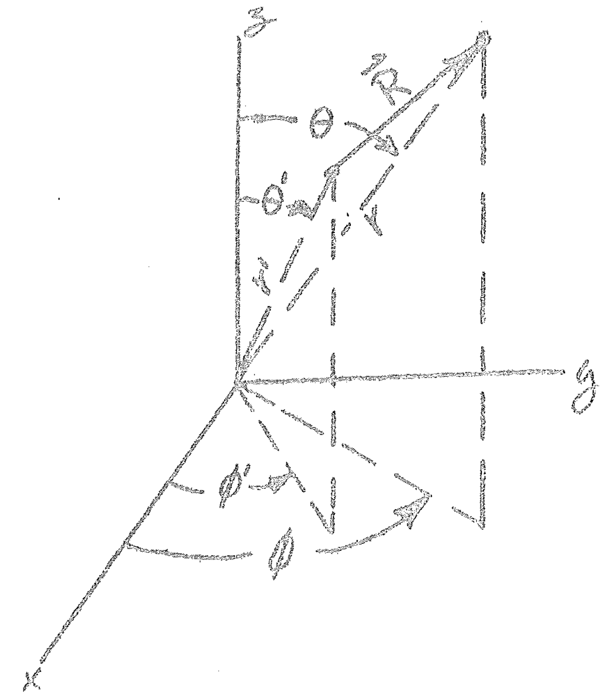
EE 342

1. Write the cylindrical coordinate unit vectors, $\vec{a}_r, \vec{a}_\phi, \vec{a}_z$, in terms of $\vec{a}_x, \vec{a}_y, \vec{a}_z$.
2. Convert your expressions in (1) so that only the rectangular variables x, y, z are used (i.e. no "r" or "ϕ").
3. Repeat (1) & (2) for the spherical coordinate unit vectors, $\vec{a}_r, \vec{a}_\theta, \vec{a}_\phi$.
4. Write the complete expression for

$$\vec{R} = (x-x')\vec{a}_x + (y-y')\vec{a}_y + (z-z')\vec{a}_z$$

in terms of (a) cylindrical coordinates and
(b) spherical coordinates





EE 342

	Problem	Subject
✓5.	1.5	Coordinate Systems
✓6.	1.8	$\oint d\vec{S} = 0$ (num)
✓7.	1.11	$\nabla^2(\frac{1}{R})$
✓8.	1.12	$\nabla\psi$ (num)
9.	1.13	$\nabla(\frac{1}{r}) = -\nabla'(\frac{1}{r})$
✓10.	1.14	Gauss' Law (num)
✓11.	1.16	Stoke's Law (num)
✓12.	1.18	Integration by parts
13.	1.19	Stoke's Law (num)
✓14.	1.22	Conservation of mass
15.	1.23	Curl (num)
16.	1.24	$\int \nabla^2(\frac{1}{r}) dv$
17.	1.25	$\vec{V} = \nabla \times \vec{A} + \nabla\phi$ (num)

18. 1.01×10^6 tons

19.

20. $-\frac{1}{2}\bar{a}_r$ (cylindrical coordinates)

21. 9.3×10^6 m/sec

22. 16.7_0 cm.

23. 0.34 mm

25.
$$\bar{E}_{axis} = \frac{\bar{P}}{\epsilon_0} \ln \frac{1 + \sqrt{1 + \left(\frac{a}{z - L/2}\right)^2}}{1 + \sqrt{1 + \left(\frac{a}{z + L/2}\right)^2}}$$

26.
$\Delta l_1 = 0.146 \times 10^{-13} R_{He}$ $\Delta l_2 = 0.234 \times 10^{-13} R_{He}$

27. $\theta_1 = 14°$

28. $C = \frac{2\pi\epsilon}{\ln \frac{b}{a}}$ $\frac{b}{a} = e$ (base of natural logs)

29.

30. $r_0 = \frac{4}{3}$ cm $\phi = 358$ Kv

31. $F = -4 \times 10^{-5}$ newtons

32.

1-5) $\bar{A} = 2\bar{a}_x + \bar{a}_y - 3\bar{a}_z$

a) FOR CYLYNDRICAL CO-ORDINATES

$\bar{a}_r = \cos\phi\,\bar{a}_x + \sin\phi\,\bar{a}_y$

$\bar{a}_\phi = -\sin\phi\,\bar{a}_x + \cos\phi\,a_y$

$\bar{A}_r = \bar{A}\cdot\bar{a}_r = (2\bar{a}_x + \bar{a}_y - 3\bar{a}_z)(\cos\phi\,\bar{a}_x + \sin\phi\,\bar{a}_y)$

$= 2\cos\phi + \sin\phi$

$\bar{A}_\phi = \bar{A}\cdot\bar{a}_\phi = (2\bar{a}_x + \bar{a}_y - 3\bar{a}_z)(-\sin\phi\,\bar{a}_x + \cos\phi\,\bar{a}_y)$

$= -2\sin\phi + \cos\phi$

$\Rightarrow \bar{A} = (2\cos\phi + \sin\phi)\bar{a}_r + (\cos\phi - 2\sin\phi)\bar{a}_\phi - 3\bar{a}_z$

b) FOR SPHERICAL CO-ORDINATES

$\bar{a}_r = \sin\theta\cos\phi\,\bar{a}_x + \sin\theta\sin\phi\,\bar{a}_y + \cos\theta\,\bar{a}_z$

$\bar{a}_\theta = \cos\theta\cos\phi\,\bar{a}_x + \cos\theta\sin\phi\,\bar{a}_y - \sin\theta\,\bar{a}_z$

$\bar{a}_\phi = -\sin\phi\,\bar{a}_x + \cos\phi\,\bar{a}_y$

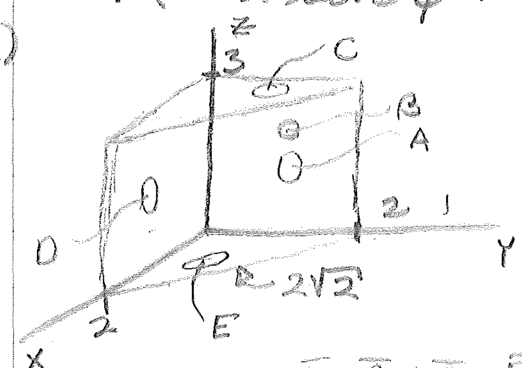
$\bar{A} = 2\bar{a}_x + \bar{a}_y - 3a_z$

$= (2\sin\theta\cos\phi + \sin\theta\sin\phi - 3\cos\theta)\bar{a}_r$

$+ (2\cos\theta\cos\phi + \cos\theta\sin\phi + 3\sin\theta)\bar{a}_\theta$

$+ (-2\sin\phi + \cos\phi)\,\bar{a}_\phi$

1-8)



$\bar{A} = 6\,i + 6\,j$

$\bar{B} = -6\,i$

$\bar{C} = +3\,\bar{k}$

$\bar{D} = -6\,j$

$\bar{E} = -3\,k$

$\bar{A} + \bar{B} + \bar{C} + \bar{D} + \bar{E} = \bar{O}$

1-11) $\frac{1}{R} = \left[(x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{-\frac{1}{2}}$

$\frac{\delta}{\delta x}\left(\frac{1}{R}\right) = 2(x-x')\left(-\frac{1}{2}\right)\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-3/2}$

$= -(x-x')\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-3/2}$

$\frac{\delta^2}{\delta x^2}\left(\frac{1}{R}\right) = -\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-3/2}$

$+ (x-x')\left(+\frac{3}{2}\right)2(x-x')\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-\frac{5}{2}}$

$= 3(x-x')^2\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-5/2}$

$-\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-3/2}$

$\nabla^2\left(\frac{1}{R}\right) = \frac{\delta^2}{\delta x^2}\left(\frac{1}{R}\right) + \frac{\delta^2}{\delta y^2}\left(\frac{1}{R}\right) + \frac{\delta^2}{\delta y^2}\left(\frac{1}{R}\right)$

$= 3\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-3/2}$

$- 3\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-3/2} = 0$

1-12) $\psi = x^2yz$

$\nabla\psi = 2xyz\bar{a}_x + x^2z\bar{a}_y + x^2y\,\bar{a}_z = 12\bar{a}_x + 4\bar{a}_y + 12a_z$

$(\nabla\psi)\cdot\left(\frac{3}{\sqrt{50}}\bar{a}_x + \frac{4}{\sqrt{50}}\bar{a}_y + \frac{5}{\sqrt{50}}\bar{a}_z\right)$

$= \frac{1}{\sqrt{50}}(36+16+60) = 112/\sqrt{50}$

1-13) $\frac{1}{R} = \left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-1/2}$

$\frac{\delta}{\delta x}\left(\frac{1}{R}\right) = -(x-x')\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-3/2}$

$\frac{\delta}{\delta x'}\left(\frac{1}{R}\right) = (x-x')\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-3/2}$

$\nabla\left(\frac{1}{R}\right) = -\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-3/2}$

$\left[(x-x')\bar{a}_x + (y-y')\bar{a}_y + (z-z')\bar{a}_z\right]$

$\nabla'\left(\frac{1}{R}\right) = \nabla\left(\frac{1}{R}\right)$

1-14) $\bar{A} = x^2\bar{a}_x + xy^2\bar{a}_y + 24x^2y^2z^3\bar{a}_z$

$\nabla\cdot\bar{A} = 2x + 2xy + 72x^2y^2z^2$

$\int_v \nabla\cdot\bar{A}\,dV = \int \nabla\cdot\bar{A}\,dx\,dy\,dz$

$\int_0^1\int_0^1\int_0^1 (2x+2xy+72x^2y^2z^2)\,dx\,dy\,dz$

$\int_0^1\int_0^1 \left[x^2+x^2y+24x^3y^2z^2\right]_0^1 dy\,dz$

$\int_0^1\int_0^1 \left[1+y+24y^2z^2\right] dy\,dz$

$\int_0^1 \left[y+\frac{1}{2}y^2+8y^3z^2\right]_0^1 dz$

$\int_0^1 \left[\frac{3}{2}+8z^2\right]dz = \left[\frac{3}{2}z+\frac{8}{3}z^3\right]_0^1 = \frac{3}{2}+\frac{8}{3}$

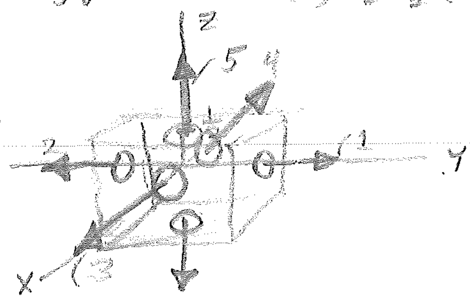
$= \frac{9}{6}+\frac{16}{6} = \frac{25}{6}$

(CONT)

1-14) $\bar{A} = x^2\bar{a}_x + x^2y^2\bar{a}_y + 24x^2y^2z^3\bar{a}_z$

GAUSS' LAW: $\int_v \nabla\cdot\bar{A}\,dv = \oint \bar{A}\cdot d\bar{s}$

$\int_v \nabla\cdot\bar{A}\,dv = \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}(2x+2x^2y+72x^2y^2z^2)\,dx\,dy\,dz = \frac{1}{24}$



FOR SIDE 1

$d\bar{s}_1 = dx\,dz\,a_y \quad ; \quad y = \frac{1}{2}$

$\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2y^2\,dx\,dz = \frac{1}{36}$

FOR SIDE 2

$d\bar{s}_2 = -dx\,dz\,\bar{a}_y \quad ; \quad y = -\frac{1}{2}$

$-\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2y^2\,dx\,dz = -1/36$

FOR SIDE 3

$d\bar{s}_3 = dy\,dz\,\bar{a}_x$

$\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2\,dy\,dz = 1/4$

FOR SIDE 4

$d\bar{s}_4 = -dy\,dz\,\bar{a}_x \quad ; \quad x = \frac{1}{2}$

$-\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2\,dy\,dz = -\frac{1}{4}$

FOR SIDE 5

$d\bar{s} = dx\,dy\,\bar{a}_z$

$\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} 24x^2y^2z^3\,dx\,dy \quad ; \qquad z = \frac{1}{2}$

$= \frac{1}{48}$

FOR SIDE 6

$d\bar{s} = -dx\,dy\,a_z = \frac{1}{48} \qquad\qquad z = -\frac{1}{2}$

$\sum_1 \int_s A\cdot ds = \frac{1}{48} + \frac{1}{48} = \frac{1}{24}$

1-18) PROVE: $\int_V \varphi \nabla\cdot\bar{F} dV = \oint \varphi F\cdot \bar{n} ds - \int_V F\cdot\nabla\varphi dV$

NOW $\nabla\cdot(\varphi\cdot\bar{F}) = \varphi\nabla\cdot\bar{F} + \bar{F}\cdot\nabla\varphi$

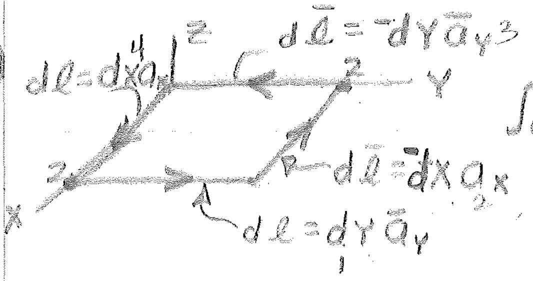
AND $\int_V \nabla\cdot\bar{A}\, dV = \oint_S \bar{A}\cdot n\, ds = \oint_S \bar{A}\cdot d\bar{S}$

LET $\bar{A} = \varphi\bar{F}$

$\Rightarrow \int_V \nabla\cdot(\varphi\bar{F})dV = \oint_S \varphi\bar{F}\cdot d\bar{S}$

NOW $\int_V \nabla\cdot(\varphi\bar{F})dV = \int_V \varphi\nabla\cdot\bar{F} dV + \int_V \bar{F}\cdot\nabla\varphi dV$

$\therefore \int_V \varphi\nabla\cdot\bar{F} dV = \oint \varphi\bar{F}\cdot d\bar{S} - \int_V \bar{F}\cdot\nabla\varphi dV$

1-16) 

$F = X\bar{a}_x + X^2Y\bar{a}_Y + Y^2X\bar{a}_z$

$\int_C F\cdot d\ell$

$\int_1 F\cdot d\ell = \int_0^2 X^2Y\,dY\,\Big|_{X=2} = \int_0^2 4Y\,dY = 2Y^2\Big|_0^2 = 8$

$\int_2 F\cdot d\ell = \int_2^0 (+X\,dX)\Big|_{Y=2} = \frac{-X^2}{2}\Big|_0^2 = -2$

$\int_3 F\cdot d\ell = \int_2^0 X^2Y\,dY\Big|_{X=0} = 0$

$\int_4 F\cdot d\ell = \int_0^2 X\,dX\Big|_{Y=0} = \frac{X^2}{2}\Big|_0^2 = 2$

FIND $\int_S (\nabla\times\bar{F})\cdot d\bar{S}$

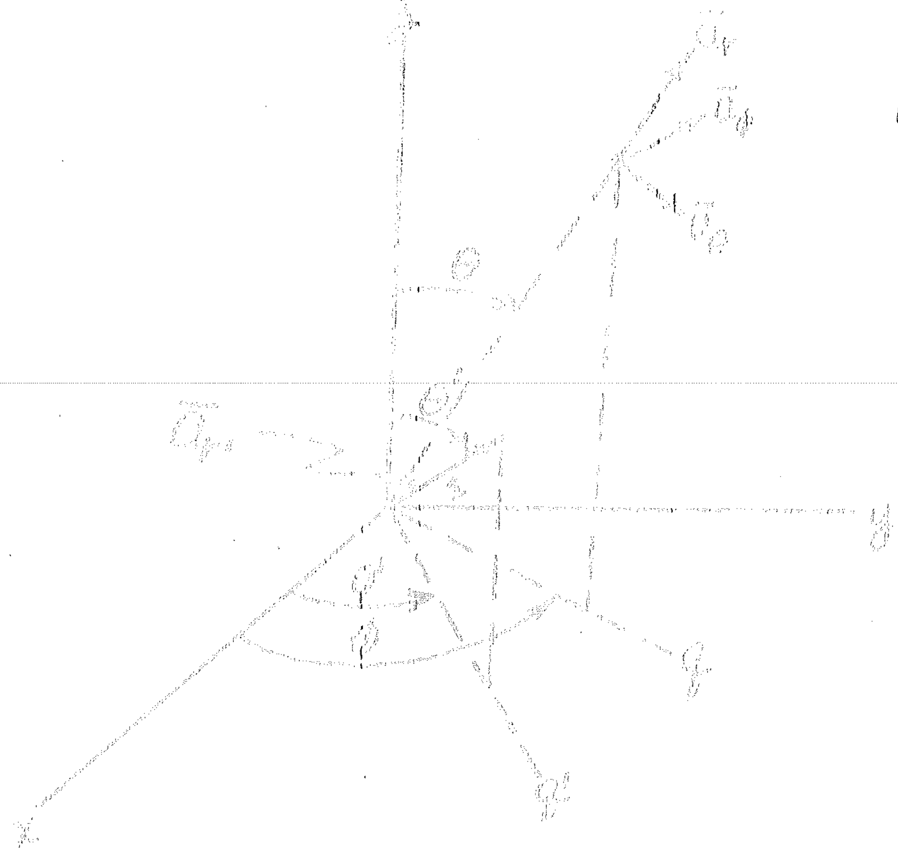
$$\nabla\times\bar{F} = \begin{vmatrix} \bar{a}_x & a_Y & a_z \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta Y} & \frac{\delta}{\delta z} \\ X & X^2Y & Y^2X \end{vmatrix} = (2YX)\bar{a}_x - (Y^2)\bar{a}_Y + 2XY\bar{a}_z$$

$d\bar{S} = dX\,dY\,\bar{a}_z$

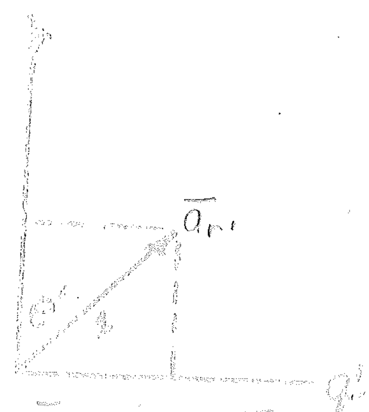
$\int_S \nabla\times\bar{F}\cdot dS = \int_S (2XY\bar{a}_x - Y^2\bar{a}_Y + 2XY\bar{a}_z)\cdot(dX\,dY\,\bar{a}_z)$

$= \int_S 2XY\,dX\,dY$

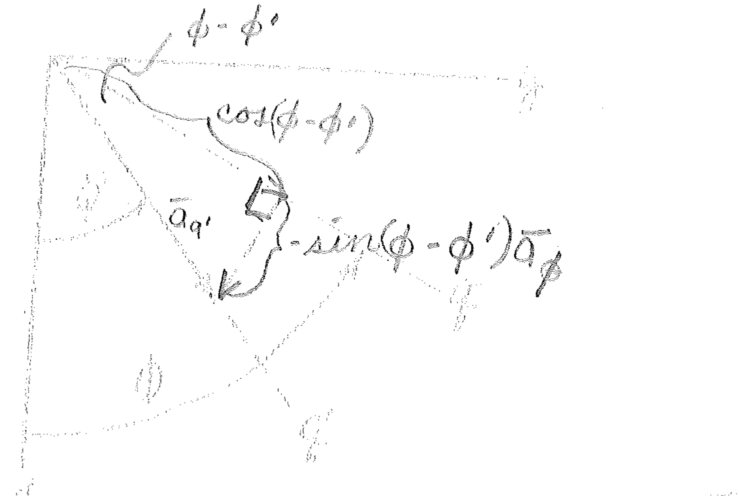
$= \int_0^2\int_0^2 2XY\,dX\,dY = 8$



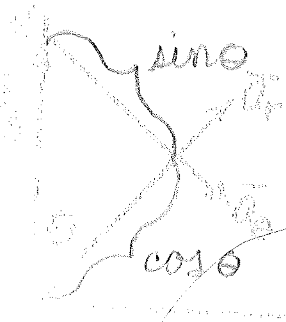
$$\vec{R} = r\bar{a}_r - r'\bar{a}_{r'}$$



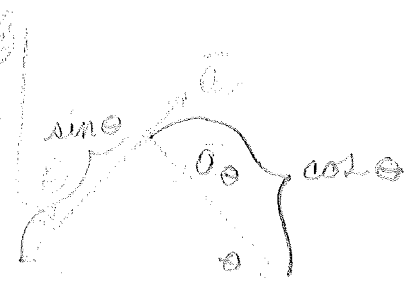
$$\bar{a}_{r'} = \sin\theta'\,\bar{a}_{q'} + \cos\theta'\,\bar{a}_z$$



$$\bar{a}_{q'} = \cos(\phi-\phi')\,\bar{a}_q - \sin(\phi-\phi')\,\bar{a}_\phi$$



$$\bar{a}_z = \cos\theta\,\bar{a}_r - \sin\theta\,\bar{a}_\theta$$



$$a_q = \sin\theta\,\bar{a}_r + \cos\theta\,\bar{a}_\theta$$

$$\bar{a}_{r'} = \cos\theta'\, \underbrace{\bar{a}_{z}}_{\cos\theta\, \bar{a}_r - \sin\theta\, \bar{a}_\theta} + \sin\theta'\, \underbrace{\bar{a}_{\rho'}}_{\cos(\phi-\phi')\, \bar{a}_\rho - \sin(\phi-\phi')\, \bar{a}_\phi}$$

$$\bar{a}_\rho = \sin\theta\, \bar{a}_r + \cos\theta\, \bar{a}_\theta$$

$$\bar{a}_{r'} = \cos\theta' \left(\cos\theta\, \bar{a}_r - \sin\theta\, \bar{a}_\theta\right)$$

$$+ \sin\theta' \left\{ \cos(\phi-\phi') \left[\sin\theta\, \bar{a}_r + \cos\theta\, \bar{a}_\theta\right] - \sin(\phi-\phi')\, \bar{a}_\phi \right\}$$

$$\bar{a}_{r'} = \left[\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi-\phi')\right] \bar{a}_r$$

$$+ \left[-\sin\theta \cos\theta' + \cos\theta \sin\theta' \cos(\phi-\phi')\right] \bar{a}_\theta$$

$$- \left[\sin(\phi-\phi')\right] \bar{a}_\phi$$

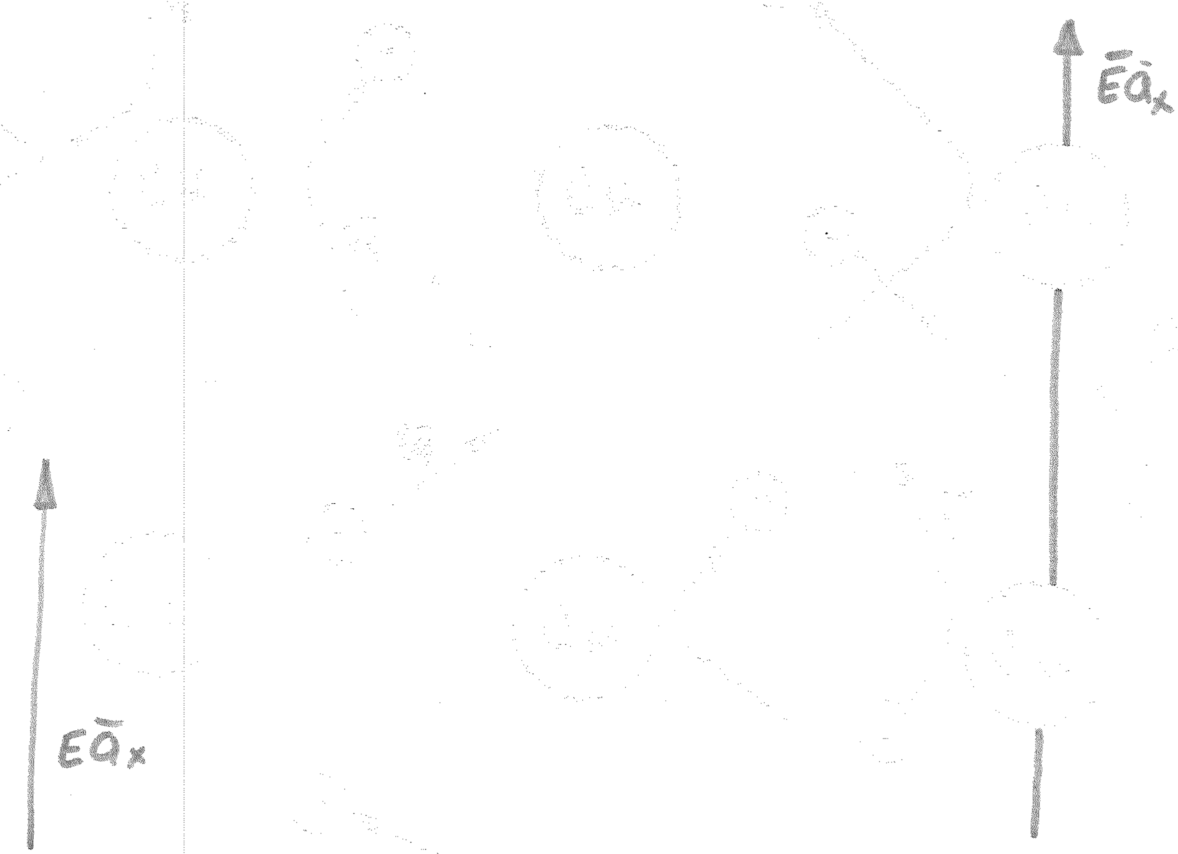
$$\bar{R} = r\, \bar{a}_r - r'\, \bar{a}_{r'}$$

$$\bar{R} = \left[r - r' \cos\theta \cos\theta' - r' \sin\theta \sin\theta' \cos(\phi-\phi')\right] \bar{a}_r$$

$$+ \left[r' \sin\theta \cos\theta' - r' \cos\theta \sin\theta' \cos(\phi-\phi')\right] \bar{a}_\theta$$

$$+ \left[r' \sin(\phi-\phi')\right] \bar{a}_\phi$$

$$= (x-x')\, \bar{a}_x + (y-y')\, \bar{a}_y + (z-z')\, \bar{a}_z$$



(=0 IF E=0)

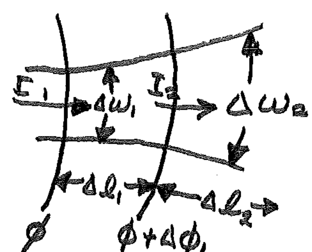
$$\bar{J}_x = \rho \langle v_x \rangle$$

DUALITY

CONDUCTOR	DIELECTRIC
$\nabla \times \vec{E} = 0$	$\nabla \times \vec{E} = 0$
$\vec{J} = \sigma \vec{E}$	$\vec{D} = \epsilon \vec{E}$
$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$	$\nabla \cdot \vec{D} - \rho = 0$
ASSUME $\nabla \cdot \vec{J} = 0$	ASSUME $\nabla \cdot \vec{D} = 0$
ALSO	ALSO
$\nabla \times \vec{J} = 0$	$\nabla \times \vec{D} = 0$
$\vec{J} = \sigma \vec{E} = -\sigma \nabla \phi$	$\vec{D} = \epsilon \vec{E} = -\epsilon \nabla \phi$
$\nabla^2 \phi = 0$	$\nabla^2 \phi = 0$

$$\vec{J} \Leftrightarrow \vec{D}$$
$$\sigma \Leftrightarrow \epsilon$$

"CURVILINEAR SQUARES"

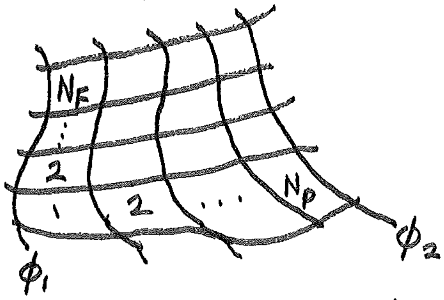


$$E \simeq \frac{\Delta\phi_1}{\Delta l_1}$$
$$J = \sigma E = -\sigma \frac{\Delta\phi_1}{\Delta l_1}$$
$$I_1 = J\,\Delta\omega_1 (1) = -\sigma \Delta\phi_1 \frac{\Delta\omega_1}{\Delta l_1}$$
$$I_2 = -\sigma \Delta\phi_2 \frac{\Delta w_2}{\Delta l_2}$$

FLOW OUT = FLOW IN $\Rightarrow I_2 = I_1$

IF WE SELECT $\frac{\Delta w}{\Delta l} = 1$ (SQUARES)

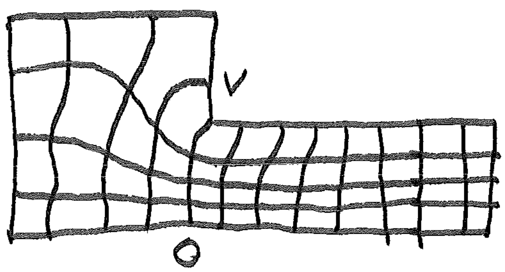
$I = -\sigma \Delta\phi$ (OR $\psi = \int \vec{D} \cdot d\vec{s} = -\epsilon \Delta\phi$)



$$I_{TOTAL} = -\sigma N_F \Delta\phi$$
$$\phi_2 - \phi_1 = V = N_p \Delta\phi$$
$$R = \frac{|V|}{|I_{TOTAL}|} = \frac{N_p}{\sigma N_F}$$

OR

$$\psi_{TOTAL} = \epsilon N_F \Delta\phi = Q$$
$$V = N_p \Delta\phi$$
$$C = \frac{Q}{V} = \frac{\epsilon N_F}{N_p}$$



$N_F = 12$
$N_p = 4$

ALUMINUM: $R = \frac{4}{(3.5\times10^7)(12)} = 0.0095 \ \frac{\mu V}{m}$

AIR: $C = \frac{10^{-9}}{36\pi}\left(\frac{12}{4}\right) = 26.6 \ \frac{pF}{m}$

IN A CONDUCTOR

$\nabla\cdot J + \frac{\delta\rho}{\delta t} = 0$ (CHARGE CONSERVED)

$\nabla\cdot\bar{D} = \rho \qquad \bar{J} = \sigma\bar{E}$

$\epsilon\nabla\cdot\bar{E} = \frac{\epsilon}{\sigma}\nabla\cdot J = \rho$

$\frac{\sigma}{\epsilon}\rho + \frac{\delta\rho}{\delta t} = 0 \ ; \quad \rho = \rho_0 e^{-\sigma t/\epsilon}$

$\tau = \frac{\epsilon}{\sigma}$ = RELAXATION TIME

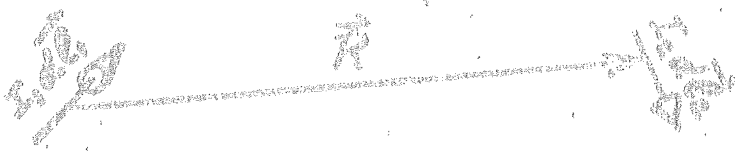
	τ
Cu	1.5×10^{-19} SEC
Ag	1.3×10^{-19} SEC
SEA H_2O	2×10^{-10} SEC
DISTILLED H_2O	10^{-16} SEC
QUARTZ	10 DAYS

FOR THERMAL CONDUCTION

$\nabla\cdot\bar{q} + S\rho\frac{\delta T}{\delta t} = 0$ (HEAT CONSERVED)

$\bar{q} = -k\nabla T$ (SEE PROB. 5.11)

MAGNETIC FIELDS



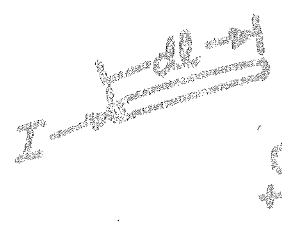
$$d\vec{F}_{21} = \frac{\mu_0}{4\pi R^2}\left[I_2\, d\vec{\ell}_2 \times \left(I_1\, d\vec{\ell}_1 \times \vec{a}_R\right)\right]$$

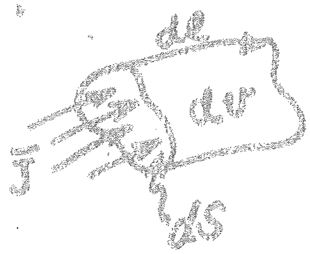
$$= I_2\, d\vec{\ell}_2 \times d\vec{B}_{21}$$

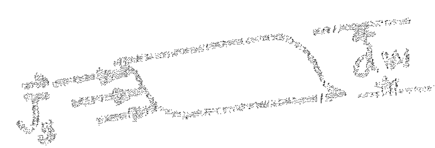
$$d\vec{B}_{21} = \frac{\mu_0}{4\pi R^2}\left(I_1\, d\vec{\ell}_1 \times \vec{a}_R\right)$$

A SMALL "CHUNK" OF CURRENT

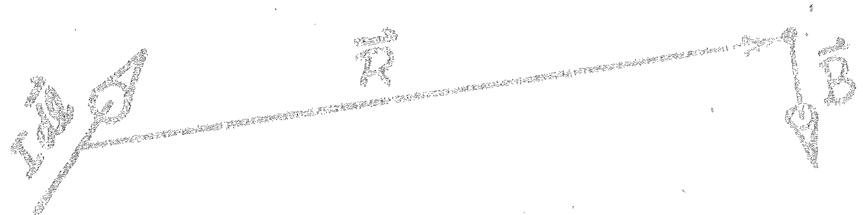
$$I\, d\vec{\ell} \rightarrow q\vec{v} \rightarrow \vec{J}\, dS\, dl = \vec{J}\, dv \rightarrow \vec{J}_s\, dw$$







$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{I\, d\vec{\ell} \times \vec{a}_R}{R^2}$$



$$\frac{\hat{a}_R}{R^2} = -\nabla\left(\frac{1}{R}\right)$$

[illegible] OF FIELD POINT

$$\frac{I\,d\vec{l}\times\hat{a}_R}{R^2} = -I\,d\vec{l}\times\nabla\left(\frac{1}{R}\right) = \nabla\times\left(\frac{I\,d\vec{l}}{R}\right) - \frac{1}{R}\nabla\times\left(I\,d\vec{l}\right)$$

$$\left(-\vec{A}\times\nabla\phi = \nabla\times\left(\phi\vec{A}\right) - \phi\,\nabla\times\vec{A}\right)$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int\left[\nabla\times\left(\frac{I\,d\vec{l}}{R}\right) - \frac{1}{R}\underbrace{\nabla\times\left(I\,d\vec{l}\right)}_{0}\right]$$

$$= \nabla\times\frac{\mu_0}{4\pi}\int\frac{I\,d\vec{l}}{R} = \nabla\times\vec{A}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int\frac{I\,d\vec{l}}{R} \Rightarrow \text{MAGNETIC VECTOR POTENTIAL}$$

$$\nabla \times \vec{B} = \nabla \times \nabla \times \frac{\mu_0}{4\pi} \int \frac{\vec{J}\, dv'}{R} = \frac{\mu_0}{4\pi} \int \nabla \times \left(\nabla \times \frac{\vec{J}}{R}\right) \ldots$$

$$= \frac{\mu_0}{4\pi} \int \Big[\nabla \underbrace{\nabla \cdot \left(\frac{\vec{J}}{R}\right)}_{0} - \nabla^2 \left(\frac{\vec{J}}{R}\right) \Big] dv$$

$$\left(\int \nabla \phi \, dv = \oint \phi \, d\vec{S} \right)$$

$$= -\frac{\mu_0}{4\pi} \int \nabla^2 \left(\frac{\vec{J}}{R}\right) dv = -\frac{\mu_0}{4\pi} \int \vec{J}\, \nabla^2 \left(\frac{1}{R}\right) dv$$

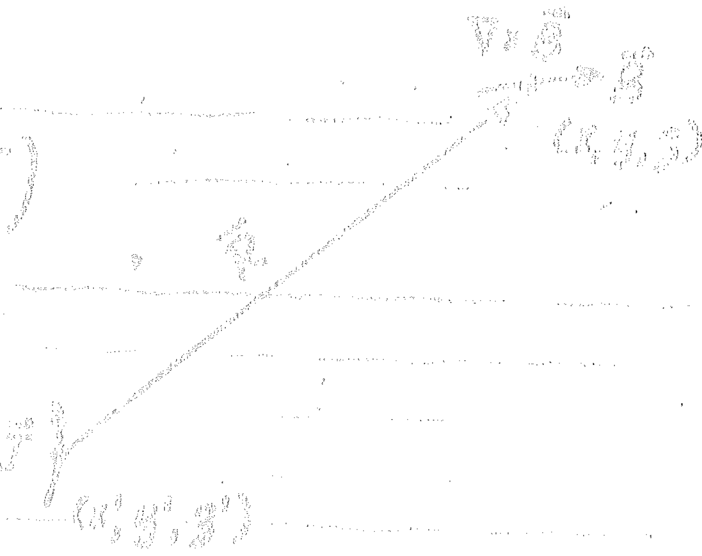
$$= -\frac{\mu_0}{4\pi} \int \vec{J} \left[-4\pi\, \delta(R) \right] dv$$

$$= \frac{\mu_0}{4\pi} \left[4\pi\, \vec{J}(R=0) \right]$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}(x,y,z)$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} \quad \left(\begin{matrix}\text{POINT} \\ \text{FORM}\end{matrix}\right)$$

$$\oint \vec{B} \cdot d\vec{l} = I \quad \left(\begin{matrix}\text{INTEGRAL} \\ \text{FORM}\end{matrix}\right)$$



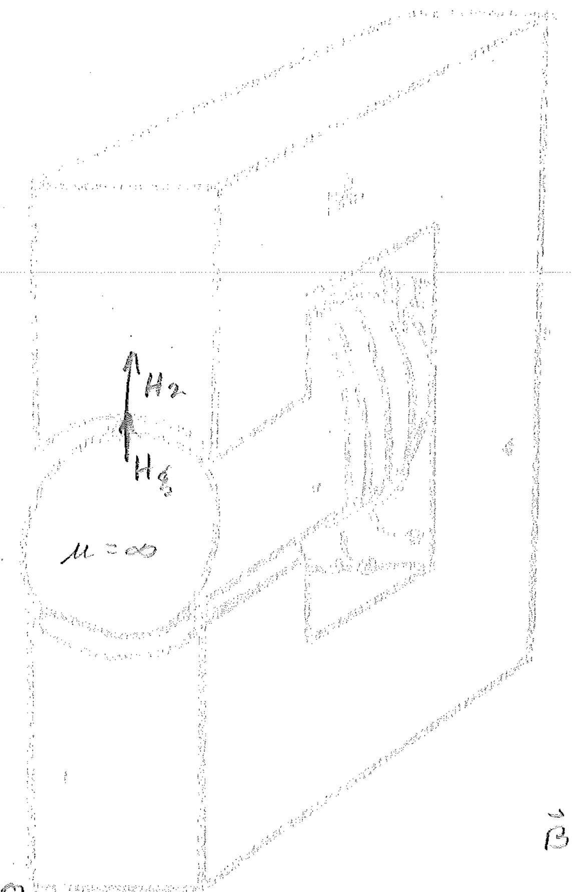
$$\int \nabla \times \vec{B} \cdot d\vec{S} = \oint \vec{B} \cdot d\vec{l} = \int \mu_0 \vec{J} \cdot d\vec{S} = \mu_0 I$$

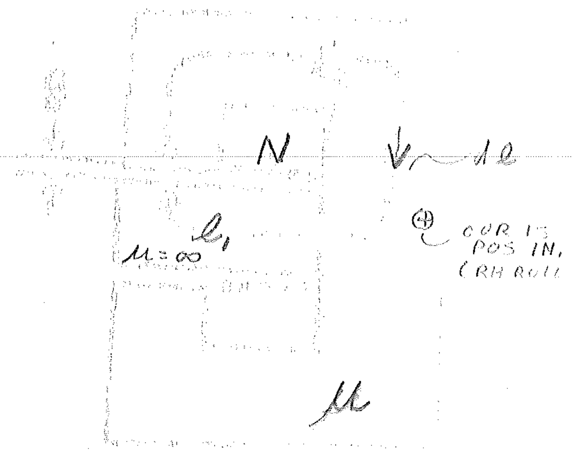
$$\oint \vec{B}\cdot d\vec{S} = 0 \quad \text{(Gauss's law)}$$

$$\int \nabla\cdot\vec{B}\, dv = 0 \qquad \nabla\cdot\vec{B} = 0$$

ELECTRIC FIELDS		MAGNETIC FIELDS	
INTEGRAL	DIFFERENTIAL	DIFFERENTIAL	INTEGRAL
$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\vec{F}}{dq}$		$d\vec{F} = I\, d\vec{l} \times \vec{B}$	
$\oint \vec{D}\cdot d\vec{S} = q$	$\nabla\cdot\vec{D} = \rho$	$\nabla\cdot\vec{B} = 0$	$\oint \vec{B}\cdot d\vec{S} = 0$
$\oint \vec{E}\cdot d\vec{l} = 0$	$\nabla\times\vec{E} = 0$	$\nabla\times\vec{B} = \mu_0 \vec{J}$	$\oint \vec{B}\cdot d\vec{l} = I\mu_0$
$\phi = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\, dv}{R}$	$\vec{E} = -\nabla\phi$	$\vec{B} = \nabla\times\vec{A}$	$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\, dv}{R}$
$\nabla^2\phi = -\rho/\epsilon_0$		$\nabla^2\vec{A} = -\mu_0\vec{J}$	
$\vec{D} = \epsilon_0\vec{E} + \vec{P}$		$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$	[illegible]
		$\vec{B} = K_m \mu_0 \vec{H} = \mu \vec{H}$	

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$
$$\vec{B} = \mu_0 (\vec{H} + \vec{M})$$





AMPERE'S LAW

$$\oint \vec{B} \cdot dl = \mu_0 I$$

OR $\boxed{\oint H \cdot dl = I}$ ← THE GOODIE

$$\vec{B} = \mu \vec{H}$$

$$\nabla \cdot B = 0$$

OR $\oint \vec{B} \cdot d\vec{S} = 0$

$\Rightarrow (B_2 - B_1) \cdot \vec{a}_n = 0$

$$H_2 l_2 + \cancelto{0}{H_1 l_1} + H_g g = NI$$

$$\mu_0 H_g = K_m \mu_0 H_2 \Rightarrow H_2 = \frac{H_g}{K_m}$$

$$\Rightarrow H_g \left(\frac{l_2}{K_m} + g \right) = NI \Rightarrow \vec{B} = \mu_0 H_g = \frac{\mu_0 N I}{\left(\frac{l_2}{K_m} + g \right)}$$

$$\nabla\cdot\bar F = \lim_{\Delta v\to 0}\frac{\oint \bar F\cdot d\bar S}{\Delta v} \qquad \nabla\times\bar F\cdot\bar a_n = \lim_{\Delta S\to 0}\frac{1}{\Delta S}\oint \bar F\cdot d\bar l$$

$$\oint \bar F\cdot d\bar S = \int \nabla\cdot\bar F\, dv \qquad \int \nabla\times\bar F\cdot d\bar S = \oint \bar F\cdot d\bar l$$

$$\frac{d\bar a_r}{dt} = \lim_{\Delta t\to 0}\frac{\bar a_r(t+\Delta t) - \bar a_r(t)}{\Delta t}$$

$$= \lim_{\Delta t\to 0}\frac{\cos\Delta\phi\,\bar a_r + \sin\Delta\phi\,\bar a_\phi - \bar a_r}{\Delta t}$$

$$= \lim_{\Delta t\to 0}\frac{(\cos\Delta\phi - 1)\bar a_r + \sin\Delta\phi\,\bar a_\phi}{\Delta t}$$

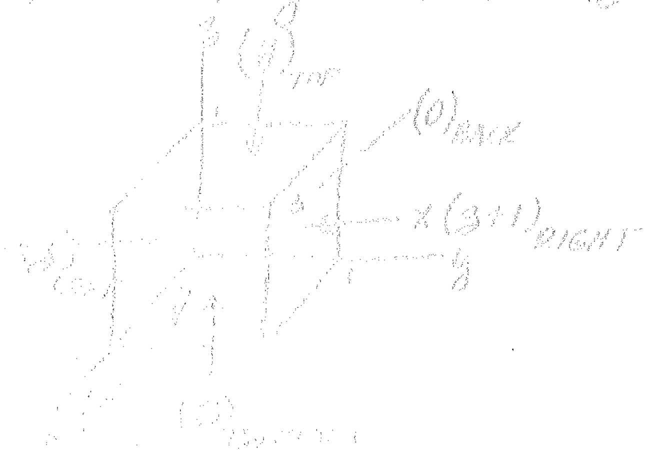
$$= \lim_{\Delta t\to 0}\frac{(1-1)\bar a_r + \Delta\phi\,\bar a_\phi}{\Delta t} = \frac{d\phi}{dt}\bar a_\phi$$

$$\frac{d\bar a_r}{dt} = \frac{d\phi}{dt}\bar a_\phi$$

$\frac{Q}{4\pi\epsilon_0 R}$ (point charge) $\quad \phi = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\frac{\rho_L\, a\,d\phi'}{\{[r - a\cos(\phi-\phi')]^2 + a^2\sin^2\ldots\}}$

$$= \frac{a\rho_L}{4\pi\epsilon_0}\int_0^{2\pi}\frac{d\phi'}{[r^2+z^2+a^2-2ar\cos(\phi-\phi')]^{1/2}}$$

$\nabla\cdot\bar F = 1+x+y \qquad \nabla\times\bar F = (z-x)\bar a_x + (y-z)\bar a_z \qquad \ldots$



$$\int \ldots\cdot d\bar S = \int_0^1\int_0^1 (y+1)\,dx\,dy$$
TOP + BOTTOM
$$+ \int_0^1\int_0^1 (1+1)\,dy\,dz$$
FRONT + BACK
$$+ \int_0^1\int_0^1 [\ldots]\,dx\ldots$$

$$[illegible] = \int (\nabla \cdot \bar{F})\, dv = \int_0^1\!\!\int_0^2\!\!\int_0^1 (1+x+y)\, dx\, dy\, dz = [illegible]\,2$$

$$\bar{A} = \bar{a}_x - 2\bar{a}_y - \bar{a}_z$$

$$\bar{a}_A = \frac{1}{\sqrt{6}} (\bar{a}_x - 2\bar{a}_y - \bar{a}_z)$$

$$\frac{x-1}{\left(\frac{1}{\sqrt{6}}\right)} = \frac{y}{\left(\frac{-2}{\sqrt{6}}\right)} = \frac{z}{\left(\frac{-1}{\sqrt{6}}\right)}$$

$$x - 1 = \frac{-y}{2} = -z$$

$$d\bar{l} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z$$

$$\int (y\,\bar{a}_x + x(y+z)\,\bar{a}_y + yz\,\bar{a}_z) \cdot (dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z)$$

$$y = 2 - 2x \qquad dy = [illegible]$$

$$\int y\,dx + x(y+z)\,dy + yz\,dz$$

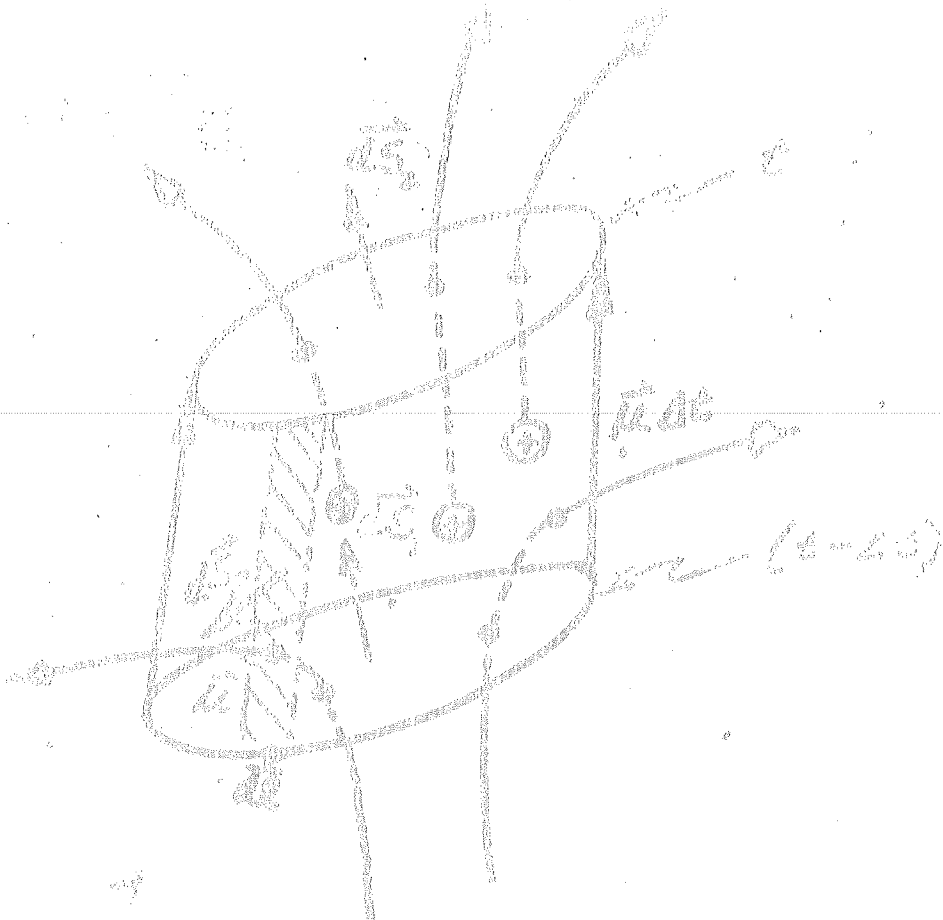
$$z = 1 - x \qquad dz = [illegible]$$

$$\int y\,dx + x\left[(2-2x) + (1-x)\right](-2dx) + (2-2x)(1-x)(-dx)$$

$$\int (9x^2 \; x - 2)\, dx = -7/6$$

Homework Assignment

Reading 25, 27, 28, 29



$$\frac{d}{dt}\int \vec{B}\cdot d\vec{S} = \int \dot{\vec{B}}\cdot d\vec{S}$$

$$\frac{d}{dt}\int \vec{B}\cdot d\vec{S} = \lim_{\Delta t\to 0}\frac{\int \vec{B}(t)\cdot d\vec{S}_2 - \int \vec{B}(t-\Delta t)\cdot d\vec{S}_1}{\Delta t}$$

APPLYING GAUSS' LAW OVER THE SURFACE AT TIME t:

$$\int \vec{B}(t)\cdot d\vec{S}_3 + \int_{S_2} \vec{B}(t)\cdot d\vec{S}_2 - \int_{S_1} \vec{B}(t)\cdot d\vec{S}_1 = \int \nabla\cdot\vec{B}\, dv$$

$(d\vec{l}\times\vec{u}\,\Delta t)$ $\qquad$ $\vec{u}\,\Delta t\cdot d\vec{S}_1$

$$\int \vec{B}(t-\Delta t)\cdot d\vec{S}_1 = \int \left[\vec{B}(t) - \frac{\partial\vec{B}}{\partial t}\Delta t\right]\cdot d\vec{S}_1$$ [illegible]

$$= \left[\int_V \nabla\cdot\vec{B}\ \vec{u}\,\Delta t \cdot d\vec{S}_2 + \int_{S_1} \vec{B}(t)\cdot d\vec{S}_1 - \oint_{S_3} \vec{B}(t)\cdot(d\vec{l}\times\vec{u}\,\Delta t) \right]$$

$$- \left[\int_{S_1} \vec{B}(t)\cdot d\vec{S}_1 - \int_{S_1} \frac{\partial\vec{B}}{\partial t}\Delta t \cdot d\vec{S}_1 \right]$$

$$= \Delta t \left[\int (\nabla\cdot\vec{B})\,\vec{u}\cdot d\vec{S} + \int \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S}_1 - \oint \vec{B}(t)\cdot(d\vec{l}\times\vec{u}) \right]$$

$$\frac{d}{dt}\int \vec{B}\cdot d\vec{S} = \int (\nabla\cdot\vec{B})\,\vec{u}\cdot d\vec{S} + \int \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S} - \oint \vec{B}\cdot(d\vec{l}\times\vec{u})$$

$$\vec{B}\cdot d\vec{l}\times\vec{u} = \vec{u}\times\vec{B}\cdot$$

$$\oint \vec{u}\times\vec{B}\cdot d\vec{l} = \int(\nabla\times\ldots$$

$$= \int \left(\vec{u}\,\nabla\cdot\vec{B} + \frac{\partial\vec{B}}{\partial t} - \nabla\times\vec{u}\times\vec{B} \right)\cdot d\vec{S}$$

$$\dot{\vec{B}} = \mu\nabla\cdot\vec{B} + \frac{\partial\vec{B}}{\partial t} - \nabla\times\vec{u}\times\vec{B}$$

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{S}$$

$$= -\int \left((\mu\nabla)\underset{0}{\vec{B}} + \frac{\partial\vec{B}}{\partial t} - \nabla\times\vec{u}\times\vec{B} \right)\cdot d$$

$$\int \nabla\times\vec{E}\cdot d\vec{S} = -\int \left(\frac{\partial\vec{B}}{\partial t} - \nabla\times\vec{u}\times\vec{B} \right)\cdot d\vec{S}$$

$$\nabla \times \vec{E} = -\left(\frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{u} \times \vec{B})\right)$$

$$\nabla \times \left(\vec{E} - \vec{u} \times \vec{B}\right) = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{F} = q\left(\vec{u} \times \vec{B}\right)$$

$$\vec{E} = \frac{\vec{F}}{q} = \vec{u} \times \vec{B}$$

VECTOR ANALYSIS

$$\nabla \times F = \begin{vmatrix} a_x & a_y & a_z \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ F_x & F_y & F_z \end{vmatrix}$$

$$\nabla^2 = \nabla \cdot \nabla = \frac{\delta^2}{\delta x} + \frac{\delta^2}{\delta y} + \frac{\delta^2}{\delta z}$$

GAUSS' LAW

$$\int_V \nabla \cdot \bar{F} dV = \oint_S \bar{F} \cdot d\bar{s}$$

STOKES LAW

$$\oint_c \bar{F} \cdot d\bar{l} = \int_S \nabla \times \bar{F} \cdot d\bar{s}$$

$\nabla \cdot \bar{F} = 0 \Rightarrow$ SOLENOIDAL

$\nabla \times \bar{F} = 0 \Rightarrow$ IRROTATIONAL

STATIONARY CURRENTS

$$I = \int_S \bar{J} \cdot d\bar{S}$$

OHM'S LAW: $\bar{J} = \sigma(E + E')$

FOR UNIFORM CYLYNDER:

$$R = \oint dl / \sigma A$$

CONSERVATION OF CHARGE:

$$\int_V (\nabla \cdot J) + \frac{\delta \rho}{\delta t}) dV = 0$$

WITH STEADY CURRENTS:

$$\nabla \cdot J = 0$$

RELAXATION TIME $\tau = \frac{\epsilon}{\sigma}$

$\mathcal{E} = \int_0 E \cdot dl$ FOR CONS $\bar{E}$

$$\Rightarrow R = \frac{\mathcal{E}}{I} = \frac{\int_c E \cdot dl}{\int_A \sigma E \cdot ds}$$

$$I^2 R = \int_V \bar{J} \cdot \bar{E} dV = \int_V \sigma \bar{E} \cdot \bar{E} dV = \frac{1}{\sigma} \int_V \bar{J} \cdot \bar{J} dV$$ JOULE LAW

FLUX PLOTTING

$$\psi_{TOT} = \epsilon N_F \Delta\phi = Q$$

$$V = N_p \Delta\phi = \phi_2 - \phi_1$$

$$R = \frac{|V|}{|I_{TOT}|} = \frac{N_p}{\sigma N_F}$$

$$C = Q/V = \epsilon N_F / N_p$$

MAGNETIC FIELD IN MATERIAL BODIES

J_m = POLARIZATION CURRENT; J_{ms} = EQ. POL. CUR

$$\nabla' \times M = J_m \; ; \; \bar{M} \times \bar{n} = J_{ms}$$

$$A = \frac{\mu_0}{4\pi} \nabla \times \frac{\bar{m}}{R} = \frac{\mu_0}{4\pi} \left(\int_V \frac{J_m}{R} dV' + \oint_S \frac{J_{ms}}{R} ds' \right)$$

$$m = M dV = I dS$$

$$I_z = -\frac{\delta M_x}{\delta x} dx dy \; ; \; J_x = \frac{-\delta M_x}{\delta y}$$

FLUX INTENSITY $\bar{H} = \frac{B}{\mu_0} - \bar{M}$

$$\nabla \times H = J \; ; \; \bar{M} = \chi_m \bar{H}$$

$$B = \mu_0 (1 + \chi_m) \bar{H} = \mu \bar{H}$$

$$\int_S \nabla \times \bar{H} \cdot d\bar{S} = \oint_c H \, dl = \int_S J \cdot ds = NI$$

$\nabla \times \bar{H} = 0 \; ; \; \nabla \cdot H = -\nabla \cdot M$ FOR A PERMANENT MAGNET

$$\mathcal{R} = l / \mu A$$

ELECTROSTATICS

COULOMB'S LAW:

$$F_{12} = \frac{q_1 q_2}{4\pi \epsilon_0 R^2} \bar{a}_r$$

$$\epsilon_0 = \frac{1}{36\pi} \times 10^{-9} \frac{F}{m}$$

$$R = [(x-x')^2 + (y-y')^2 + (z-z')^2]^{\frac{1}{2}}$$

ELECRIC FIELD (DEF)

$$\bar{E} = \lim_{\Delta q \to 0} \bar{F} / \Delta q$$

CHARGE DENSINTY

$$\rho = \lim_{\Delta V \to 0} \Delta q / \Delta V$$

GAUSS' FLUX THEM

$$\oint E \cdot ds = q / \epsilon_0$$

$$\nabla \times \bar{E} = 0$$

POISSON'S EQUATION

$$\oint_S E \cdot ds = \frac{1}{\epsilon_0} \int_V \rho dV$$

$$\nabla^2 \phi = -\rho / \epsilon_0$$

DUALITY TWIXT $\bar{J}$ AND $\bar{D}$

($\bar{D}$ = DISPLACEMENT FLUX DENSITY)

CONDUCTING	DIELECTRIC
$\nabla \times \bar{E} = 0$	$\nabla \times \bar{E} = 0$
$\bar{J} = \sigma \bar{E}$	$\bar{D} = \epsilon \bar{E}$
$\nabla \cdot \bar{J} + \frac{\delta \rho}{\delta t} = 0$	$\nabla \cdot \bar{D} - \rho = 0$
IF $\nabla \cdot \bar{J} = 0$	IF $\nabla \cdot \bar{D} = 0$
$\Rightarrow \bar{J} = \sigma \bar{E} = -\sigma \nabla \phi$	$\bar{D} = \epsilon E = \epsilon \nabla \phi$
$\nabla^2 \phi =$	$\nabla^2 \phi = 0$

$$\bar{J} \Leftrightarrow \bar{D} \; ; \; \epsilon \Leftrightarrow \sigma$$

$$C = \int_S \epsilon \bar{E} \cdot d\bar{S} \Big/ \int_L \bar{E} \cdot d\bar{l}$$

NEGLECTING FRINGE EFFECTS

$$RC = \epsilon / \sigma = C/G$$

CONVECTION CURRENT

$$dI = \frac{\rho ds dl}{dt} = \rho v ds$$

CONV. CURR. DENSITY

$\bar{J} = \rho \bar{V} \ni V$ = VELOCITY

QUASI-STATIONARY MAGNETIC FIELDS

FARADAY'S LAW:

$$\oint_c \bar{E} \cdot dl = -\frac{\delta}{\delta t} \int_S \bar{B} \cdot d\bar{S} = -\frac{\delta}{\delta t} \int_S \nabla \times \bar{A} \cdot d\bar{S} = -\frac{\delta}{\delta t} \oint_c \bar{A} \cdot dl$$

$$\Rightarrow \int_S (\nabla \times E + \frac{\delta B}{\delta t}) \cdot ds = 0$$

INDUCED FIELD DUE TO MOTION

$$\bar{F} = q \bar{V} \times \bar{B} \Rightarrow \bar{E} = \bar{F}/q = \bar{V} \times \bar{B}$$

$$V_{IND} = \int_S \frac{\delta B}{\delta t} \cdot ds + \oint_c V \times B \cdot dl$$

$\psi_{21} = \int_{S_1} \bar{B}_2 \cdot ds$ (LENZ'S LAW)

$$L_{11} = \psi_{11} / I_1 \; ; \; L_{21} = \psi_{21} / I_2$$

"ELECTROSTATIC" FIELDS IN MATERIAL BODIES

$$\bar{P} = N(\alpha_e + \alpha_2 + \frac{p^2}{3kT}) \bar{E} = \epsilon_0 \chi_e E$$

χ_e = ELECTRIC SUSCEPTIBILITY

$$\bar{D} = \epsilon_0 \bar{E} + \bar{P} = \epsilon \bar{E}$$

$$\nabla \cdot \bar{D} = \rho$$

$$\oint_S D \cdot ds = \int_V \rho dV = Q$$

$E_a = K E_d \ni K$ IS DIELECTRIC

E FIELDS	H FIELDS
$\bar{E} = \lim_{\Delta q \to 0} F/\Delta q$	$d\bar{F} = I d\bar{l} \times \bar{B}$ DEF
$\oint D \cdot ds = q \; ; \; \nabla \cdot D = \rho$	$\nabla \cdot B = 0 \; ; \; \oint B \cdot ds = 0$
$\oint E \cdot dl = 0 \; ; \; \nabla \times \bar{E} = 0$	$\nabla \times \bar{B} = \mu_0 J \quad \oint B \cdot dl = I\mu_0$ FUND LAW
$\phi = \frac{1}{4\pi} \int \frac{\rho dV}{R} \; ; \; \bar{E} = -\nabla \phi$	$\nabla \times \bar{B} = \mu_0 \bar{J} \; ; \; \oint B \cdot dl = I\mu_0$
	$\bar{B} = \nabla \bar{A} \quad \bar{A} = \frac{\mu_0}{4\pi} \int \bar{J} \, dV/R$
$\nabla^2 \phi = \rho / \epsilon_0$	$\nabla^2 A = -\mu_0 J$
$\bar{D} = \epsilon_0 E + \bar{P}$	$\bar{H} = B/\mu_0 - M$
	$B = K_m \mu_0 H = \mu \bar{H}$

STATIC MAGNETIC FIELD IN A VACUUM

AMPERE'S LAW

$$dF_{21} = \oint_{c_2}\oint_{c_1} \frac{\mu_0}{4\pi R^2} [I_2 dl_2 \times (I_1 dl_1 \times \bar{a}_n)]$$

μ_0 = PERMEABILITY (F/M)

$$B_{21} = \frac{\mu_0}{4\pi} \oint_{c_1} I_1 d\bar{l}_1 \times \bar{a}_r / R^2$$

LORENTZ FORCE: $\bar{F} = q \bar{V} \times \bar{B}$

$$\Rightarrow B = \frac{\mu_0}{4\pi} \int_S \frac{J_s \times \bar{a}_r}{R^2} ds'$$

(B - WEB/M^2 = V-SEC/M^2)

$$A = \frac{\mu_0}{4\pi} \int_V \frac{J(x', y', z')}{R} dV'$$

$$\Rightarrow \bar{B} = \nabla \times \bar{A}$$

MAGNETIC DIPOLE

$$M = \frac{I}{2} \oint r \times dr = \frac{1}{2} \int_V \bar{r} \times \bar{J} dV$$

T (TORQUE) $= \bar{M} \times \bar{B}$ $\quad (\int dM \times B)$

MAGNETIC FLUX AND $\nabla \cdot B$

$$\nabla \cdot \bar{B} = 0$$

$$\psi = \int_S \bar{B} \cdot d\bar{S} = \int_S \nabla \times \bar{A} \cdot d\bar{S} = \oint_c \bar{A} \cdot d\bar{l}$$

AMPERE'S CIRCUITAL LAW

$$\nabla \times \bar{B} = \mu_0 \bar{J}(x, y, z)$$

$$\int_S \nabla \times B \cdot ds = \int_S \mu_0 \bar{J} \cdot d\bar{S} = \oint_c \bar{B} \cdot d\bar{l}$$

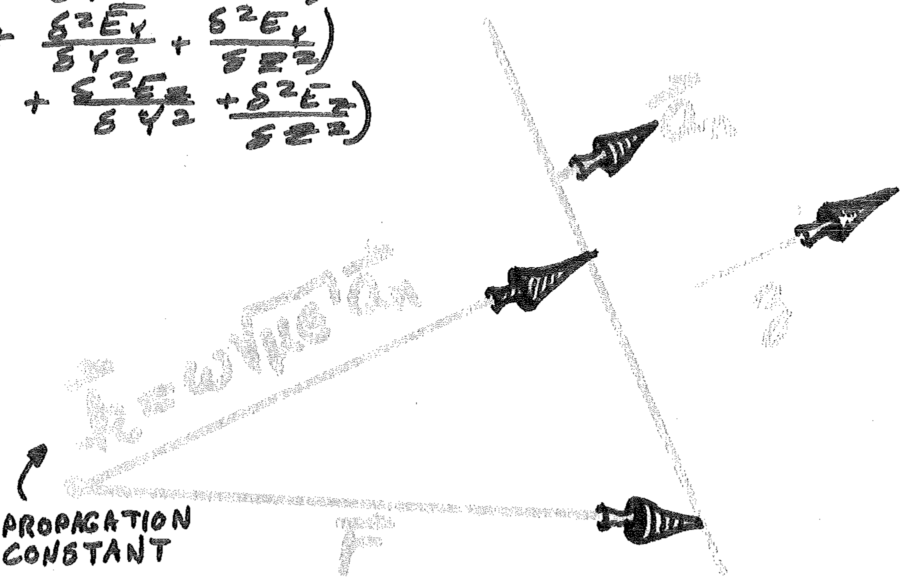
$$\nabla \cdot D = \rho \; ; \; \nabla \cdot \bar{E} = \rho / \epsilon_0 \; ; \; \nabla \times B = \mu_0 \bar{J}$$

$$\nabla \cdot \bar{D} = \nabla \times \bar{P}$$

$$q \, dV = I d\bar{l}$$

FIELD VARIATION IN A PLANE NORMAL TO DIRECTION OF PROPAGATION

$$\nabla^2\vec{E} = \bar{a}_x\left(\frac{\delta^2 E_x}{\delta x^2} + \frac{\delta E_x}{\delta y^2} + \frac{\delta^2 E_x}{\delta z^2}\right)$$
$$+ \bar{a}_y\left(\frac{\delta^2 E_y}{\delta x^2} + \frac{\delta^2 E_y}{\delta y^2} + \frac{\delta^2 E_y}{\delta z^2}\right)$$
$$+ a_y\left(\frac{\delta^2 E_z}{\delta x^2} + \frac{\delta^2 E_z}{\delta y^2} + \frac{\delta^2 E_z}{\delta z^2}\right)$$



$$e^{j(\omega t - kz)} = e^{j(\omega t - \vec{k}\cdot\vec{r})}$$

$$\nabla^2\vec{E} - \mu\epsilon\frac{\delta^2 \vec{E}}{\delta t^2} = 0$$
$$\vec{E}_0 = E_0(x, y, z, t)$$
$$= \vec{E}(x, y, z)e^{j\omega t}$$
$$\Rightarrow \nabla^2\vec{E} + \omega^2\mu\epsilon\vec{E} = 0$$

PHASOR $\quad \nabla^2\vec{E} + k^2\vec{E} = 0$

$E_x(z)$

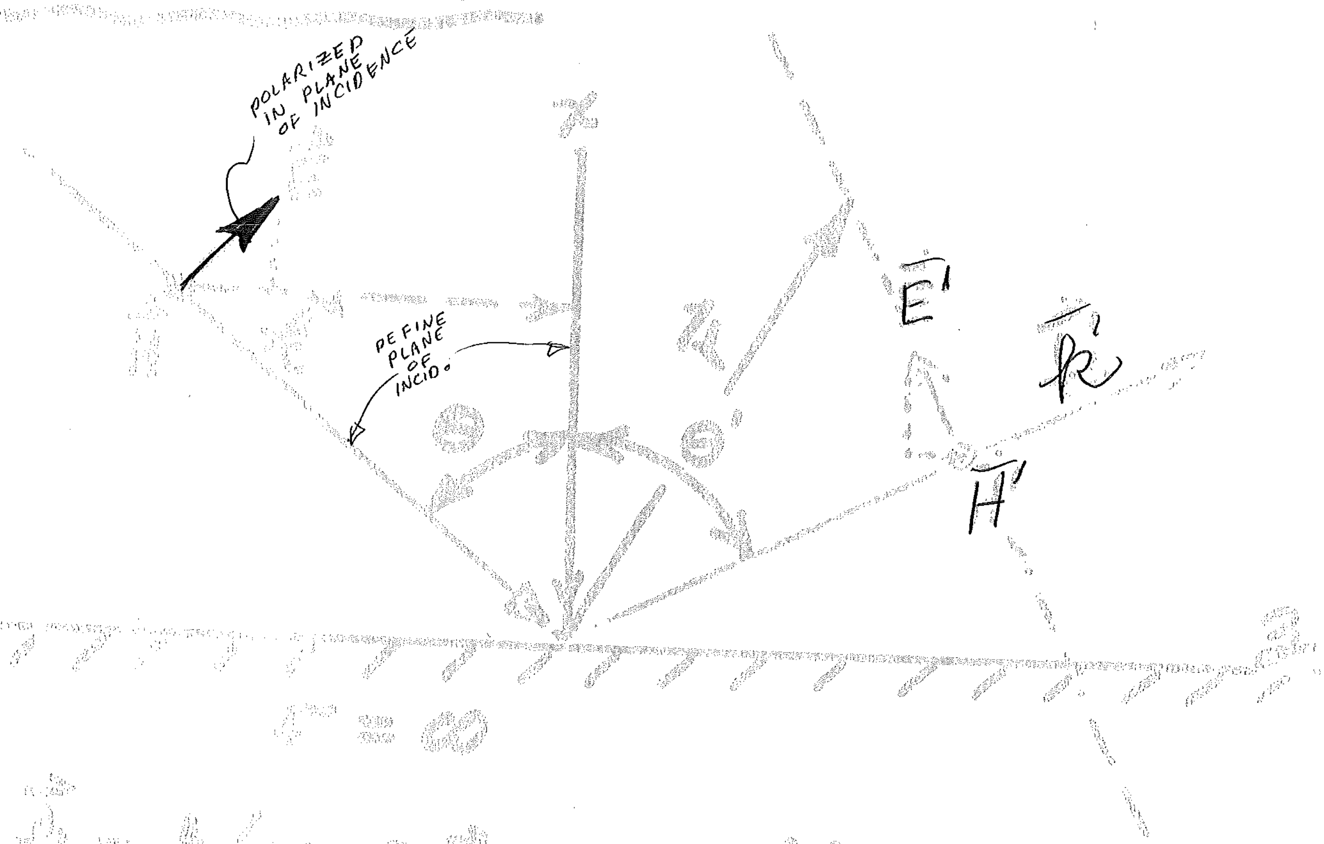
ASSUME ONLY X COMPONENT VARYING WITH $z \Rightarrow \nabla^2\bar{E} = \frac{\delta^2 E_x}{\delta z^2}$

$$\frac{\partial^2 \bar{E}}{\partial \xi^2} + k^2 \bar{E} = 0$$

(UNIFORM PLANE WAVE)

$$\bar{E}_i(\xi, t) = \bar{E}_0 e^{j(\omega t - k\xi)} = \bar{E}_0 e^{j(\omega t - \bar{k}\cdot\bar{r})}$$

REFLECTION FROM PERFECT CONDUCTOR



$$\bar{k} = k(\sin\theta\, \bar{a}_z - \cos\theta\, \bar{a}_x)$$

$$\bar{k}' = k(\sin\theta'\, \bar{a}_z + \cos\theta'\, \bar{a}_x)$$

$$\bar{r} = x\bar{a}_x + z\bar{a}_z$$

$$\vec{E} = E_0(\cos\theta\,\vec{a}_z + \sin\theta\,\vec{a}_x)\,e^{j(\omega t - \vec{k}\cdot\vec{r})}$$

$$\vec{E}' = E_0'(-\cos\theta'\,\vec{a}_z + \sin\theta'\,\vec{a}_x)\,e^{j(\omega t - \vec{k}'\cdot\vec{r})}$$

$$E_x = E_0 \sin\theta\, e^{-jk(-x\cos\theta + z\sin\theta)}$$
$$+ E_0' \sin\theta'\, e^{-jk(x\cos\theta' + z\sin\theta')}$$

$$E_z = E_0 \cos\theta\, e^{-jk(x\cos\theta + z\sin\theta)}$$
$$- E_0' \cos\theta'\, e^{-jk(x\cos\theta' + z\sin\theta')}$$

$$H_y = \frac{E_0}{\eta} e^{-jk(x\cos\theta + z\sin\theta)} + \frac{E_0'}{\eta} e^{-jk(x\cos\theta' + z\sin\theta')}$$

$\eta = \sqrt{\frac{\mu}{\epsilon_0}}$ ← CHARACTERISTIC Z OF FREE SPACE

<u>At $x = 0$, $E_z = 0$</u>

$$0 = E_0 \cos\theta\, e^{-jkz\sin\theta} - E_0' \cos\theta'\, e^{-jkz\sin\theta'}$$

$$\sin\theta = \sin\theta' \qquad \theta = \theta'$$
$$\rightarrow E_0 = E_0'$$

$k = \sqrt{\mu\epsilon}\,\omega$

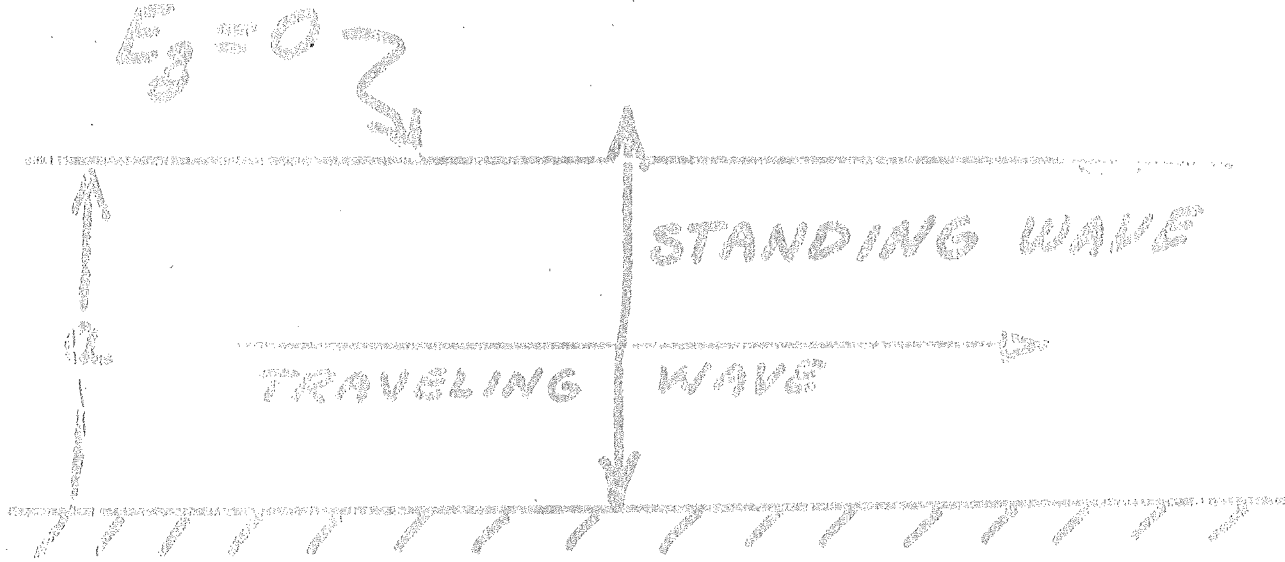
~~FOR PLANE WAVE~~

$$E_x = 2E_0 \sin\theta \cos(kx\cos\theta)\, e^{-jkz\sin\theta}$$

$$E_z = 2jE_0 \cos\theta \sin(kx\cos\theta)\, e^{-jkz\sin\theta}$$

$$H_y = \frac{2E_0}{\eta} \cos(kx\cos\theta)\, e^{-jkz\sin\theta}$$

$$v_p = \frac{\omega}{k\sin\theta} = \frac{c}{\sin\theta}$$ — PHASE VELOCITY



$$ka\cos\theta = n\pi$$

$$a = \frac{n\pi}{k\cos\theta}$$

$$\cos\theta = \frac{n\pi}{ka}$$

$$\sin\theta = \sqrt{1-\left(\frac{n\pi}{ka}\right)^2}$$

$$k\sin\theta = \sqrt{k^2-\left(\frac{n\pi}{a}\right)^2} = \beta$$

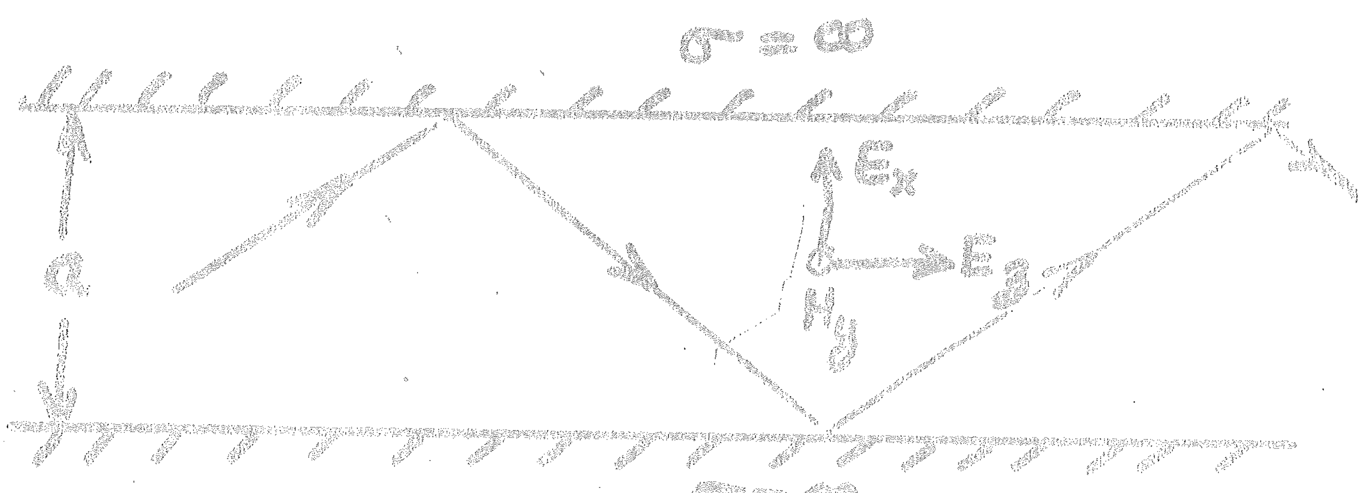
$$E_x = \frac{2E_0\beta}{k}\cos\left(\frac{n\pi x}{a}\right)e^{-j\beta z}$$

$$E_z = 2jE_0\frac{n\pi}{ka}\sin\left(\frac{n\pi x}{a}\right)e^{-j\beta z}$$

$$H_y = \frac{2E_0}{\eta}\cos\left(\frac{n\pi x}{a}\right)e^{-j\beta z}$$

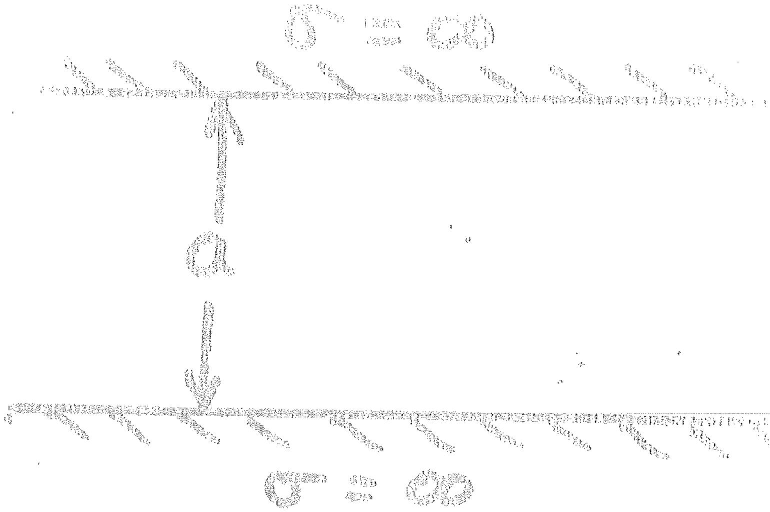
$$Z = \left|\frac{E_x}{H_y}\right| = \frac{\frac{2E_0\beta}{k}}{\frac{2E_0}{\eta}} = \eta\frac{\beta}{k} = \frac{\eta}{k}\sqrt{k^2-\left(\frac{n\pi}{a}\right)^2}$$

$$Z = \eta\sqrt{1-\left(\frac{n\pi}{ka}\right)^2}$$



TM · ALL MAGNETIC VECTORS LIE IN PLANE TRANSVERSE TO PROPAGATION

ALTERNATE METHOD of SOLUTION



$$\frac{\partial^2 E_z}{\partial x^2} + (k^2 - \beta^2) E_z = 0 \qquad \vec{E} = \left[E_z(x)\hat{a}_z + E_x(x)\hat{a}_x\right] e^{-j\beta z}$$

Let $l^2 = (k^2 - \beta^2)$

$$E_z = A \sin lx + B \cos lx$$

$E_z = 0$ at $x = 0$:

$$0 = A(0) + B \rightarrow B = 0$$

$E_z = 0$ at $x = a$:

$$0 = A \sin la \longrightarrow la = n\pi$$

$$l = \frac{n\pi}{a}$$

$$\beta = \sqrt{k^2 - l^2} = \sqrt{k^2 - \left(\frac{n\pi}{a}\right)^2}$$

…ndition, Let $A = 2jE_0 l/k$

$$= \frac{2jE_0 l}{k} \sin lx \, e^{-j\beta z}$$

…ROM HANDY-DANDY DITTO SHEET

$$E_x = -\frac{1}{k^2-\beta^2}\left(j\beta \frac{\partial E_z}{\partial x}\right) = \frac{-j\beta l}{l^2} \frac{2jE_0 l}{k} \cos$$

$$= \frac{2E_0\beta}{k} \cos lx \, e^{-j\beta z}$$

$$E_y = \frac{1}{k^2-\beta^2}\left(-j\beta \frac{\partial E_z}{\partial y}\right) = 0$$

$$H_x = \frac{1}{k^2-\beta^2}\left(j\omega\epsilon \frac{\partial E_z}{\partial y}\right) = 0$$

$$H_y = \frac{-1}{k^2-\beta^2}\left(j\omega\epsilon \frac{\partial E_z}{\partial x}\right) = \frac{-j\omega\epsilon}{l^2} \frac{2jE_0 l^2}{k} \cos$$

$$= \frac{2E_0}{\eta} \cos lx \, e^{-j\beta z}$$

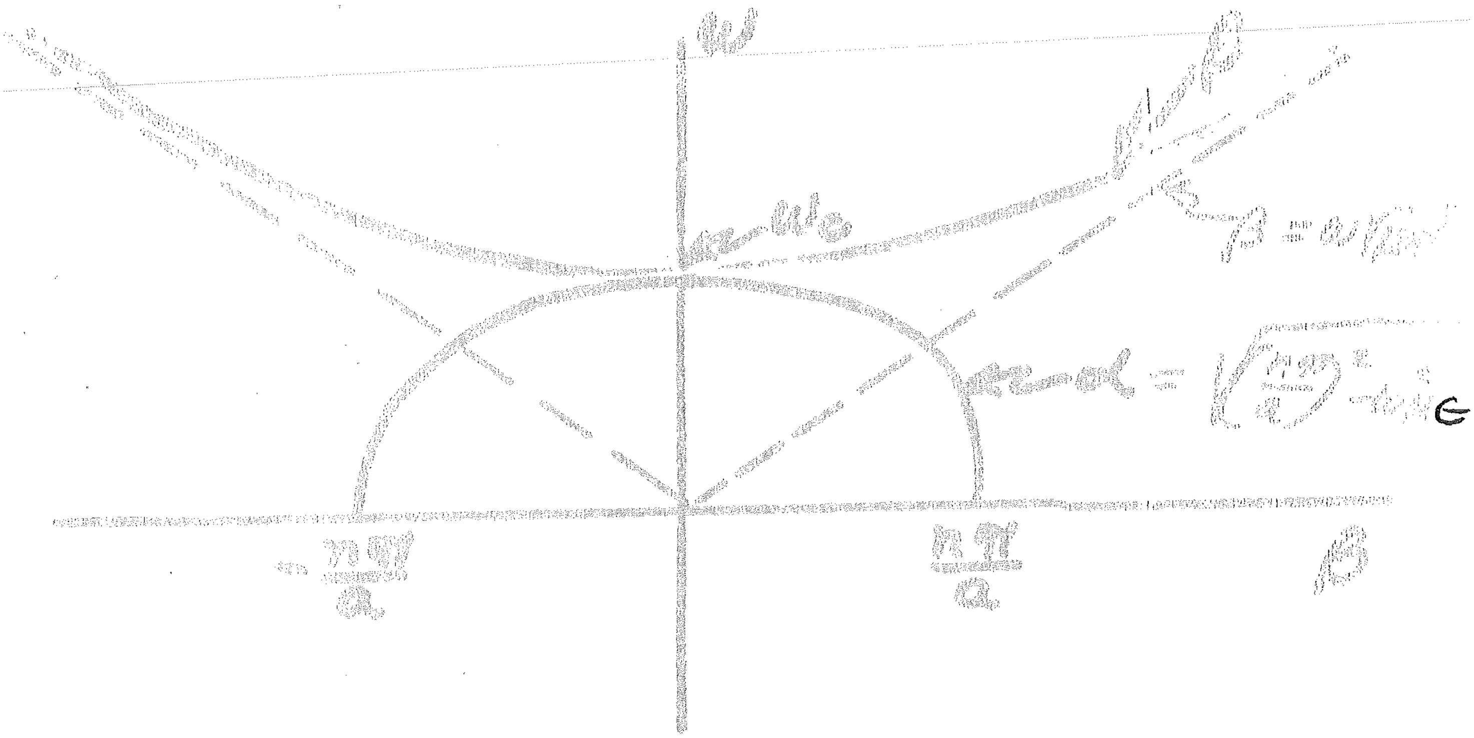
Characteristic Impedance

$$Z_{TE} = \left|\frac{E_x}{H_y}\right| = \frac{\left(\frac{\eta E_0 \beta}{k}\right)}{\left(\frac{k E_0}{\eta}\right)} = \eta\sqrt{1-\left(\ldots\right.}$$

$$v_p = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{k^2-\left(\frac{n\pi}{a}\right)^2}} = \frac{c}{\sqrt{1-\left(\frac{\lambda}{\lambda_c}\right)^2}}$$

ω–β DIAGRAM

$$\beta^2 = k^2 - l^2 = \omega^2\mu\epsilon - \left(\frac{n\pi}{a}\right)^2$$

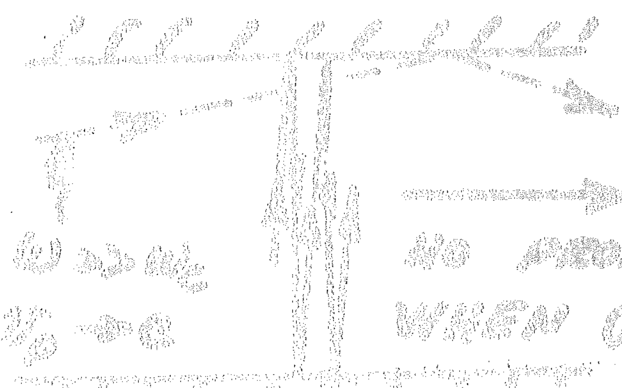


$$\beta^2 = 0 = \omega_c^2\mu\epsilon - \left(\frac{n\pi}{a}\right)^2$$

$$\omega_c^2 = \left(\frac{n\pi}{a}\right)^2 c^2 \qquad \omega_c = \frac{n\pi c}{a}$$

$$2\pi f_c = \frac{n\pi f_c \lambda_c}{a}$$

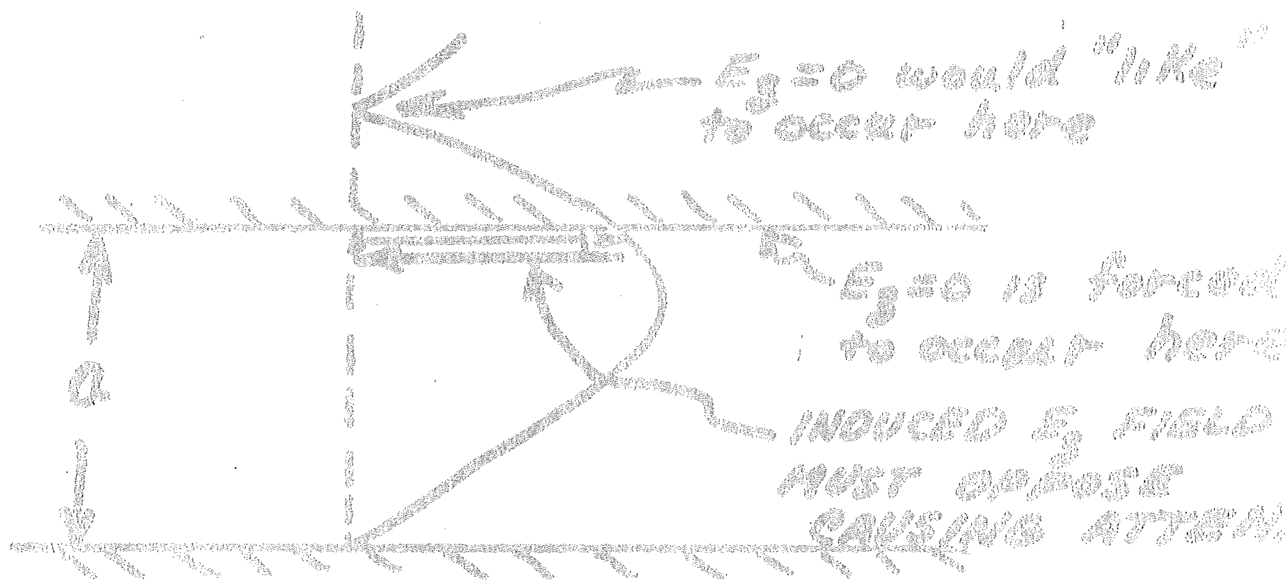
$$\lambda_c = \frac{2a}{n} \quad \underline{OR} \quad a = \frac{n\lambda}{2}$$

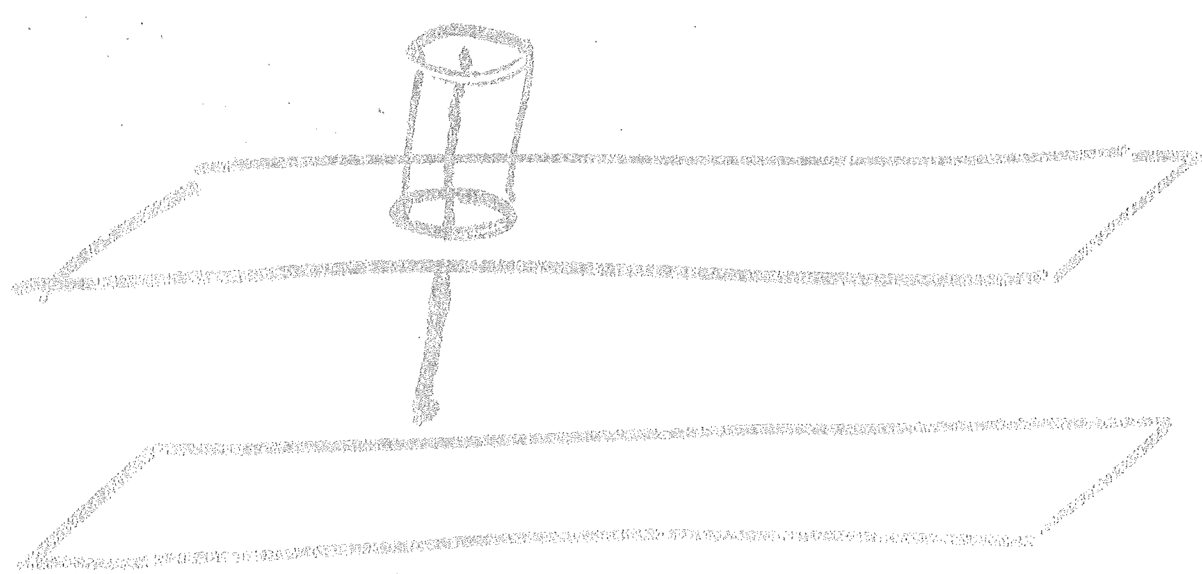


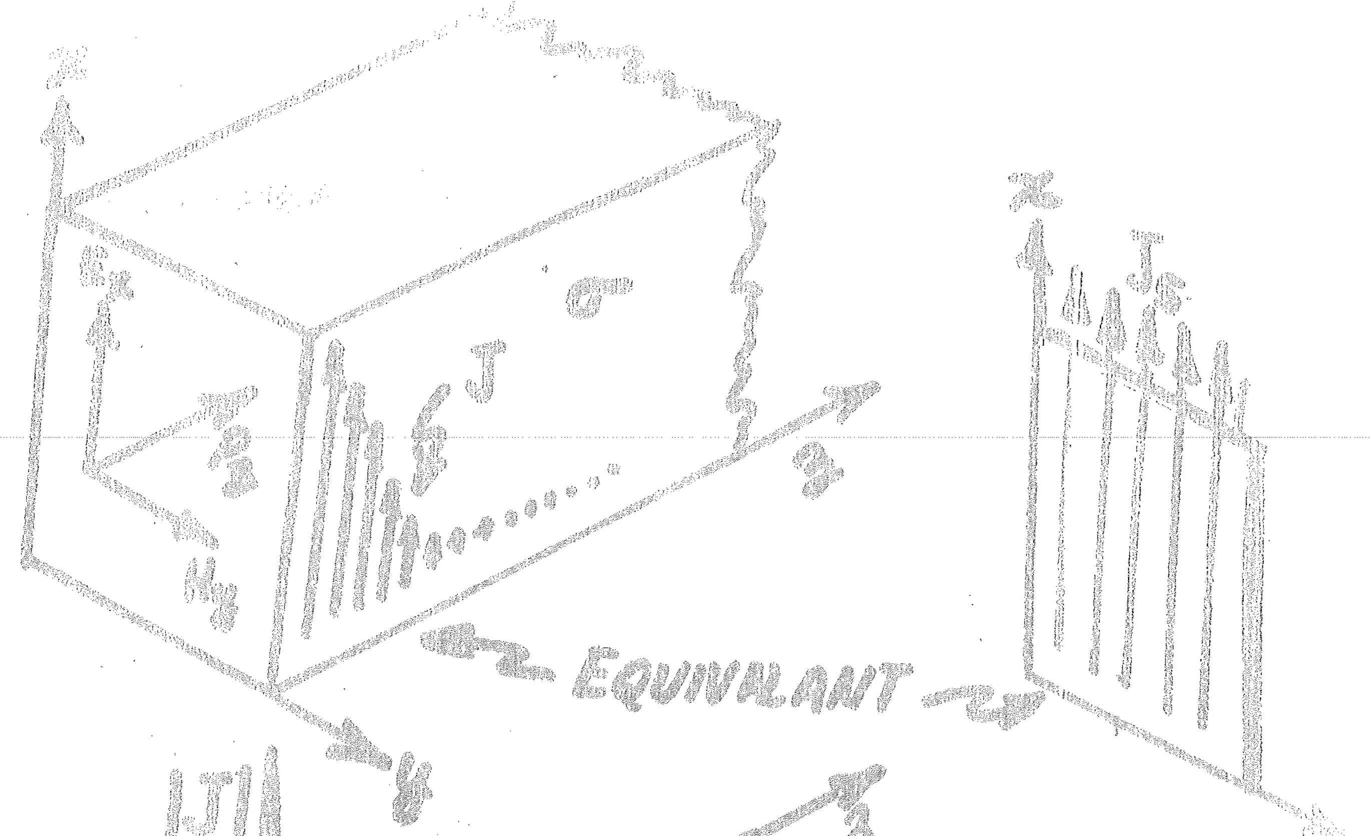
For $\omega < \omega_c$, β becomes imaginary,

$\beta = -j\alpha$ $\qquad e^{j(\omega t - \beta z)} = e^{-\alpha z} e^{j\omega t}$

$$\alpha^2 = \left(\frac{n\pi}{a}\right)^2 - \omega^2 \mu \epsilon$$







$$J_0 = \sigma E_0$$

$$\vec{E} = E_0(x,y)\, e^{-(\alpha+j\beta)z}\, \vec{a}_x$$

$$J_s = \int_0^{\infty} J_0\, e^{-(\alpha+j\beta)z}\, dz = \frac{J_0}{(\alpha+j\beta)} = \frac{\sigma E_0}{(\alpha+j\beta)}$$

$$P_{loss} = \frac{1}{2}\int_0^{\infty} \sigma\, \vec{E}\cdot\vec{E}^*\, dz = \frac{\sigma}{2}\int_0^{\infty} E_0^2\, e^{-2\alpha z}\, dz = \frac{\sigma E_0^2}{4\alpha}$$

DEFINE R_s

$$P_{loss} = \frac{1}{2}\vec{J}_s\cdot\vec{J}_s^{*} R_s \qquad \frac{\sigma E_0^2}{4\alpha} = \frac{\sigma^2 E_0^2}{2(\alpha^2+\beta^2)} R_s \qquad R_s = \frac{\alpha}{\sigma}$$

PLANE WAVES IN GOOD CONDUCTORS

$$\epsilon \rightarrow \epsilon\left(1+\frac{\sigma}{j\omega\epsilon}\right) \qquad e^{j(\omega t - kz)}$$

$$k = \omega\sqrt{\mu\epsilon\left(1+\frac{\sigma}{j\omega\epsilon}\right)} \qquad \frac{\sigma}{\omega\epsilon} >> 1$$

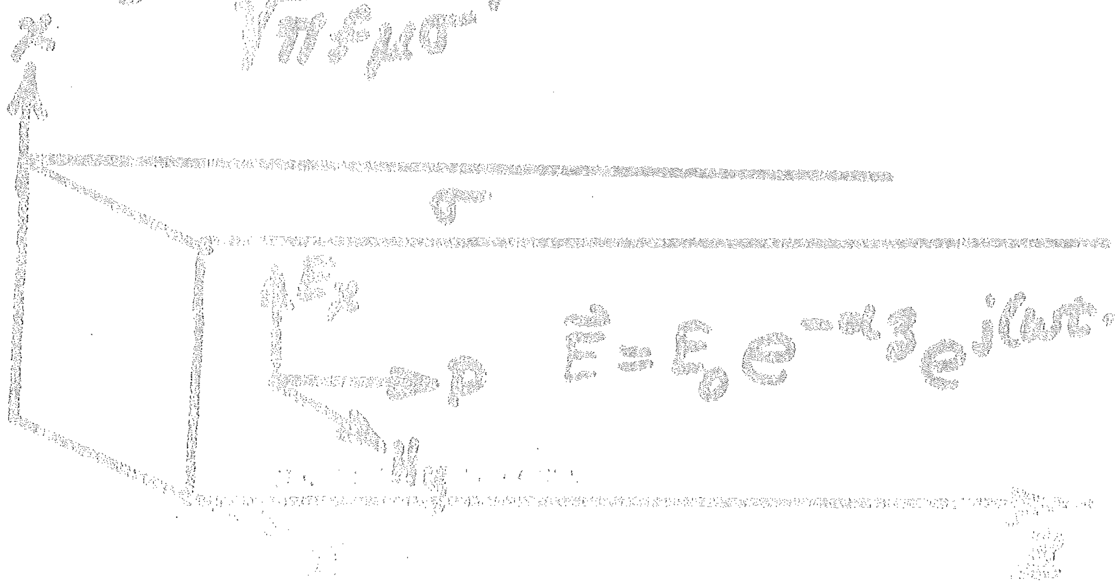
$$\simeq \omega\sqrt{\frac{\mu\sigma}{j\omega}} = \sqrt{\frac{\omega\mu\sigma}{2}}\,(1-j1)$$

$$-jkz = -\sqrt{\frac{\omega\mu\sigma}{2}}\,(1+j1)z = -(\alpha+j\beta)z$$

$$\alpha = \beta = \sqrt{\pi f \mu\sigma}$$

$\delta = \dfrac{1}{\alpha} =$ "TIME CONSTANT" FOR SPACE VARIATION

$\delta = \dfrac{1}{\sqrt{\pi f \mu \sigma}} =$ SKIN DEPTH



$$\vec{E} = E_0 e^{-\alpha z} e^{j(\omega t - \beta z)} \vec{a}_x$$

$$Z = \left|\frac{E_x}{H_y}\right| = \sqrt{\frac{\mu}{\epsilon\left(1 + \frac{\sigma}{j\omega\epsilon}\right)}} \approx \sqrt{\frac{j\omega\mu}{\sigma}}$$

$$= \sqrt{\frac{\omega\mu}{2\sigma}}\,(1+j)$$

$$R_s = \frac{1}{\sigma\delta} = \sqrt{\frac{\pi f \mu}{\sigma}}$$

[illegible] DUE TO FINITE CONDUCTIVITY OF WALLS

$$\vec{E} = E_x(x,y)\, e^{-\alpha z}\, e^{j(\omega t - \beta z)}\, \vec{a}_x$$

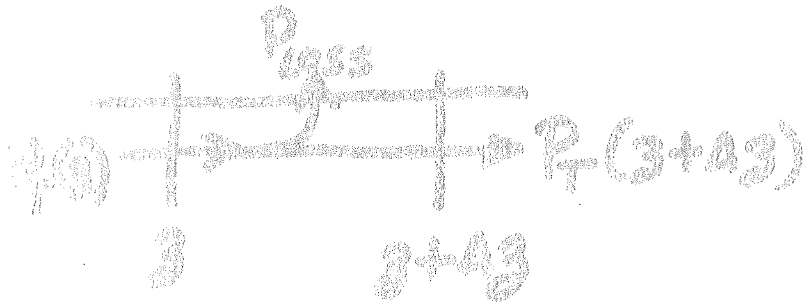
$$\vec{H} = H_y(x,y)\, e^{-\alpha z}\, e^{j(\omega t - \beta z)}\, \vec{a}_y$$

POWER TRANSFER PAST A POINT

$$P_T = \frac{1}{2} \mathrm{Re} \int \vec{E} \times \vec{H}^* \cdot d\vec{S}$$

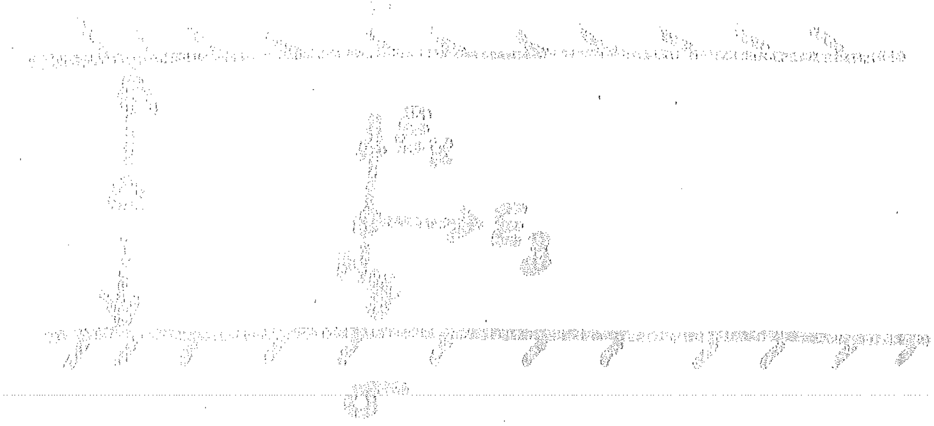
$$= \frac{1}{2} \int E_x H_y\, e^{-2\alpha z}\, dS$$

POWER LOSS PER UNIT LENGTH



$$P_{LOSS} = \lim_{\Delta z \to 0} \frac{P_T(z) - P_T(z+\Delta z)}{\Delta z} = -\frac{\partial P_T}{\partial z} = 2\alpha P_T$$

$$\alpha = \frac{P_{LOSS}}{2 P_T}$$



$$H_y = \frac{2E_0}{\eta} \cos lx \, e^{-j\beta z}$$

$$E_x = \frac{2E_0 \beta}{k} \cos lx \, e^{-j\beta z}$$

$$P_{loss} = \frac{1}{2} J_s^2 R_s (1) = \text{POWER LOSS PER UNIT LENGTH}$$

1 meter width

$$|J_s| = |H_y| \qquad \vec{a}_n \times (\vec{H}_2 - \vec{H}_1) = \vec{J}_s$$

$$P_{loss} = \left\{ \frac{1}{2} \left[\frac{2E_0}{\eta} \cos lx \right]^2_{x=0} \sqrt{\frac{\pi f \mu}{\sigma}} \, e^{-2\alpha z} \right\} 2$$

$$P_T = \frac{1}{2} \int E_x H_y \, dS = \frac{1}{2} \int_0^a \frac{4E_0^2 \beta}{\eta k} \cos^2 lx \, e^{-2\alpha z} (1)$$

$$= \frac{E_0^2 \beta}{\eta k} e^{-2\alpha z}$$

$$\alpha = \frac{P_{loss}}{2P_T} = \frac{\frac{k^2E_0^2}{\eta^2}\sqrt{\frac{\omega\mu}{\sigma}}\, e^{-2\alpha z}}{\frac{2E_0^2\beta}{\eta k}\, e^{-2\alpha z}}$$

$$= \frac{2k}{\beta}\sqrt{\frac{\omega\mu}{2\sigma}}\sqrt{\frac{\epsilon}{\mu}}$$

$$\alpha = \frac{\sqrt{\frac{2\omega\epsilon}{\sigma}}}{\sqrt{1-\left(\frac{\lambda}{\lambda_c}\right)^2}}$$

$$k = \omega\sqrt{\mu\epsilon\left(1+\frac{\sigma}{j\omega\epsilon}\right)} \qquad \frac{\sigma}{\omega\epsilon} \ll 1$$

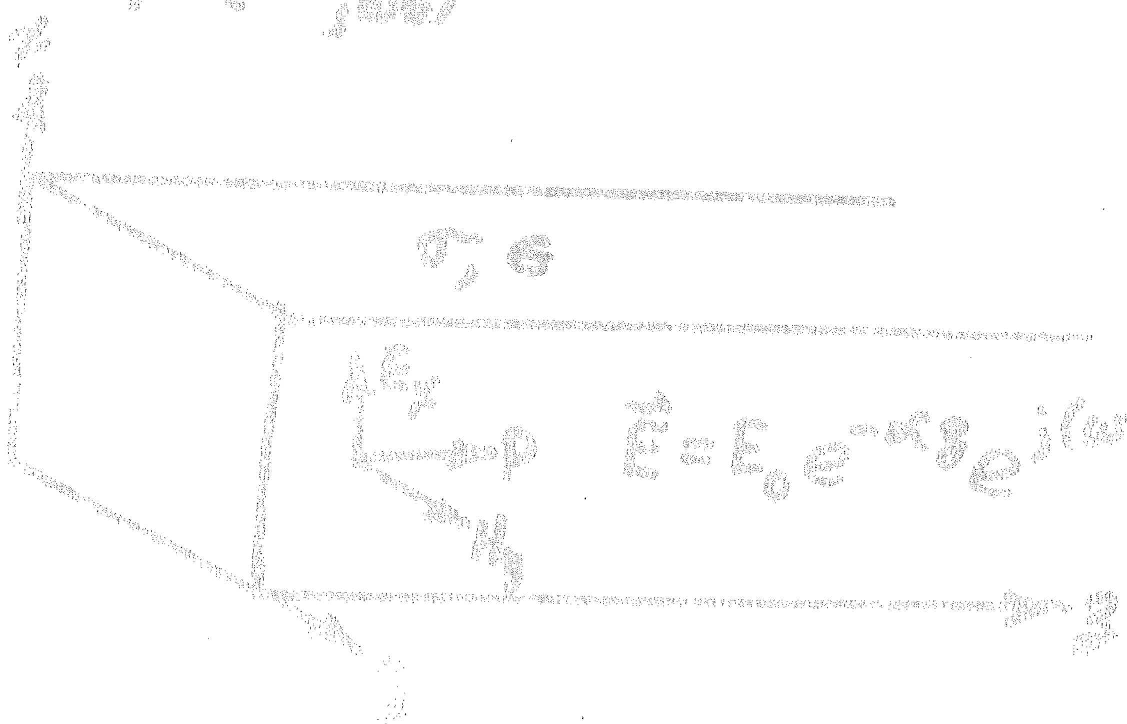
$$\sqrt{1+x} \approx 1 + \frac{x}{2} + \dots$$

$$k \approx \omega\sqrt{\mu\epsilon}\left(1+\frac{\sigma}{j2\omega\epsilon}\right)$$

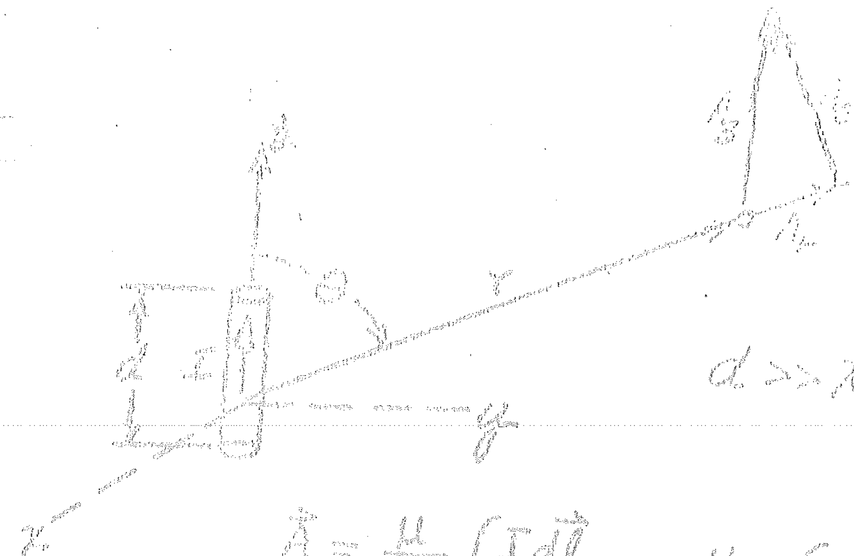
$$-jkz = \left(-j\omega\sqrt{\mu\epsilon} - \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}\right)z = -(\alpha + j\beta)z$$

$$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} \qquad \beta = \omega\sqrt{\mu\epsilon}$$

$$Z = \sqrt{\frac{\mu}{\epsilon\left(1+\frac{\sigma}{j\omega\epsilon}\right)}} \approx \sqrt{\frac{\mu}{\epsilon}}$$



$$\vec{E} = E_0 e^{-\alpha z} e^{j(\omega t - \beta z)}\hat{a}_x$$



$d << \lambda \qquad r >> d$

$$\vec{A} = \frac{\mu}{4\pi}\int \frac{I\,d\vec{l}}{R} = \frac{\mu}{4\pi}\int \frac{I_0 e^{j(\omega t - kr)}\,d\vec{z}}{r}$$

$$\vec{A} \approx \frac{\mu I_0 d}{4\pi} e^{j(\omega t - kr)}\,\vec{a}_z = A_z \vec{a}_z$$

$$A_r = A_z \cos\theta \qquad A_\theta = -A_z \sin\theta$$

$$\vec{B} = \nabla \times \vec{A}$$

$$B_\phi = \frac{1}{r}\frac{\partial}{\partial r}(r A_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta} \qquad B_r = B_\theta = 0$$

$$B_\phi = \mu H_\phi = \frac{\mu I_0 d}{4\pi}\sin\theta\left[\frac{1}{r^2} + \frac{jk}{r}\right] e^{j(\omega t - kr)}$$

$$\nabla \times \vec{H} = j\omega\epsilon \vec{E}$$

$$\vec{a}_r \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta H_\phi) - \vec{a}_\theta \frac{1}{r}\frac{\partial}{\partial r}(r H_\phi) = j\omega\epsilon(\vec{a}_r E_r + \vec{a}_\theta E_\theta)$$

$$E_r = \frac{1}{j\omega\epsilon}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, H_\phi)\right] = \frac{I_0 d\ell\, \eta}{2\pi}\cos\theta\left[\frac{1}{r^2} + \frac{1}{jkr^3}\right]e^{j(\omega t \ldots}$$

$$E_\theta = \frac{-1}{j\omega\epsilon}\left[\frac{1}{r}\frac{\partial}{\partial r}(r H_\phi)\right] = \frac{I_0 d\ell\, \eta}{4\pi}\sin\theta\left[\frac{jk}{r} + \frac{1}{r^2} + \frac{1}{jkr^3}\right]e^{j(\omega t \ldots}$$

$$H_\phi = \frac{I_0 d\ell}{4\pi}\sin\theta\left[\frac{1}{r^2} + \frac{jk}{r}\right]e^{j(\omega t - kr)}$$

$\frac{1}{r^3}$ → ELECTROSTATIC DIPOLE FIELD

$\frac{1}{r^2}$ → INDUCTION (BIOT-SAVART) FIELD

$\frac{1}{r}$ → RADIATION FIELD

at $\left|\frac{1}{r^2}\right| = \left|\frac{k}{r}\right| \qquad r = \frac{1}{k} = \lambda/2\pi$

$$\vec{S} = \frac{1}{2}\,Re\,(\vec{E} \times \vec{H}^*) = \frac{1}{2}Re\left[(\vec{a}_r E_r + \vec{a}_\theta E_\theta) \times \vec{a}_\phi H_\phi^*\right]$$

$$= \frac{1}{2}\,Re\left[-\vec{a}_\theta \underbrace{E_r H_\phi^*}_{\text{PURE IMAGINARY}} + \vec{a}_r E_\theta H_\phi^*\right]$$

$$E_\theta H_\phi^* \sim \eta\left[\frac{1}{r^2} + j\left(\frac{k}{r} - \frac{1}{kr^3}\right)\right]\left[\frac{1}{r^2} - j\frac{k}{r}\right] = \frac{k^2}{r^2}$$

$$\vec{\mathcal{P}}_{av} = \frac{1}{2}\left(\frac{I_0 d k}{4\pi r}\right)^2 \eta \sin^2\theta\, \vec{a}_r$$

$$P_r = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left[\frac{1}{2}\left(\frac{I_0 d k}{4\pi r}\right)^2 \eta \sin^2\theta\, \vec{a}_r\right]\cdot\left[r^2 \sin\theta\, d\theta\, d\phi\, \vec{a}_r\right]$$

$$P_r = \frac{\pi\eta}{3}\left(\frac{I_0 d}{\lambda}\right)^2 = \frac{1}{2} I_0^2 R_{rad} \qquad \int_0^{\pi} \sin^3\theta\, d\theta =$$

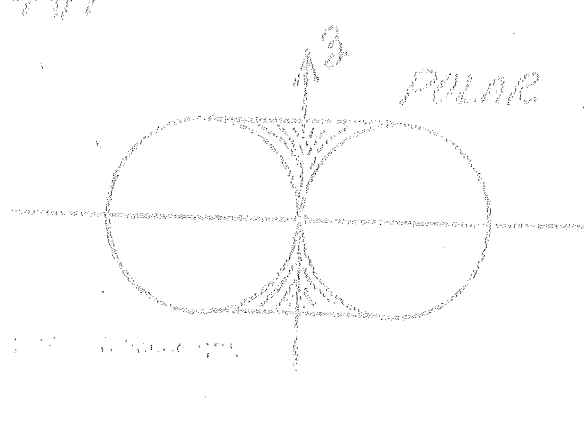
$$R_{rad} = \frac{2\pi\eta}{3}\left(\frac{d}{\lambda}\right)^2 = 80\pi^2\left(\frac{d}{\lambda}\right)^2 \qquad \left\{\begin{array}{l}\text{FREE SPACE} \\ \eta = 120\pi\end{array}\right.$$

$$\left[\, d/\lambda = 0.01, \quad R_{rad} \approx 0.08\,\Omega \,\right]$$

<u>FOR</u> <u>LARGE</u> <u>r</u>.

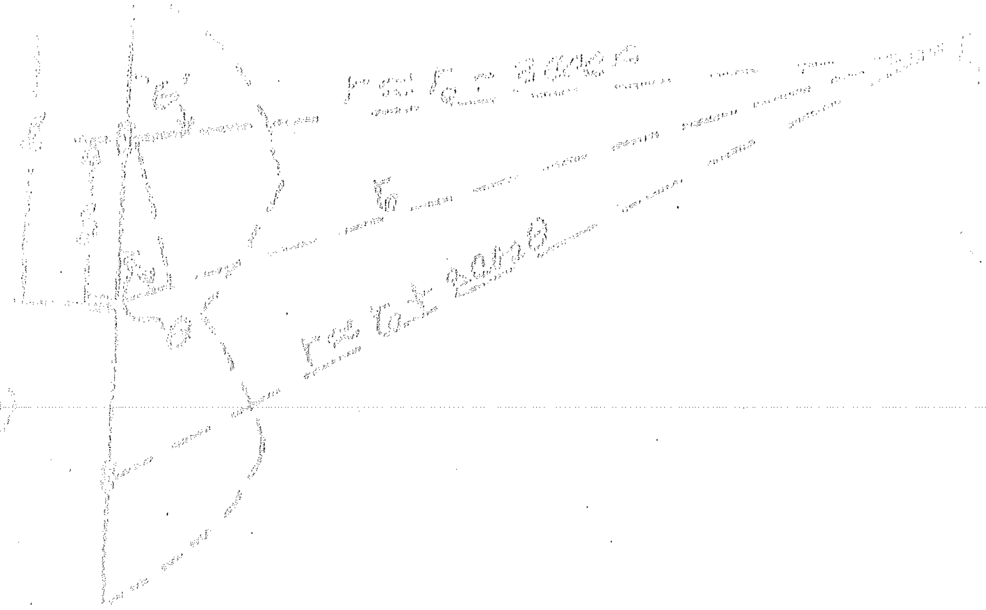
$$E_\theta = \frac{j I_0 k d \eta}{4\pi r} \sin\theta\, e^{j(\omega t - kr)}$$

$$H_\phi = \frac{j I_0 k d}{4\pi r} \sin\theta\, e^{j(\omega t - kr)}$$



$I = I_m \sin k(l-z)$
$(z>0)$

$I = I_m \sin k(l+z)$
$(z<0)$



$$dE_\theta = \frac{jI k\, dz}{4\pi r}\, \eta \sin\theta\, e^{j(\omega t - kr)}$$

MAGNITUDE: $\frac{1}{r} \approx \frac{1}{r_0}$ PHASE: $r \approx r_0 - z\cos\theta$

$$E_\theta = \eta H_\phi = \int_{-l}^{l} dE_\theta$$

$$E_\theta = \frac{jI_m k\eta}{4\pi r_0} \sin\theta\, e^{j(\omega t - kr_0)} \left[\int_{-l}^{0} e^{jkz\cos\theta} \sin k(l+z)\,dz + \int_{0}^{l} e^{jkz\cos\theta} \sin k(l-z)\,dz \right]$$

$$\int e^{ax} \sin(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin(bx+c) - b\cos(bx+c)\right]$$

$$E_\theta = \frac{jI_m \eta}{2\pi r_0} e^{j(\omega t - kr_0)} \left[\frac{\cos(kl\cos\theta) - \cos kl}{\sin\theta}\right]$$

[illegible] ($\frac{\lambda}{2}$ dipole)

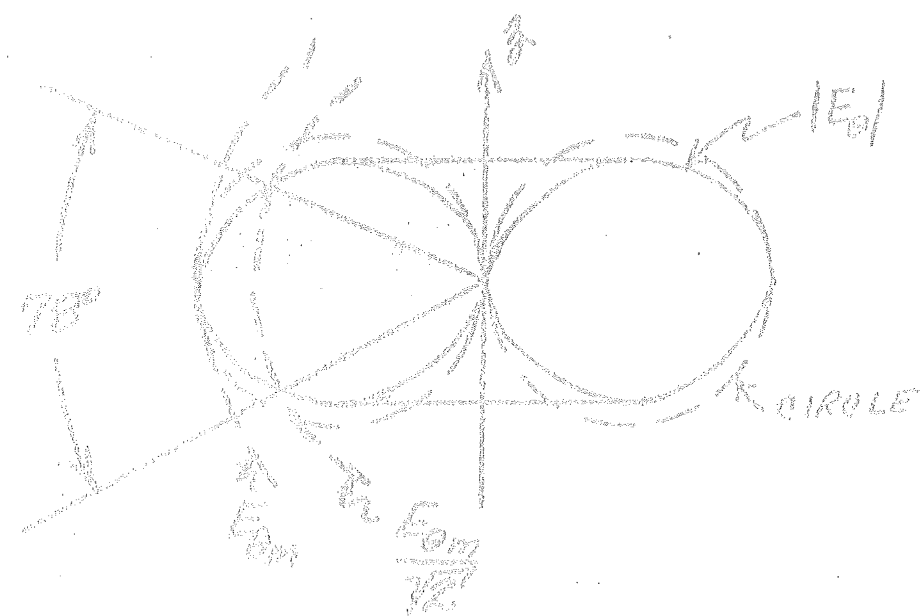
$$E_\theta = \frac{j I_m \eta}{2\pi r_0}\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right] e^{j(\omega t - k r_0)}$$

$$P_T = \int_0^{2\pi}\int_0^{\pi}\left[\frac{1}{2}\,\mathrm{Re}\left(E_\theta H_\phi^*\right)\right]\left[r_0^2 \sin\theta\, d\theta\, d\phi\right]$$

$$= 2\pi \int_0^{\pi} \frac{E_\theta^2}{2\eta}\, r_0^2 \sin\theta\, d\theta = \frac{\eta I_m^2}{4\pi} \underbrace{\int_0^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\, d\theta}_{(1.22)}$$

$$= \frac{1}{2} I_m^2 R_{rad}$$

$$R_{rad} = \frac{\eta}{2\pi}(1.22) = 73.1\ \Omega \quad \text{(FREE SPACE)}$$



Antenna Gain

$$g = \frac{4\pi r_0^2 \left|\bar{P}\right|_{max}}{P_T}$$

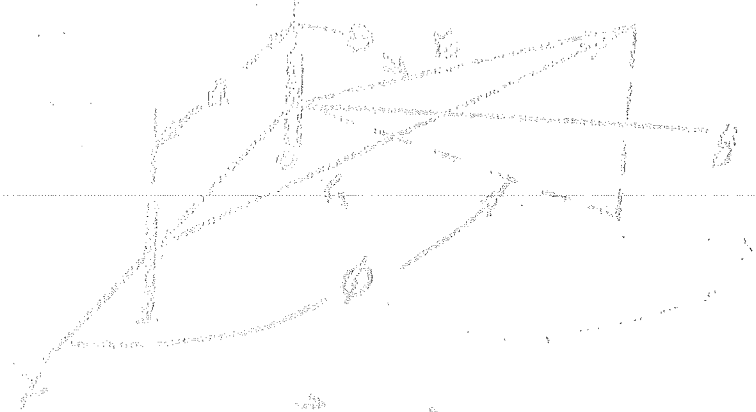
$$\left|\bar{P}\right|_{max} = \frac{E_{\theta m}^2}{2\eta} \qquad E_{\theta m} = \frac{j I_m \eta}{2\pi r_0}[1]$$

$$P_T = \frac{1}{2} I_m^2 R_{rad}$$

$$g = \frac{4\pi r_0^2 \left(\frac{I_m \eta}{2\pi r_0}\right)^2 \frac{1}{2\eta}}{\frac{1}{2} I_m^2 \left[\frac{\eta}{2\pi}(1.22)\right]} = \frac{2}{1.22} = 1.64$$

$$g\,(db) = 10 \log_{10} 1.64 = 2.15\ db$$

ARRAYS



$$\vec{E}_0 = \vec{a}_\theta K_0(\theta) I_0 e^{-jkr_0}$$

$$\vec{E}_1 = \vec{a}_\theta K_1(\theta) I_1 e^{-jkr_1}$$

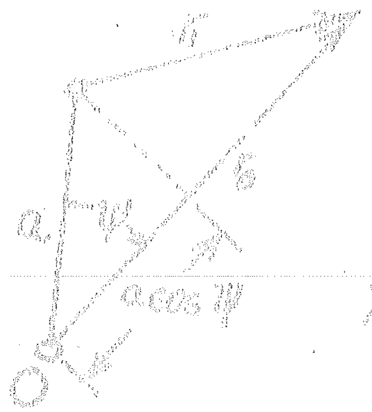
FOR $\frac{\lambda}{2}$ DIPOLE: $K(\theta) = \frac{j\eta}{2\pi r_0}\left[\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right]$

SUPPOSE

$$I_0 = I_0 e^{j0} \qquad I_1 = C_1 I_0 e^{j\alpha_1}$$

$$r_1 \approx r_0 - a\sin\theta\cos\phi$$

$$\vec{E} = \vec{a}_\theta \underbrace{K(\theta)}_{\text{ANTENNA FACTOR}} |I_0| e^{-jkr_0} \underbrace{\left[1 + C_1 e^{j(\alpha_1 + ka\sin\theta\cos\phi)}\right]}_{\text{ARRAY FACTOR}}$$



$I\,d\vec{l} \to \vec{J}_s\,dA \to \vec{J}\,dv$
(WHICHEVER APPLIES)

$$\vec{A} = \frac{\mu}{4\pi}\int \frac{I_a\, e^{j(\omega t - kr)}}{r_0}\,d\vec{l}$$

$$\vec{A} = \frac{\mu}{4\pi r_0} e^{j(\omega t - kr_0)} \int I_a\, e^{jka\cos\psi}\,d\vec{l}$$

$$\vec{N} = \int I_a\, e^{jka\cos\psi}\,d\vec{l} = N_r\vec{a}_r + N_\theta\vec{a}_\theta + N_\phi\vec{a}_\phi$$

= RADIATION VECTOR (FN. OF θ & φ ONLY, NOT r)

$$\nabla\times\vec{A} = \frac{\vec{a}_r}{r_0\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right] + \frac{\vec{a}_\theta}{r_0}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r_0}(r_0A_\phi)\right] + \frac{\vec{a}_\phi}{r_0}\left[\frac{\partial}{\partial r_0}(r_0A_\theta) - \frac{\partial A_r}{\partial\theta}\right]$$

FOR TERMS WHICH WILL VARY AS $1/r_0$:

$$\vec{B} = \nabla\times\vec{A} = \frac{-\vec{a}_\theta}{r_0}\frac{\partial}{\partial r_0}(r_0A_\phi) + \frac{\vec{a}_\phi}{r_0}\frac{\partial}{\partial r_0}(r_0A_\theta)$$

$$= \frac{\mu}{4\pi r_0}\left(jk\,e^{-jkr_0}N_\phi\vec{a}_\theta - jk\,e^{-jkr_0}N_\theta\vec{a}_\phi\right)e^{j\omega t}$$

$$\vec{H} = \frac{\vec{B}}{\mu} = \frac{jk}{4\pi r_0}e^{j(\omega t - kr_0)}\left(N_\phi\vec{a}_\theta - N_\theta\vec{a}_\phi\right)$$

FROM $\vec{E} = \frac{-j\omega}{k^2}\nabla(\nabla\cdot\vec{A}) - j\omega\vec{A}$

AFTER A LABORIOUS EXPANSION AND INVESTIGATION, WE GET:

$$\vec{E} = \frac{-j\omega\mu}{4\pi r_0} e^{j(\omega t - k r_0)}\left(N_\theta \vec{a}_\theta + N_\phi \vec{a}_\phi\right) \quad \left\{ \text{FOR } \frac{1}{r_0} \text{ TERMS} \right.$$

$$\vec{P}_{AVG} = \frac{\eta}{8\lambda^2 r_0^2}\left[|N_\theta|^2 + |N_\phi|^2\right]\vec{a}_r$$

$$P_T = \frac{\eta}{8\lambda^2}\int_0^{\pi}\int_0^{2\pi}\left[|N_\theta|^2 + |N_\phi|^2\right]\sin\theta \, d\theta \, d\phi$$

RECALL THAT FROM:

$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$, WE OBTAINED $\vec{J}_s = \hat{a}_n \times \vec{H}$.

IF $\vec{H}_1 = 0$, $\vec{H}_2$

#2 $\hat{a}_n$

#1

IF: $\nabla \times \vec{E} = -\vec{J}_m - \frac{\partial \vec{B}}{\partial t}$, THEN $\vec{J}_{ms} = -\hat{a}_n \times \vec{E}$

$\vec{J}_m$ ← MAGNETIC CURRENT DENSITY

AND:

$$\vec{A} = \frac{\mu}{4\pi}\int \frac{\vec{J}_s \, dA}{R} e^{j(\omega t - kR)} \qquad \vec{F} = \frac{\epsilon}{4\pi}\int \frac{\vec{J}_{ms} \, dA}{R} e^{j(\omega t - kR)}$$

$\vec{F}$ ← "ELECTRIC VECTOR POTENTIAL"

WE MAY ALSO OBTAIN:

$$\vec{E} = \frac{-j\omega}{k^2}\nabla(\nabla\cdot\vec{A}) - j\omega\vec{A} - \frac{1}{\epsilon}\nabla\times\vec{F}$$

$$\vec{H} = \frac{-j\omega}{k^2}\nabla(\nabla\cdot\vec{F}) - j\omega\vec{F} + \frac{1}{\mu}\nabla\times\vec{A}$$

AND FOR $\frac{1}{r_0}$ RADIATION FIELDS

$$\vec{F} = \frac{\epsilon}{4\pi r_0} e^{j(\omega t - kr_0)} \int I_m e^{jka\cos\psi}\, \vec{dl} =$$

$$\vec{L} = \int I_m e^{jka\cos\psi}\, \vec{dl} = L_r\vec{a}_r + L_\theta\vec{a}_\theta + L_\phi\vec{a}_\phi$$

= MAGNETIC RADIATION VECTOR

THE FIELDS BECOME:

$$\vec{E} = \frac{-j e^{j(\omega t - kr_0)}}{2\lambda r_0}\left[(\eta N_\theta + L_\phi)\vec{a}_\theta + (\eta N_\phi - L_\theta)\vec{a}_\phi\right]$$

$$\vec{H} = \frac{1}{\eta}\left[E_\theta\vec{a}_\phi - E_\phi\vec{a}_\theta\right]$$

$$\vec{P}_{AVG} = \frac{\eta}{8\lambda^2 r_0^2}\left[\left|N_\theta + \frac{L_\phi}{\eta}\right|^2 + \left|N_\phi - \frac{L_\theta}{\eta}\right|^2\right]\vec{a}_r$$

WAVE PROPAGATING IN z DIRECTION $e^{j(\omega t - \beta z)}$

$$k = \omega\sqrt{\mu\epsilon}$$

$$\nabla \times \vec{E} = -j\omega\mu \vec{H} \qquad \nabla \times \vec{H} = j\omega\epsilon \vec{E}$$

$$\frac{\partial E_z}{\partial y} + j\beta E_y = -j\omega\mu H_x \qquad \frac{\partial H_z}{\partial y} + j\beta H_y = j\omega\epsilon E_x$$

$$-j\beta E_x - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y \qquad -j\beta H_x - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \qquad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\epsilon E_z$$

$$E_x = \frac{-1}{k^2 - \beta^2}\left(j\beta \frac{\partial E_z}{\partial x} + j\omega\mu \frac{\partial H_z}{\partial y}\right)$$

$$E_y = \frac{1}{k^2 - \beta^2}\left(-j\beta \frac{\partial E_z}{\partial y} + j\omega\mu \frac{\partial H_z}{\partial x}\right)$$

$$H_x = \frac{1}{k^2 - \beta^2}\left(j\omega\epsilon \frac{\partial E_z}{\partial y} - j\beta \frac{\partial H_z}{\partial x}\right)$$

$$H_y = \frac{-1}{k^2 - \beta^2}\left(j\omega\epsilon \frac{\partial E_z}{\partial x} + j\beta \frac{\partial H_z}{\partial y}\right)$$

$$\nabla^2 \vec{E} = \nabla_{xy}^2 \vec{E} + \frac{\partial^2 \vec{E}}{\partial z^2} = \nabla_{xy}^2 \vec{E} - \beta^2 \vec{E}$$

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0 \;\Rightarrow\; \nabla_{xy}^2 \vec{E} + (k^2 - \beta^2)\vec{E} = 0$$

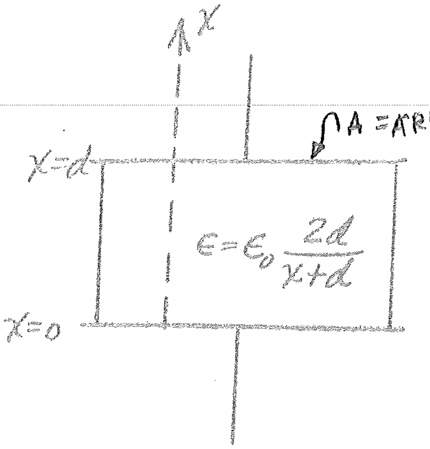
BOB MARKS

21

EE 342 Feb 72
CCR

"The amount of noise which anyone can bear undisturbed stands in inverse proportion to his mental capacity and may therefore be regarded as a fair measure of it."
---Arthur Schopenhauer
19th Century German Philosopher

1.



A parallel plate capacitor with an inhomogeneous dielectric is shown. Determine (Book's sheet)

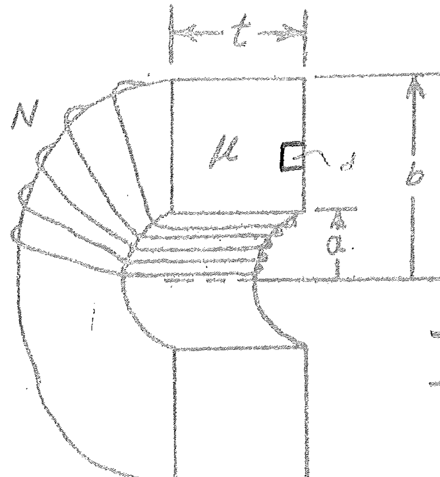
(a) the polarization charge density in the dielectric,

(b) the surface polarization charge density at x = d,

(c) the capacitance.

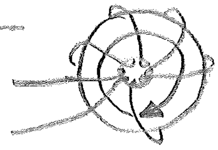
3

2.



For the system shown, determine the self inductance. (Assume the toroid to be complete...only a section is illustrated.) $\bar{B} = \mu \bar{H}$

FIND $\vec{B}$



$$\oint_c \bar{H} \cdot d\bar{l} = NI$$

$$\int_a^b \int_0^{2\pi} H_\phi \cdot r\, d\phi\, dr\, \bar{a}_\phi = NI$$

$$\int_a^b 2\pi r H_\phi\, dr = NI$$

$$\pi(b^2 - a^2) H_\phi = NI \Rightarrow \cancel{\vec{H} = \frac{NI}{\pi(b^2-a^2)} \bar{a}_\phi}$$

$$\bar{B} = \mu \vec{H} = \mu \frac{NI}{\pi(b^2 - a^2)} \bar{a}_\phi$$

3

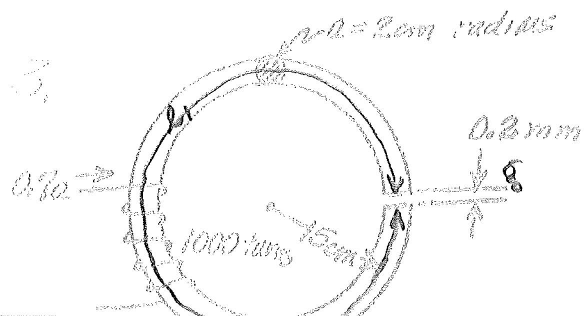
$$\psi = \int_S B \cdot ds$$

$$= \int_S (B_\phi)(ds_\phi)$$

$$= \frac{\mu NI}{\pi(b^2 - a^2)} ab$$

$$\Rightarrow L = \frac{\psi}{I} = \frac{\mu N ab}{\pi(b^2 - a^2)}$$

3.



The toroid shown has a circular cross section of 2 cm radius and the B-H curve indicated on the page attached. Determine

(a) The magnetic ~~fmx~~ flux density in the gap.

(b) The dipole moment per unit volume in the iron, M.

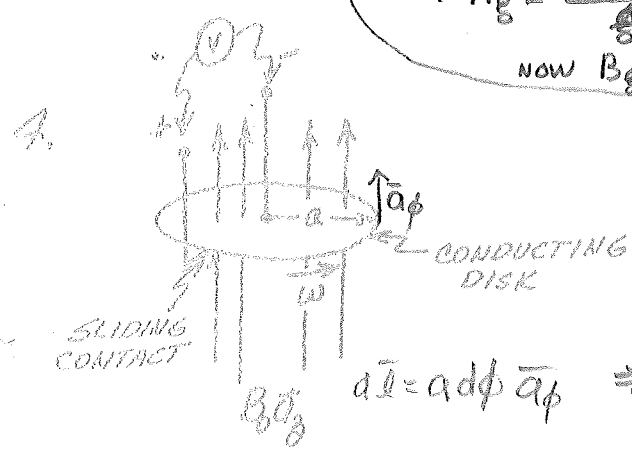
$\oint H \cdot d\ell = NI$

$H_1 \ell_1 + H_g g = NI$

$\Rightarrow H_g = \dfrac{NI - H_1 \ell_1}{g_1} = \dfrac{(700) - H_1(2\pi(.15) - .002)}{.002}$

NOW $B_g = B_1 = \mu_0 H_g = \mu H_1$ (OVER) (TO BACK)

4.



Determine the reading of an ideal voltmeter attached to the terminals.

$V_{IND} = \int_S \cancel{\frac{\delta B}{\delta t} \cdot d\bar{S}} + \oint_C \bar{V} \times \bar{B} \cdot d\ell$

$\vec{V} = (\omega a)\bar{a}_\phi$

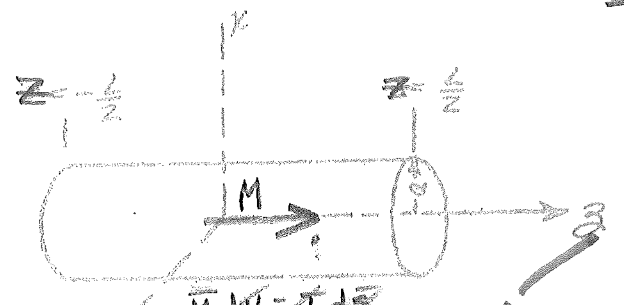
$d\bar{\ell} = a\,d\phi\,\bar{a}_\phi$

$d\bar{\ell} = a\,d\phi\,\bar{a}_\phi \quad \Rightarrow V = \oint_C (\omega a \bar{a}_\phi) \times (B_0 \bar{a}_z) \cdot (a\,d\phi\, a_\phi)$

$= \oint_C \omega a B_0 \bar{a}_r \cdot a\,d\phi\,\bar{a}_\phi = 0$

$=$

5.



$\vec{M} = M_0 (a-r) z^2 \vec{a}_z$

For the system shown, determine

(a) all of the equivilant magnetic charges, and (Back)

(b) the equivilant electric currents.

~~$\bar{M}dV = I d\bar{S}$~~

~~$I = \rho V S$~~

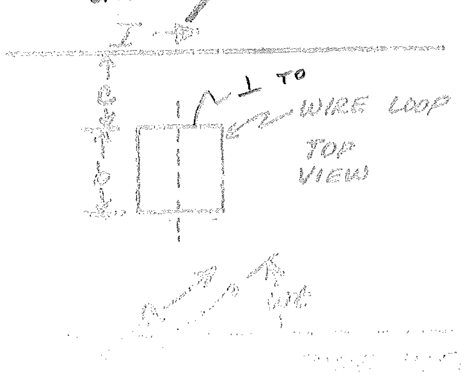
~~$\Rightarrow \bar{M} dV = (\rho V S) d\bar{S}$~~

~~$\Rightarrow \rho = \dfrac{\bar{M}\,dV}{\rho V S\,dS}$~~

~~$V = \pi a^2 L$; $S = 2\pi a L + 2\pi a^2$~~

~~$\dfrac{dV}{da} = 2\pi a L$~~

6.

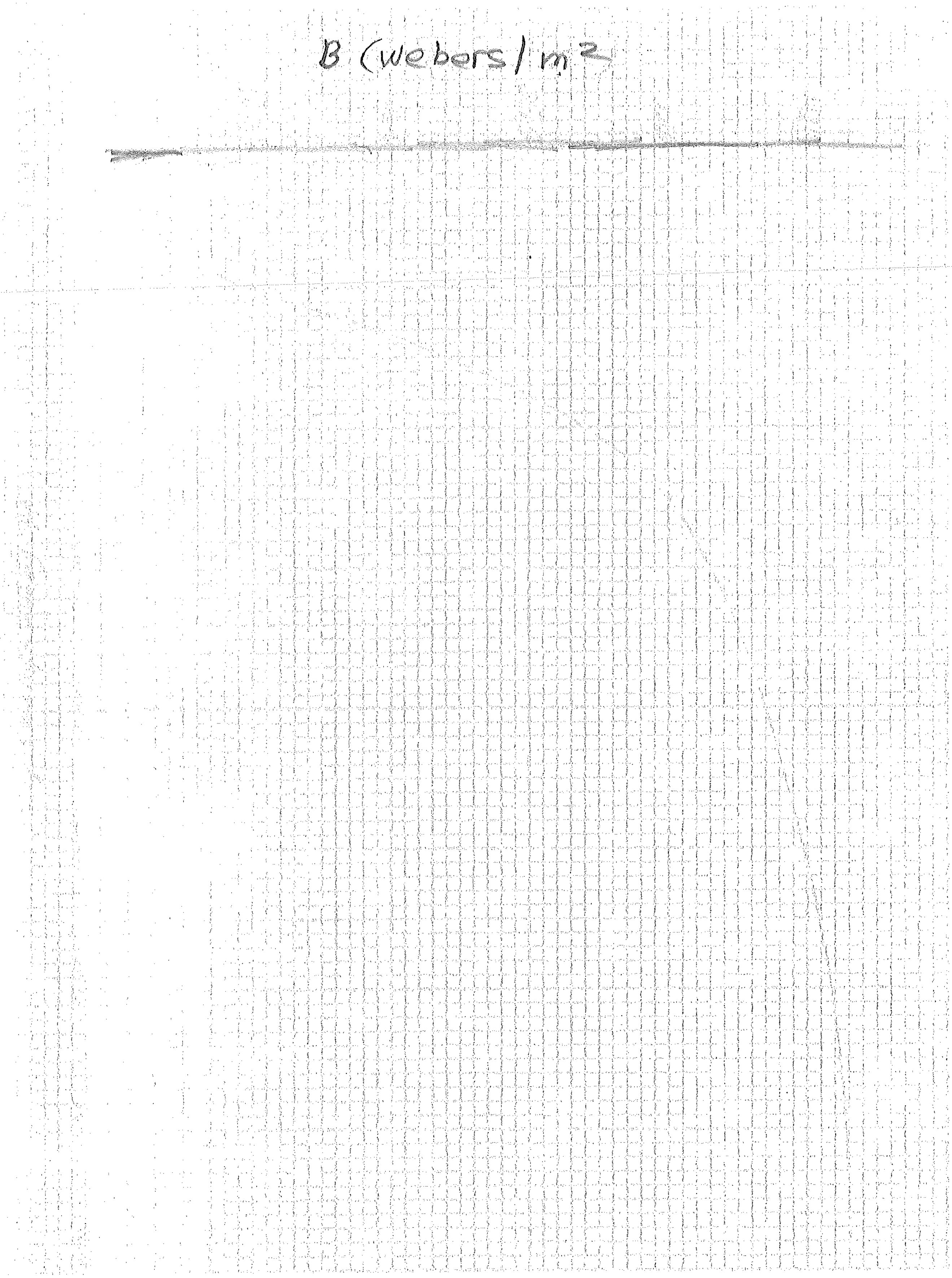


In the system shown, a perfectly conducting rectangular loop is rotating on its axis in the field of a DC current I. Determine the voltage induced in the loop.

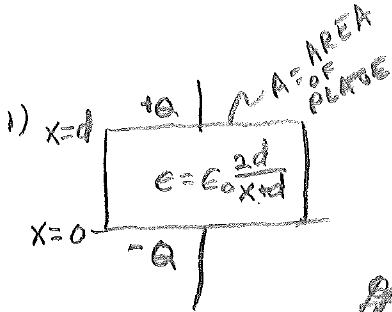
$V_{IND} = \int_S \frac{\delta B}{\delta t} \cdot dS + \oint_C V \times B \cdot d\ell$

$\bar{F} = q\bar{V} \times \bar{B} \Rightarrow \bar{E} = \bar{F}/q = \bar{V} \times \bar{B}$

$= (\omega a)\bar{a}_\phi \times \bar{B} = \vec{J}/\sigma$


B (webers/m^2

1)



$\vec{D} = \epsilon \vec{E}$

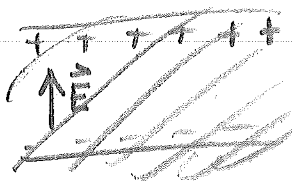
$= \left(\epsilon_0 \frac{2d}{x+d}\right) \vec{E}$

$\cancel{\rho = \nabla \cdot \vec{D} = \nabla \cdot \left(\epsilon_0 \frac{2d}{x+d}\right) \vec{E} = \nabla \times \vec{P}}$

$\cancel{\oint_S \vec{D} \cdot d\vec{S} = Q}$

$\cancel{\vec{D} \cdot (Ad)\bar{a}_x = Q}$

$\cancel{\Rightarrow DAd = Q}$



$\cancel{\left(\vec{E} = \lim_{\Delta Q \to 0} F/\Delta Q\right)}$

$\cancel{\oint \vec{E} \cdot d\vec{l} = 0}$

$\vec{D} = \epsilon \vec{E} = \left(\epsilon_0 \frac{2d}{x+d}\right) \vec{E} = \epsilon_0 \bar{E} + \bar{P}$

[redacted]

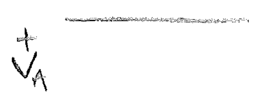
b) $R = \left|\frac{E}{I}\right| = \frac{\int_c \bar{E} \cdot d\bar{l}}{\int_A \sigma \bar{E} \cdot d\bar{S}}$

c) $C = \frac{dQ}{V} = \frac{\int_S \epsilon \bar{E} \cdot d\bar{S}}{\int_L \bar{E} \cdot d\bar{l}}$

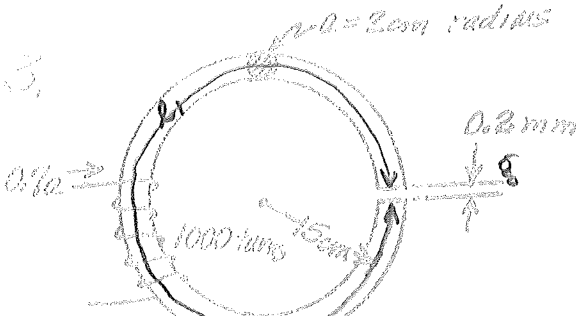
$\Updownarrow$ (NEGLECTING FRINGE EFFECTS)

$\frac{\epsilon}{\sigma} = CR$

$C = \frac{Q}{V_1}$



3.



The toroid shown has a circular cross section of 2 cm radius and the B-H curve indicated on the page attached. Determine

(a) The magnetic ~~fmx~~ flux density in the gap.

(b) The dipole moment per unit volume in the iron, M.

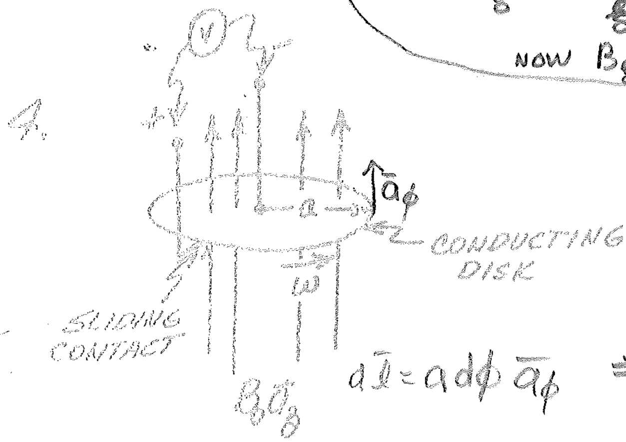
$\oint H\cdot d\ell = NI$

$H_1 \ell_1 + H_g g = NI$

$\Rightarrow H_g = \frac{NI - H_1 \ell_1}{g} = \frac{(700) - H_1(2\pi(.15) - .002)}{.002}$

now $B_g = B_1 = \mu_0 H_g = \mu H_1$ (OVER) (TO BACK)

4.



Determine the reading of an ideal voltmeter attached to the terminals.

$V_{IND} = \cancel{\int_S \frac{\delta B}{\delta t}\cdot dS}^{\,0} + \oint_c \bar V \times \bar B \cdot d\ell$

$\bar V = (\omega a)\bar a_\phi$

$d\bar l = a\,d\phi\,\bar a_\phi$

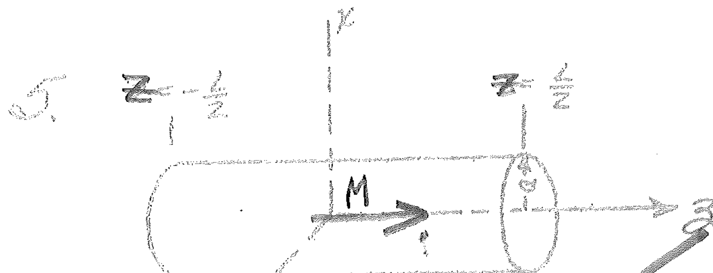
$d\bar l = a\,d\phi\,\bar a_\phi$

$\Rightarrow V = \oint_c (\omega a \bar a_\phi)\times(B_0 \bar a_z)\cdot(a\,d\phi\, a_\phi)$

$= \oint_c \omega a B_0 \bar a_r \cdot a\,d\phi\,\bar a_\phi = 0$

$=$

5.



$\vec M = M_0 (a-r) z^2 \vec a_z$

For the system shown, determine

(a) all of the equivilant magnetic charges, and (Back)

(b) the equivilant electric currents.

$\cancel{\bar M dV = I d\bar s}$

$\cancel{I = \rho V S}$

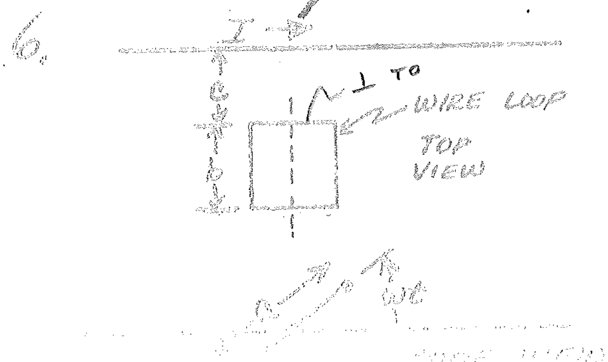
$\cancel{\Rightarrow \bar M dV = (\rho V S) d\bar s}$

$\cancel{\Rightarrow \rho = \frac{\bar M dV}{\rho V S d\bar s}}$

$\cancel{V = \pi a^2 L ; \bar S = 2\pi a L + 2\pi a^2}$

$\cancel{\frac{dV}{da} = 2\pi a L}$

6.



In the system shown, a perfectly conducting rectangular loop is rotating on its axis in the field of a DC current I. Determine the voltage induced in the loop.

$V_{IND} = \int_s \frac{\delta B}{\delta t}\cdot ds + \oint_c V \times B \cdot d\ell$

$\bar F = q\bar V \times \bar B \Rightarrow \bar E = \bar F/q = \bar V \times \bar B$

$= (\omega a)\bar a_\phi \times \bar B = \vec J/\sigma$

$$H_g = \frac{700 - H_1(.3)\pi}{2\times 10^{-3}}$$

$$\mu_0 H_g = \mu H_1$$

$$\Rightarrow H_g = \frac{\mu}{\mu_0} H_1 = \frac{700 - H_1(.3)\pi}{2\times 10^{-3}}$$

$$\mu H_1 = \frac{(700 - H_1(.3)\pi)\, 4\pi\times 10^{-7}}{2\times 10^{-3}}$$

$$B = \frac{700 - H_1(.94)}{.157} \times 10^{-4}$$

$$= \frac{700}{(.157)}\times 10^{-4} - \frac{(.94)}{(.157)} H_1 \times 10^{-4}$$

$$= 4.46 - 6\times 10^{-4} H_1$$

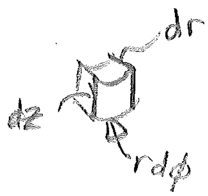
why?

(UNREALISTING RELATION. HAS ALGEBRA MISTAKE.

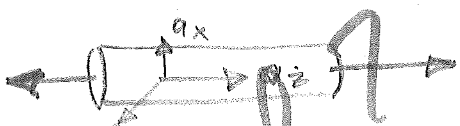
~~Procedure is to construct load line~~ on graph, find ~~μ and solve for B~~ intersection on B-H curve and read off corresponding B value

---

5a) $\bar{A} = \frac{\mu_0}{4\pi}\int_V \frac{J(x',y',z')}{R}\, dV' =$



~~$I = \int_S J\cdot ds = \frac{2\bar{M}}{\oint r\times dr}$~~



$$\vec{M} = \frac{1}{2}\int_V \bar{r}\times\bar{J}\,dV$$

$$= \frac{1}{2}\pi a^3 L\,\bar{a}_z \Rightarrow \vec{J} = \frac{2\vec{M}}{\pi a^2 L}$$

why

$$\Rightarrow \vec{A} = \frac{2\mu_0}{4\pi^2 a^2 L}\int_V \frac{M_0(a-r)Z^2\bar{a}_z}{}\, dV$$

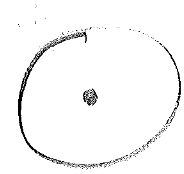
$$= \frac{\mu_0 M_0 \bar{a}_z}{2\pi^2 a^2 L}\int_V (a-r)z^2\, r\,dr\,d\phi\,dz$$

$$= \frac{\mu_0 M_0 \bar{a}_z}{2\pi^2 a^2 L_0}\int_{-L/2}^{L/2}\int_0^{2\pi}\int_0^a (ar - r^2) z^2\, dr\,d\phi\,dz$$

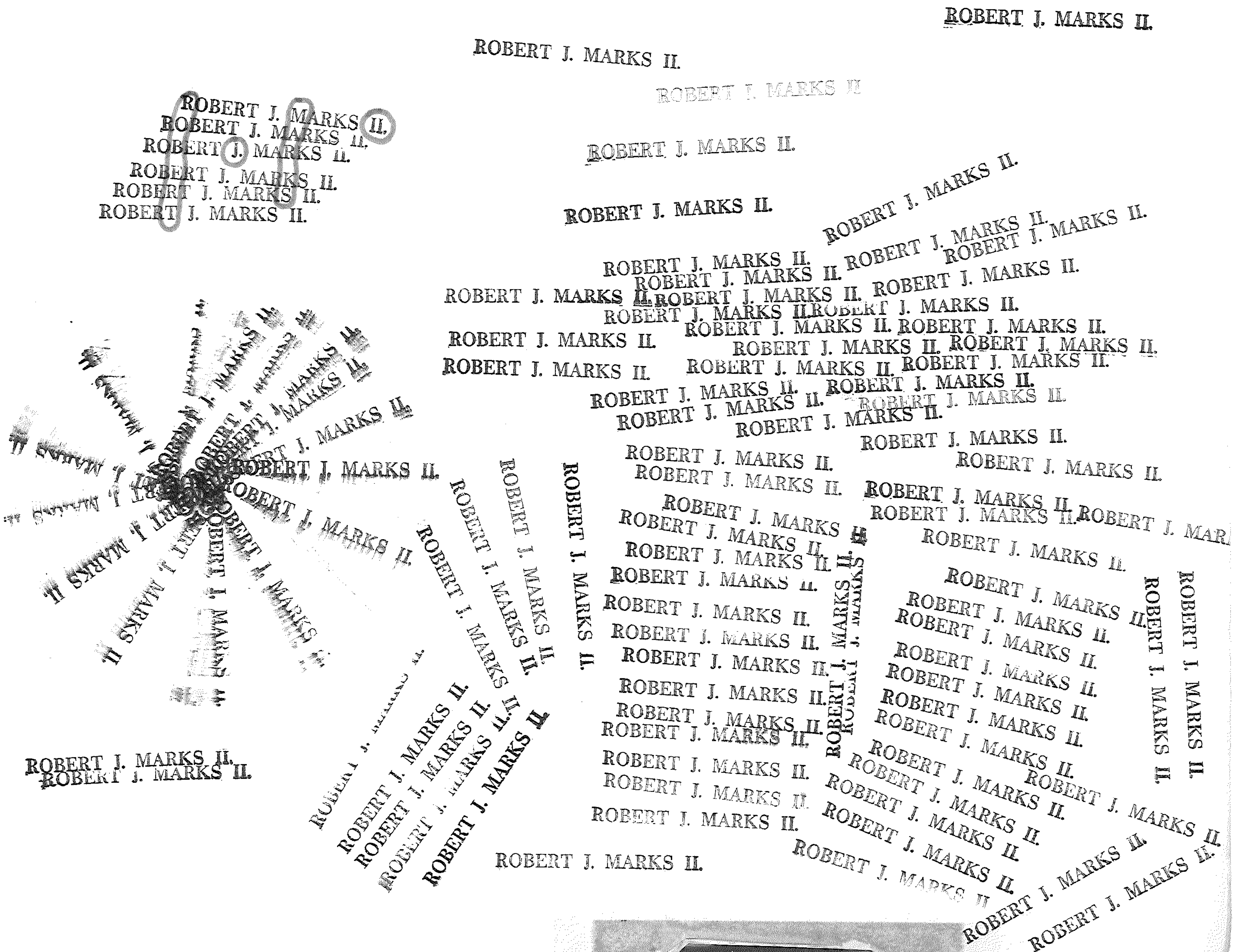
---

b) $$\bar{J} = \frac{2\bar{M}}{\pi a^2 L} = \frac{2M_0(a-r)z^2 a_z}{\pi a^2 L}$$

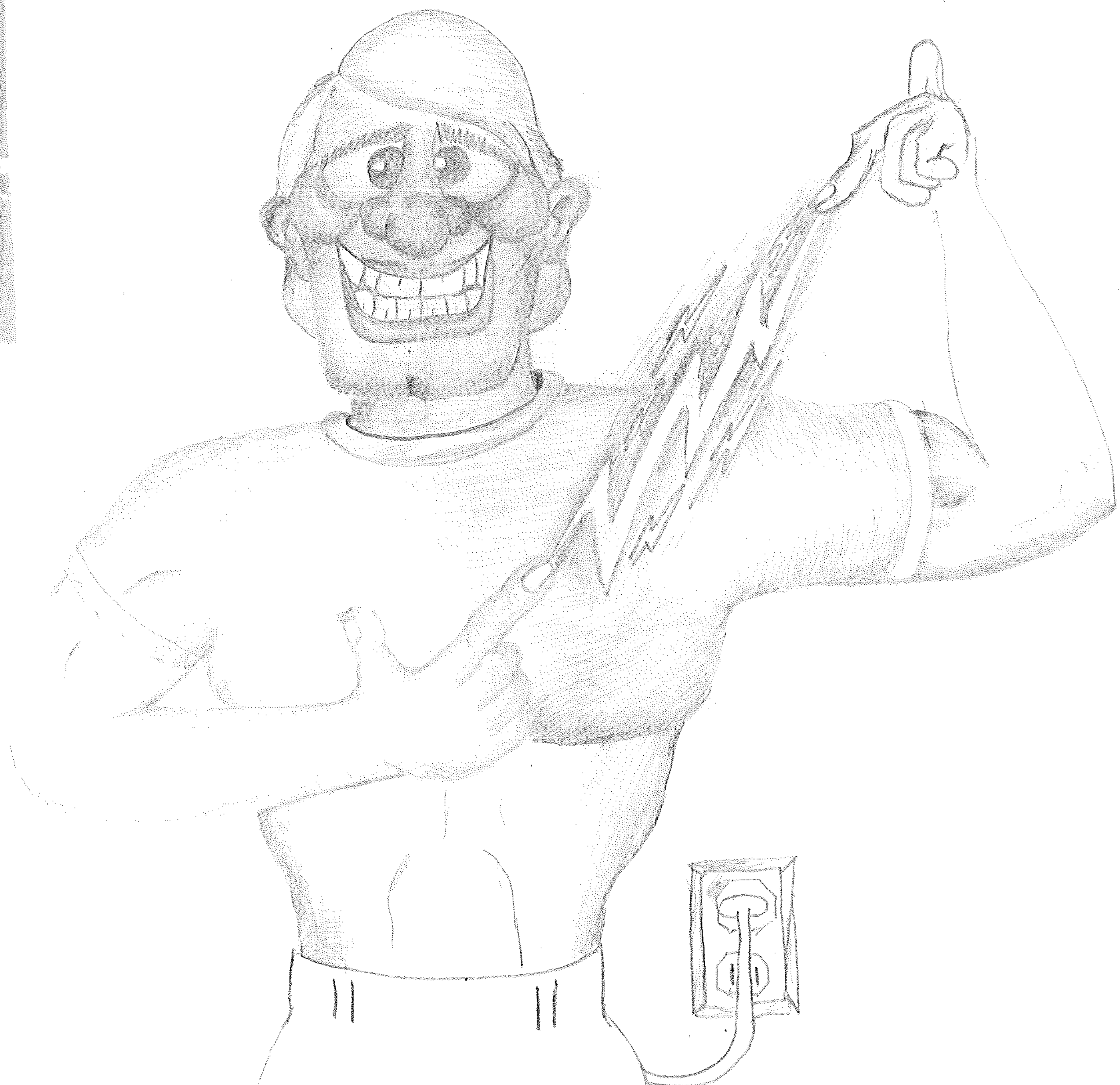
$$I = \int_S \bar{J}\cdot d\bar{s} \Rightarrow ds = r\,d\phi\,dr$$



$$I = \int_0^a\int_0^{2\pi} \frac{2M_0(a-r)z^2}{\pi a^2 L}\, r\,d\phi\,dr$$



EE 5341
ADV. FIELDS



1-20-16

INHOMOGENEOUS (DRIVEN) WAVE EQ.

$$\nabla^2 \psi(\vec{x},t) - \frac{\delta^2\psi}{\delta t^2} = -4\pi f(\vec{x},t)$$

VECTOR GENERALIZATION

$$\nabla^2 \vec{E} - \frac{1}{c^2}\,\delta^2\vec{E}/\delta t^2 = -4\pi \vec{f}(\vec{x},t)$$

IN CARTESIAN COORDINATES, MAY BREAK UP

$$\nabla^2 E_x - \frac{1}{c^2}\frac{\delta E_x}{\delta x^2} = -4\pi f_x(\vec{x},t)$$

CONCENTRATE ON ONE DIMENSION:

$$\nabla^2 \psi(\vec{x},t) - \frac{\delta^2\psi}{\delta t^2} = -4\pi f(\vec{x},t)$$

WILL FIND GENERAL SOLN' THRU GREENS FUNCTION.

DEFINE; $\Box_x = \nabla^2 - \frac{1}{c^2}\frac{\delta^2}{\delta t^2}$ = D'ALAMBERTIAN

(NAMED, AFTER D'ALAMBERT)

THEN $\Box_{x'}\,\psi(\vec{x}',t') = -4\pi f(\vec{x}',t')$ (1)

$\Box_x\,\psi(\vec{x},t) = -4\pi f(\vec{x},t)$ ⇐ HOMOGENEOUS & ISOTROPIC

CONSIDER

$$\Box_{x'}\, G(\vec{x}, t; \vec{x}', t') = -4\pi\,\delta^3(\vec{x}-\vec{x}')\,\delta(t-t')$$ (2)

$\vec{x}, t \Rightarrow$ OBSERVER COMPONENTS

$x', t' \Rightarrow$ SOURCE COMPONENTS

IN A HOMOGENEOUS ISOTROPIC MEDIA:

$$G(\vec{x}, t; \vec{x}', t') = G(\vec{x}-\vec{x}', t'-t)$$

TO SEE HOW G IS USEFUL, TAKE

$$\int_{t_1'}^{t_2'} dt' \int_{V'} d^3x' \left[(1)\times G(\vec{x},t;\vec{x}',t) - (2)\,\psi(\vec{x}';t')\right]$$

$$\Rightarrow \int dt' \int d^3x' \left[G\,\Box_{x'}\psi - \psi\,\Box_{x'}G\right]$$

$$= -4\pi \int dt' d^3x'\, G f + 4\pi \int dt' d^3x'\, \psi(\vec{x}';t') \times \delta^3(\vec{x}-\vec{x}')\,\delta(t-t')$$

WE HAVE YET TO SPECIFY BOUNDARY CONDITIONS WHICH WILL SIMPLIFY.

NOW TO FIND G.

RECALL

$$\Box_{x'} G(\vec{x}'-\vec{x};t'-t) = -4\pi\,\delta^3(\vec{x}'-\vec{x})\,\delta(t'-t)$$

$$\Rightarrow \Box_{x'} G(\vec{x}';t') = -4\pi\,\delta^3(\vec{x}')\,\delta(t')$$

$$= \frac{-4\pi}{(2\pi)^4}\int d\omega \int dk\; e^{i(\vec{k}\cdot\vec{x}'-\omega t')}$$

$$\left[\begin{array}{l} \text{FROM } \delta(t') = \frac{1}{2\pi}\int d\omega\, e^{-i\omega t'}\,dt' \\ \text{AND } \delta(x') = \frac{1}{2\pi}\int d n_x\, e^{i n_x x'}\,dx' \end{array}\right]$$

$$\Rightarrow \delta(x')\delta(y')\,dz' = \int dn_x \int dn_y \int dn_z \times e^{i(n_x x' + n_y y' + n_z z')}$$

$$= \int d^3k\; e^{i\vec{k}\cdot\vec{x}}$$

DEFINE $g(\vec{k},\omega) \ni$

$$G(x;t') = \frac{1}{(2\pi)^4}\int d\omega\, dk^3\; e^{i(\vec{k}\cdot x-\omega t')}\, g(\vec{k},\omega)$$

LET $k = |\vec{k}|$

$$\nabla e^{i\vec{k}\cdot\vec{x}} = e^{i\vec{k}\cdot\vec{x}}\, i\,\nabla(\vec{k}\cdot\vec{x})$$

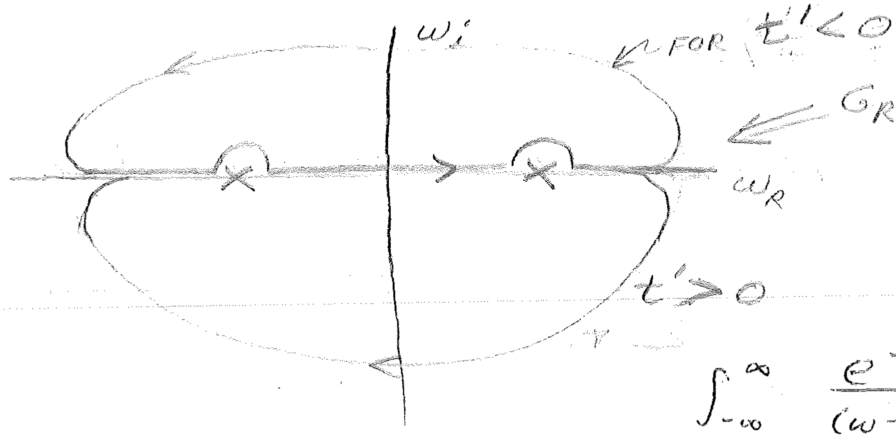
$$= i\vec{k}\, e^{i\vec{k}\cdot\vec{x}}$$

$$\nabla^2 e^{i\vec{k}\cdot\vec{x}} = -k^2 e^{i\vec{k}\cdot\vec{x}} \qquad \Rightarrow k^2 = \vec{k}\cdot\vec{k}$$

$$\Box_{x'}\int d\omega\, d^3k\; e^{i(\vec{k}\cdot\vec{x}-\omega k')}\, g(\vec{k},\omega)$$

$$= \frac{-4\pi}{(2\pi)^4}\int d\omega \int d^3k\; e^{i\vec{k}\cdot\vec{x}'-\omega t'}$$

$$\int d\omega \int d^3k\left(-k^2 + \frac{\omega^2}{c^2}\right) e^{i(\vec{k}\cdot\vec{x}'-\omega t')} \times g(\vec{k},\omega) = \frac{-1}{(4\pi)^3}\int d\omega\int d^3k\; e^{i(k\cdot\vec{x}'-\omega t')}$$



$$\int_{-\infty}^{\infty} \frac{e^{-j\omega t}\, d\omega}{(\omega-kc)(\omega+kc)} = 0 \quad t<0\ ?$$

1-22-76 (THURS) TEXT IS SMYTHE

$G_R = 0$ FOR $t<0$

FOR $t'>0$, MUST EVALUATE RESIDUES, THEN

$$G_R(\vec{x}',t') = \frac{-c^2}{4\pi^3}\int d^3k\, e^{i\vec{k}\cdot\vec{x}'} \times (-i2\pi)\left[\left\{\frac{e^{-i\omega t'}}{\omega+ck}\right\}\Bigg|_{\omega=ck} + \left\{\frac{e^{-i\omega t'}}{\omega-ck}\right\}\Bigg|_{\omega=-ck}\right]$$

$$= \frac{-ic^2}{2\pi^2}\int d^3k\, \frac{e^{i\vec{k}\cdot\vec{x}'}}{2ck}\,(i\,2\sin ckt')$$

$$= \frac{c}{2\pi^2}\int d^3k\, e^{i\vec{k}\cdot\vec{x}'}\,\frac{\sin ckt'}{k}\ ;\ t'>0$$

TO EVALUATE THIS, USE SPHERICAL COORDINATES IN K SPACE. CHOOSE IT SUCH THAT THE POLAR AXIS LIES ALONG $\vec{x}'$.

THUS

$$G_R = \frac{c}{2\pi^2}\int_0^\infty k^2dk\int_0^\pi \sin\theta\, d\theta\int_0^{2\pi} d\phi\, e^{ik|\vec{x}|\cos\theta} \times \frac{\sin ckt'}{k}$$

$$= \frac{c}{\pi}\int_0^\infty \frac{k^2dk}{k}\, e^{ik|\vec{x}'|\cos x}\Big|_0^\pi \frac{-1}{ik|\vec{x}'|} \times \sin ckt'$$

PLUGGING BACK IN GIVES

$$\psi(\vec{x},t) = \int_V d^3x' \int dt' \, f(\vec{x}',t') \frac{\delta\left[\frac{|\vec{x}-\vec{x}'|}{c} - (t-t')\right]}{|\vec{x}-\vec{x}'|}$$

$$- \frac{1}{4\pi c^2} \int_V d^3x' \left[G \frac{\delta\psi}{\delta t'} - \psi \frac{dG}{dt'} \right]_{t_1}^{t_2} \quad \sim \text{TRANSIENT}$$

$$+ \frac{1}{4\pi} \int_{t_1}^{t_2} dt' \oint_S dS' \, \vec{n} \cdot (G\nabla'\psi - \psi\nabla' G)$$

$$= \int_V d^3x' \, \frac{f(\vec{x}, t'_{RET})}{|\vec{x} - \vec{x}'|} + \text{OTHER TERMS}$$

STEADY STATE TERM

WHERE

$$t'_{RET} = t - \frac{|\vec{x} - \vec{x}'|}{c}$$

t'_{RET} IS THE TIME THE RADIATION TAKES TO GET FROM THE SOURCE TO OBSERVER.

RETARDED GREENS FUNCTIONS IS KIND OF EQUIVALENT TO HUYGEN'S PRINCIPLE

FROM OUR RESULT, WE MAY DERIVE KIRCHOFF'S FORMULA FOR SCALER OPTICS. ASSUME

$f = 0$ (ie, NO SOURCES)

$\frac{\delta\psi}{\delta t'}, \psi \to \infty$ AS $t \to \infty$

SO WE ARE LEFT WITH ONLY THE THIRD TERM $\Longrightarrow$

DIGRESSION:

IF, IN $\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -4\pi f(\vec{x},t)$ ①

WE ASSUME $f(\vec{x},t) = f(\vec{x},\omega)e^{-i\omega t}$

$\psi(\vec{x},t) = \psi(\vec{x},\omega)e^{-i\omega t}$

THEN WE GET

$$\nabla^2\psi(\vec{x},\omega) + k^2\psi(\vec{x},\omega) = -4\pi f(\vec{x},\omega) \quad ②$$

WHICH IS THE INHOMOGENEOUS FORM OF HELMHOLTZ'S EQUATION.

THE GREENS FUNCTION FOR THIS IS MERELY THE FOURIER TRANSFORM OF THE GREENS FUNCTION OF EQ. 1: $G_R = \frac{1}{R}e^{ikR}$. EQ. 2 IS THE FOURIER TRANSFORM OF EQ 1.

DIVY UP S

LET $S = S_1 + S_2$

AT INFINITY, ψ BEHAVES AS $f(\theta,\phi)\,e^{ikr}/r$

AND $\nabla\psi$ AS $\psi\left(1 - \frac{1}{ikr}\right)$

AS $S_2 \to \infty$, THE SURFACE INTEGRAL OVER S_2 VANISHES AND

$$\psi(\vec{x}) = \frac{+1}{4\pi}\int_{S_1}\frac{e^{ikR}}{R}\,\vec{n}\cdot\left[\nabla'\psi + ik\left(1+\frac{i}{kR}\right)\frac{\vec{R}}{R}\psi\right]ds'$$

PLUG 'EM IN ψ:

$$\psi(\vec{x}) = \frac{1}{4\pi}\int_{S_1} \frac{e^{ikr}}{r\left[1-\frac{\vec{n}_x\cdot\vec{x}'}{r}\right]}\, e^{ik\vec{n}_x\cdot x'}$$

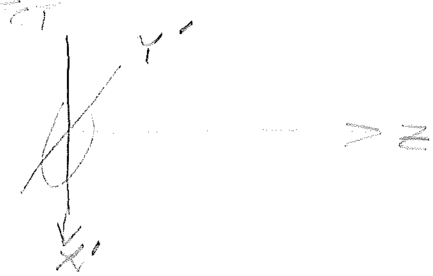
$$\times\ \vec{n}\cdot\left[\nabla'\psi + k\left(1+\frac{i}{kr\left[1-\frac{\vec{n}_x\cdot\vec{x}'}{r}\right]}\right)\frac{\vec{R}}{R}\psi\, ds'\right.$$

NEGLECT

$\frac{\vec{R}}{R} \simeq \vec{n}_x \rightarrow$ NOTE

$\vec{n} = \hat{z} \rightarrow$ ACCOMPLE

$\vec{n}_x \simeq \hat{z} \rightarrow$ NOTE



WE WILL LOOK AT FIELD ONLY NEAR Z AXIS

GIVES

$$\psi(\vec{x}) = \frac{e^{ikr}}{4\pi r}\int_{S_1} e^{i(k_x X' + k_Y Y')}$$

$$\times\left[\frac{\delta\psi}{\delta z'} + ik\psi\right] dx'dy'$$

WHERE

$k_x = k\,\vec{n}_x|_{x \text{ DIRECTION}}$

$k_Y = k\,\vec{n}_Y|_{Y \text{ DIRECTION}}$

SUPPOSE INCIDENT WAVE IS A QUASI PLANE WAVE:

$$\psi_{INCIDENT} = f(x,y)\, e^{ik_z z} \simeq f(x,y)\, e^{ikz}$$

PLUG UP IN:

$$\psi(\vec{x}') = \frac{e^{ikr}}{r}\, i2k\int_A e^{-i(k_x X' + k_Y Y')}\, \psi_{INC}(x', Y')\, dx'dY'$$

THE FAR FIELD IS THUS THE TWO DIMENSIONAL FOURIER TRANSFORM OF THE APERTURE.

MAY SIMILARLY USE LENS

$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\delta \vec{B}}{\delta t}$ (FARADAY'S LAW) $\Leftarrow$ A MAXWELL EQ. IN GAUSSIAN CGS

EQUIVALENTLY

$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{d}{dt}\vec{\nabla} \times \vec{A} = 0$

OR

$\vec{\nabla} \times \left[\vec{E} + \frac{1}{c}\frac{\delta \vec{A}}{\delta t}\right] = 0$

THUS

$\vec{E} + \frac{1}{c}\frac{\delta \vec{A}}{\delta t} = -\nabla\phi \Rightarrow \phi$ IS A SCALER

$\therefore \begin{cases} \vec{B} = \nabla \times \vec{A} \\ \vec{E} = -\nabla\phi - \frac{1}{c}\frac{\delta A}{\delta t} \end{cases}$ (E & B FUNDEMENTAL, H & D SECONDARY)

NOTE: NEED 4 SCALER FUNCTIONS TO SPECIFY $\vec{B}$ & $\vec{H}$

IN GAUSSIAN CGS UNITS IN FREE SPACE

$\vec{B} = \vec{H}$; $\vec{D} = \vec{E}$

$\nabla \times \vec{B} = \frac{1}{c}\frac{d\vec{E}}{dt} + \frac{4\pi}{c}\vec{J}$

$\nabla \times \nabla \times \vec{B} = \frac{1}{c}\frac{d}{dt}\left(-\frac{1}{c}\frac{\delta \vec{A}}{\delta t} - \nabla\phi\right) + \frac{4\pi}{c}\vec{J}$

IDENTITY

$\nabla\left(\nabla\cdot\vec{A} + \frac{1}{c}\frac{d\phi}{dt}\right) - \nabla^2\vec{A} + \frac{1}{c^2}\frac{\delta^2 \vec{A}}{\delta t^2} = \frac{4\pi}{c}\vec{J}$

OR $-\square\vec{A} + \nabla\chi = \frac{4\pi}{c}\vec{J}$

$\Rightarrow \chi = \nabla\cdot\vec{A} + \frac{1}{c}\,\delta\phi/\delta t$

$\square = \nabla^2 - \frac{1}{c^2}\frac{\delta^2}{\delta t^2}$

RECALL: $\nabla\cdot\vec{D} = 4\pi\rho$

SINCE $\vec{E} = \vec{D}$, $\nabla\cdot\vec{E} = 4\pi\rho$

THEN

$\nabla\cdot\left(-\frac{1}{c}\frac{\delta\vec{A}}{\delta t} - \nabla\phi\right) = 4\pi\rho$

$= -\nabla^2\phi + \frac{1}{c^2}\frac{\delta^2\phi}{\delta t^2} - \frac{1}{c}\frac{\delta}{\delta t}\left(\frac{1}{c}\frac{\delta\phi}{\delta t} + \nabla\cdot\vec{A}\right)$

$\Rightarrow -\square\phi - \frac{1}{c}\frac{\delta\chi}{\delta t} = 4\pi\rho$

1-29-76

WE HAD

$$-\Box A + \nabla \chi = \frac{4\pi}{c}\vec{J}$$

$$-\Box\phi - \frac{1}{c}\frac{\delta\chi}{\delta t} = 4\pi\rho \Rightarrow \chi = \nabla A + \frac{1}{c}\frac{d\phi}{dt},\ \Box = \nabla^2 - \frac{1}{c^2}\frac{\delta^2}{\delta t^2}$$

LET'S GO TO A NEW GAUGE. WANNA GET RID OF χ's

$$-\Box\vec{A}' + \nabla\chi' = \frac{4\pi}{c}\vec{J}\ ;\ A' = A + \nabla\Lambda,\ \phi' = \phi - \frac{1}{c}\frac{\delta\Lambda}{\delta t}$$

$$-\Box\phi' - \frac{1}{c}\frac{\delta\chi}{\delta t} = 4\pi\rho\ ;\ \chi' = \nabla_1\vec{A}' + \frac{1}{c}\frac{d\phi}{dt}$$

LETS TRY TO CHOOSE Λ TO MAKE $\chi' = 0$

$$\chi' = \nabla\vec{A}' + \frac{1}{c}\frac{\delta\phi'}{\delta t}$$

$$= \nabla(\vec{A} + \nabla\Lambda) + \frac{1}{c}\frac{\delta}{\delta t}\left(\phi - \frac{1}{c}\frac{\delta\Lambda}{\delta t}\right)$$

$$= \chi + \Box\Lambda$$

IF WE CHOOSE $\Lambda \ni \Box\Lambda = -\chi$, THEN $\chi' = 0$. THAT IS

$$\nabla^2\Lambda - \frac{1}{c^2}\frac{\delta^2\chi}{\delta t^2} = -\chi \leftarrow \text{LORENZ GAUGE}$$

THIS IS AN INHOMOGENEOUS (DRIVEN) WAVE EQUATION. USUALLY, THERES SOLN. MAKING THIS SOLN GIVES

$$\begin{cases} \Box\vec{A} = -\frac{4\pi}{c}\vec{J} \\ \Box\phi = -4\pi\rho \end{cases}$$

EQUIVALENT TO MAXWELL'S EQUATION.

$\vec{J}$ DRIVES $\vec{A}$, ρ DRIVES ϕ

FOUR DRIVEN WAVE EQUATIONS

EX: SINGLE CHARGED PARTICLE

$$\rho(\vec{x}, t) = q\,\delta(\vec{x} - \vec{r}(\vec{x})t)$$

$$J(\vec{x}, t) = q\vec{v}\,\delta(\vec{x} - \vec{r}(\vec{x})t)$$

2-3-75 (TUES)

DR. BYSZEWSKI

SMYTHE - "STATIC & DYNAMIC ELECTRICITY

SUPLEMENTARY: JACKSON "CLASSICAL ELECTRODYNAMICS"

GRADING : HOMEWORK - 50 POINTS

TEST - 20 POINTS

FINAL - 30 POINTS

A 100-90 B 90-80 C 80-70

COULOMB'S LAW

$q \circ \leftarrow r \rightarrow \circ q'$

$$\vec{F} = \frac{1}{4\pi\epsilon} \frac{q \cdot q'}{r^3} \vec{r}$$

ϵ = CAPACITIVITY

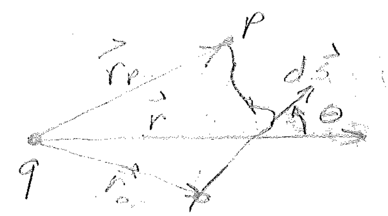
$$\epsilon_{VACUUM} = 8.85 \times 10^{-12}\ F/m$$

$$K = \epsilon/\epsilon_V$$

$$q \bullet \quad \rightarrow \quad \overset{\vec{F}}{\underset{q'}{\rightarrow}} \left(\vec{E}_p \overset{\Delta}{=} \frac{\vec{F}}{q'} \right)$$

$$\vec{E}_p = \frac{1}{4\pi\epsilon} \sum \frac{q_i \vec{r_i}}{r_i^3} \leftarrow \text{NOTE: NO TEST CHARGE}$$

POTENTIAL



$$\underline{dV \overset{\Delta}{=} -\vec{E} d\vec{s}} = -E\, ds \cos\theta$$

$$V_p - V_0 = \int_0^p dV$$

$$= \frac{q}{4\pi\epsilon} \left(\frac{1}{r_p} - \frac{1}{r_0} \right)$$

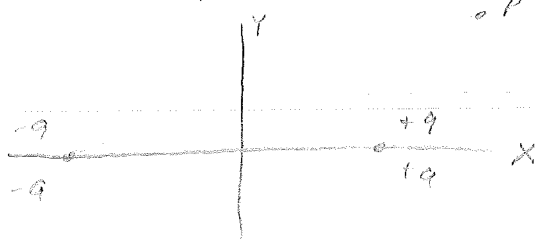
AS $r_0 \rightarrow \infty$, $V_p = \frac{q}{4\pi\epsilon r_p}$

NOTE THAT RESULT IS IND. OF PATH

$$\oint \vec{E} \cdot d\vec{s} = 0$$

$$\boxed{d\vec{s} = \lambda \vec{E}}$$

$$\frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z}$$



$$E_x = \frac{q}{4\pi\epsilon}\left[\frac{x-a}{\sqrt{y^2+(x-a)^2}^{\,3}} \pm \frac{x+a}{\sqrt{y^2+(x+a)^2}^{\,3}}\right]$$

$$E_y = \frac{q}{4\pi\epsilon}\left[\frac{y}{\sqrt{y^2+(x-a)^2}^{\,3}} \pm \frac{y}{\sqrt{y^2+(x+a)^2}^{\,3}}\right]$$

$$\frac{dy}{dx} = \frac{E_y}{E_x}$$

SOLVE THIS DIFFERENTIAL EQ.

$$U = \frac{x+a}{y} \quad ; \quad V = \frac{x-a}{y}$$

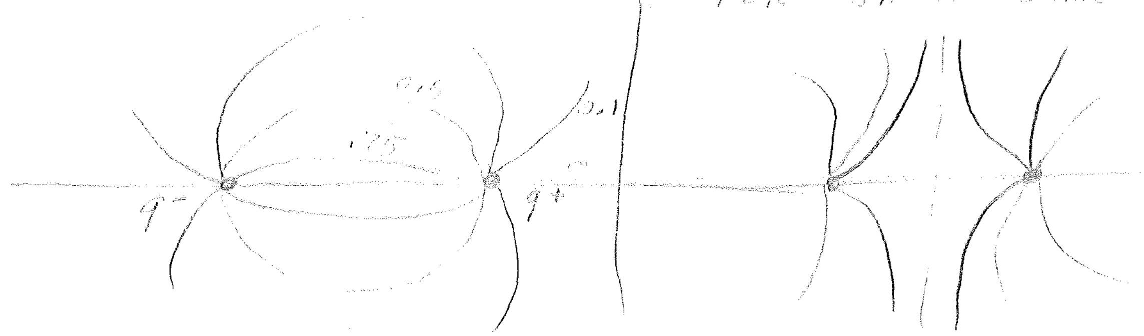
GIVES

$$(x+a)\left[(x+a)^2+y^2\right]^{-1/2} \pm (x-a)\left[(x-a)^2+y^2\right]^{-\frac{1}{2}} = C$$

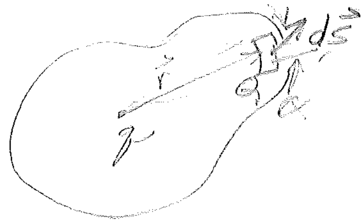
DIFFERENT C'S WILL GIVE DIFFERENT LINES OF THE $\vec{E}$ FIELD.

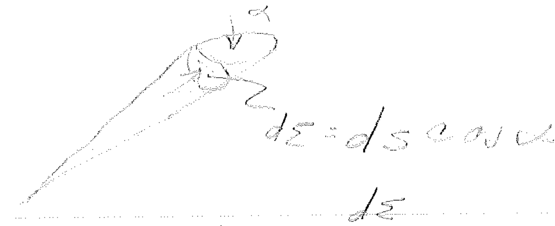
[HOMEWORK:
FIND THIS SAME EQ. USING GAUSS' THEOREM]

FOR SAME CHARGES



GAUSS' ELECTRIC FLUX THEOREM





$$d\Omega = \frac{d\Sigma}{r^2}$$

$dN = \epsilon E_n ds$ = FLUX OUTTA ds PERPENDICULARLY

$$E_n = \frac{q}{4\pi\epsilon r^3}\,\vec{r}\cdot\vec{n} = \frac{q r\cos\alpha}{4\pi\epsilon r^2}$$

THEN

$$dN = \frac{q(\cos\alpha\, ds)}{4\pi\, r^2} = \frac{q}{4\pi}\frac{d\Sigma}{r^2} = \frac{q\, d\Omega}{4\pi}$$

INTEGRATE OVER WHOLE SURFACE

$$4\pi \oint_S dN = q\int_0^{4\pi} d\Omega$$

$N = \oint_S dN = q$ ← GAUSS FLUX THEOREM

FOR NO CHARE, NO FLUX

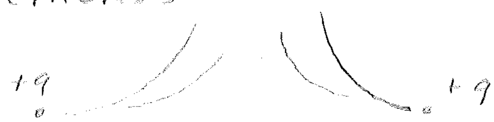
WE MAY EQUIVALENTLY WRITE

$$\oint_S \epsilon\vec{E}\cdot\vec{n}\,ds = q$$ GAUSS'S FLUX THEOREM

$$= \oint_S \vec{D}\cdot\vec{n}\,ds = q \Rightarrow \vec{D} = \epsilon\vec{E}$$

$[E] \sim V/m$ $\quad$ $[D] \sim \text{coul}/m^2$

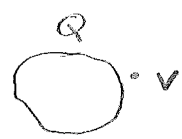
2-5-75 (THURS)



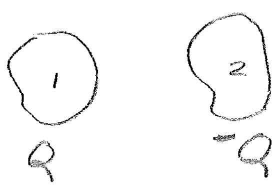
DOUBLING CHARGES DOES NOT CHANGE LINES.
POTENTIAL ON LINE, THOUGH, WILL BE DOUBLED
GENERALIZE

DOUBLING q_1, q_2, q_3 WILL DOUBLE V @ P

$$\Rightarrow \frac{Q}{V} = \text{CONST} = \text{CAPACITANCE}$$



$\underline{C = Q/V}$

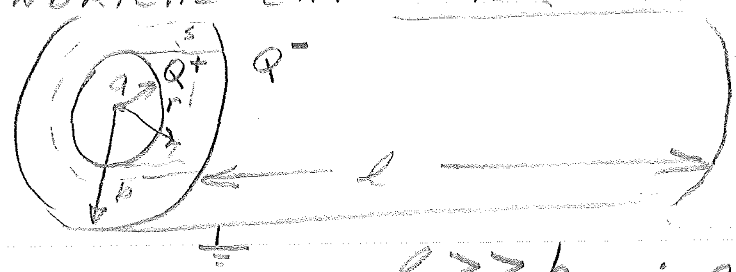


⇐ CAPACITOR

$$C = \frac{Q}{V_1 - V_2}$$

$C = \sum_{i=1}^{n} C_i$ (PAR) $\quad \frac{1}{C} = \sum_{i=1}^{n} \frac{1}{C_i}$ (SERIES)

CYLYNDRICAL CAPACITOR



$$\ell >> b \; ; \; a < r < b$$

$$\oint_s \bar{D} \cdot \vec{n} \, ds = Q$$

$$\Rightarrow \quad 2\pi r \cdot \ell \varepsilon E = Q$$

$$E = -\frac{\delta V}{\delta r}$$

$$V_a - V_b = \frac{Q}{2\pi \ell \epsilon} \ln b/a$$

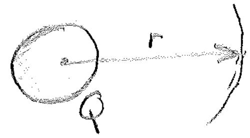
σ = AREA CHARGE DENSITY $= \frac{Q}{2\pi b \ell}$

$$\Rightarrow V_a - V_b = \frac{\epsilon \sigma}{b} \ln b/a$$

$$\boxed{C = \frac{2\pi \ell \epsilon}{\ln(b/a)}}$$

BACK TO SPHERICAL CAPACITOR, LET $r\,(b) \to \infty$

$$\Rightarrow V = \frac{Q}{4\pi \epsilon a}$$



$$V = \frac{Q}{4\pi \epsilon r}$$

NOTE: CANNOT DO THIS FOR CYLLANDER SINCE WE RESTRICTED $\ell >> b$

LOOK AT ENERGY

$$W_j = q_j V_j = q_j \sum_{i=1}^{n} \frac{q_i}{4\pi\epsilon r_{ij}} \quad ; i \neq j$$

$$q_j \circ \longrightarrow \begin{Bmatrix} q_1 & q_2 \\ q_3 & \end{Bmatrix}$$

$$W = \sum_{j=1}^{n} W_j \quad \Leftarrow \text{DOUBLE SUM}$$

1 • $W = 1(2+3+4) = 12 + 13 + 14$

2 • $W = 2(1+3+4) = 12 + 23 + 24$

3 • $W = 3(1+2+4) = 13 + 23 + 34$

4 • $W = 4(1+2+3) = 14 + 24 + 34$

$$\Rightarrow W = \frac{1}{2}\sum_{j=1}^{n} W_j = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{q_i q_j}{4\pi\epsilon r_{ij}}$$

$$= \frac{1}{2}\sum_{j=1}^{n} q_j V_j$$

FOR CONDUCTING MATERIAL ALL V IS SAME EVRYWHERE

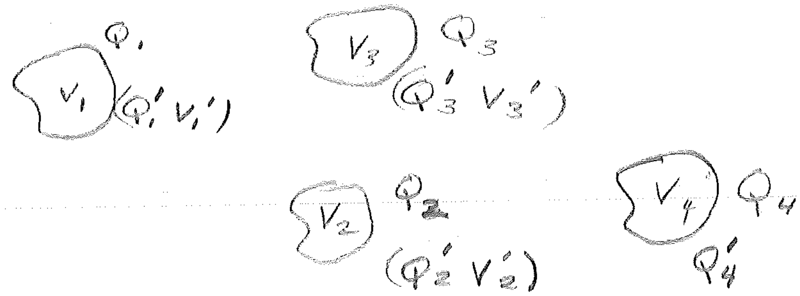
$$= \frac{1}{2} V \sum_{j=1}^{n} q_j$$

$$= \frac{1}{2} VQ = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} C V^2$$

$$V = V_1 - V_2$$

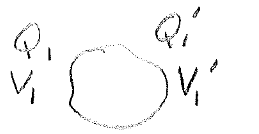
$$\Rightarrow \underline{W = \frac{1}{2} Q (V_1 - V_2)}$$

GREEN'S RECIPROCATION THEOREM



$$\sum_{i=1}^{n} Q_i V_i' = \sum_{i=1}^{n} Q_i' V_i$$

CONSIDER $n = 1$



$$; \quad QV_1' = Q_1' V_1 \Rightarrow \frac{Q_1}{V_1} = \frac{Q_1'}{V_1'} = C$$

PROOF:

$$Q_1' V_1 = 0 + Q_1' \frac{Q_2}{4\pi\epsilon r_{21}} + Q_1' \frac{Q_3}{4\pi\epsilon r_{13}} + \dots + Q_1' \frac{Q_n}{4\pi\epsilon r_{n1}}$$

$$Q_2' V_2 = Q_2' \frac{Q_1}{4\pi\epsilon r_{12}} + 0 + Q_2' \frac{Q_3}{4\pi\epsilon r_{23}} + \dots$$

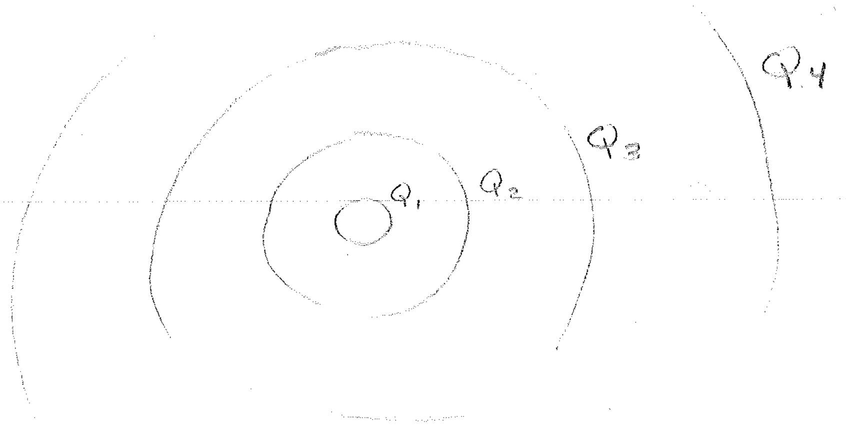
$$\vdots$$

$$\Rightarrow \sum_{i=1}^{n} Q_i' V_i = (\quad) + \overline{Q_3 \sum_{i=1}^{n} \frac{Q_i'}{4\pi\epsilon r}} \; ; \; i \neq 3$$

$$\overline{} \; Q_3 V_3'$$

$$= \sum_{i=1}^{n} Q_i V_i'$$

EXAMPLE (FROM TEXT)



H.W.

FIND DISPLACEMENT $\vec{D}$ ON SURFACE OF PLANE CONDUCTOR OF ZERO THICKNESS & ON THE SHEET OF CONDUCTING MATERIAL OF THICKNESS d.

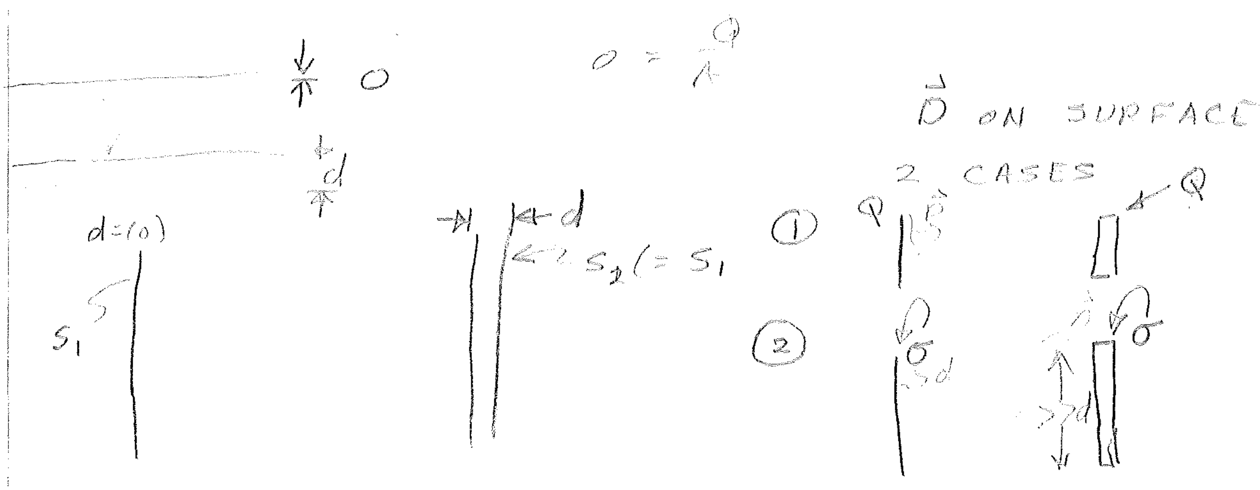
SAME GEOMETRY. FOR TWO CASES,

1) BOTH CONDUCTORS CONTAIN SAME CHARGE Q 2) BOTH CONDUCTORS HAVE SAME σ.

IS IT POSSIBLE THAT $\vec{D}$ IS NOT PARR. TO $\vec{E}$. IF YES, WHY, & EXAMPLE.

FROM CHAPT: 2

4, 13, 14, 15, 49C



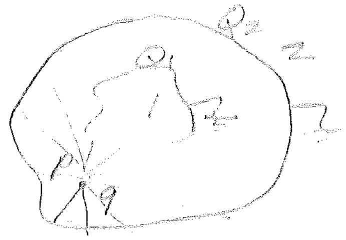
APPLY TO PROBLEM

$$V_P' = \frac{Q'}{4\pi\epsilon r} \qquad V' = \frac{Q'}{4\pi\epsilon a}$$

GIVES Q = INDUCED CHARGE ON SPHERE

$$\boxed{Q = -\frac{a}{r}\, q}$$

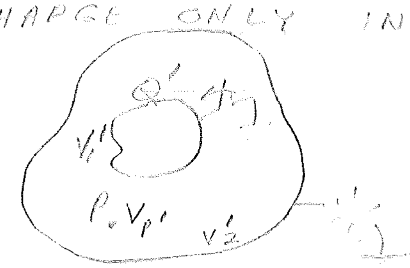
[illegible]



WHAT IS INDUCED CHARGES Q_1 & Q_2

$$q = -(Q_1 + Q_2)$$

CHARGE ONLY INNER SURFACE



$$q V_P' + Q_1 V_1' + Q V_2' = 0$$

GIVES
$$Q_1 = \frac{V_2' - V_P'}{V_1' - V_2'}\, q$$

$$Q_2 = \frac{V_1' - V_P'}{V_2' - V_1'}\, q$$

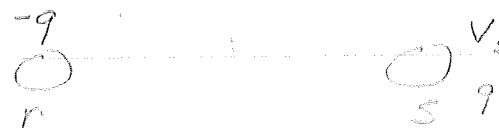
MAY TAKE SURFACES AS SPHERES (HOMEWORK ?)

WITH APPROPRIATE GROUNDING

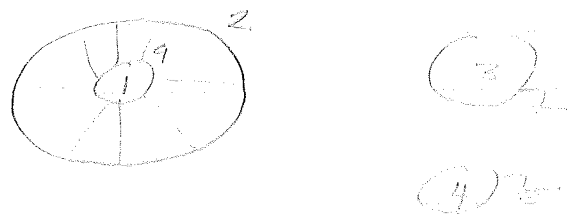
$C_{rr}\ Q_r/V_r$; $Q_r = C_{sr} V_s$

INDUCED ON r

VOLTAGE ON S



THUS Q_r HAS OPPOSITE SIGN OF V



SURFACE 1 CANNOT INDUCE ON <u>3</u> OR <u>4</u>

RECALL $\bar{E}$ & $\bar{D}$ W = ENERGY

$$\frac{dW}{dV} = \frac{1}{2}\ \frac{\bar{D}\cdot\bar{E}}{2} \Rightarrow W = \frac{1}{2}\int_V \bar{D}\cdot\bar{E}\,dV$$

RECALL, EACH BRINGING IN Q FROM ∞:

$$W_i = V_i \cdot Q_i \Rightarrow \boxed{W = \frac{1}{2}\sum_{i=1}^{n} V_i Q_i}$$

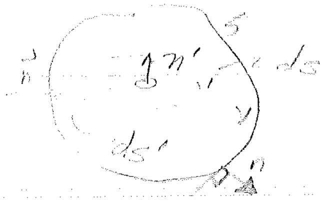
MAY USE MATRIX RELATIONSHIP TO MAKE THIS <u>ALL</u> VOLTAGE OR <u>ALL</u> CHARGE.

$$F = -\frac{\delta W}{\delta \eta}$$

$$\hat{F}_p = \int_s \frac{\bar{D}\cdot\bar{E}}{2}\,\bar{p}\cdot\bar{n}\,ds$$

STOKE'S THEOREM

$$\vec{A} = \vec{n} \times \vec{F}$$



$$\oint_S (\vec{n} \times \vec{F})\, \vec{n}\, ds = \int_V \vec{\nabla} \cdot (\vec{n} \times \vec{F})\, dV$$

$$\vec{\nabla} \cdot \vec{n} \times \vec{F} = \vec{F} \cdot \vec{\nabla} \times \vec{n} - \vec{n} \cdot \vec{\nabla} \times F$$

$$F \cdot \vec{\nabla} \times \vec{n} = 0$$

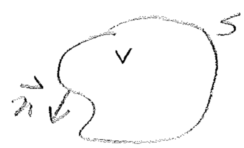
$$(\vec{n} \times \vec{F})\, \vec{n} = \begin{vmatrix} \bar{a}_x & a_y & a_z \\ 1 & 0 & 0 \\ F_x & F_y & F_z \end{vmatrix} \cdot \vec{n} = 0$$

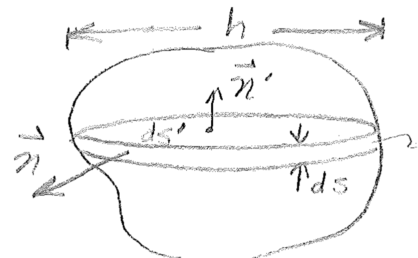
2-12-76 (THURS.)

RECALL GAUSS' THEOREM

$$\oint_S \vec{A} \cdot \vec{n}\, ds = \int_V \vec{\nabla} \cdot \vec{A}\, dV$$

LET $\vec{A} = \vec{n} \times \vec{F}$





$$\oint_S \vec{n} \times \vec{F} \cdot \vec{n}\, ds = \int_V \vec{\nabla} \cdot \vec{n} \times \vec{F}\, dV$$

$$\vec{\nabla} \cdot \vec{n} \times \vec{F} = -\vec{n} \cdot \vec{\nabla} \times \vec{F}$$

$$(\vec{n} \times \vec{F})\, \vec{n} = 0$$

$$\Rightarrow \quad \vec{n} \times \vec{F} \cdot \vec{n}' = \vec{F} \cdot \vec{n}' \times \vec{n}$$

(NOTE $\vec{n}' \perp \vec{n}$)

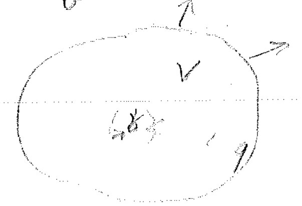
$$dV = h\, ds \quad ; \quad ds' = h\, dl$$

$$\bar{n}_1 \times \bar{n}\, dl = -\bar{n} \times n'\, dl = d\vec{l}$$

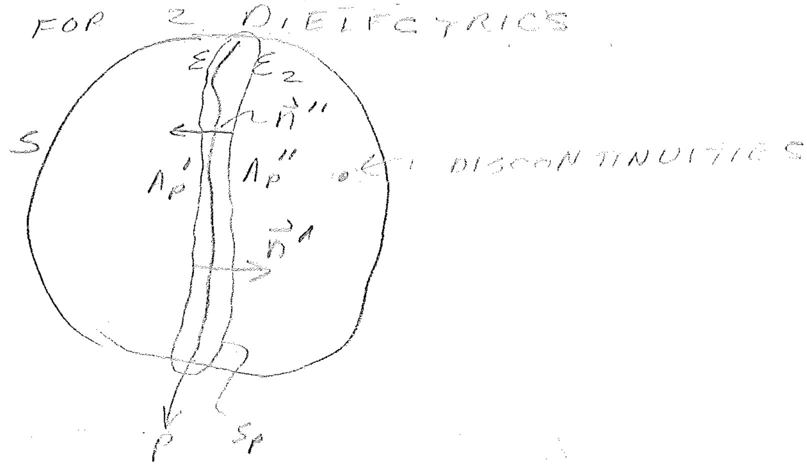
$$\oint \bar{A}\cdot d\bar{n}\, ds = \int \bar{\nabla}\cdot\bar{A}\, dv$$

REWRITE AS

$$\sum_{j=1}^{m} \oint_{S_j} \bar{A}\cdot\bar{n}\, dS_j = \int_V \bar{\nabla}\cdot\bar{A}\, dV$$



FOR 2 DIELECTRICS



$$\oint_S (A'_p\cdot n'_p + A''_p\cdot n''_p)\, ds \Rightarrow \sum_{j=1}^{m} \oint \bar{A}_j\cdot\bar{n}_j\, ds_j + \oint (A'_p\, n'_p + A''_p\, n''_p)\, dS_p$$

$$= \int_V \bar{\nabla}\cdot\bar{A}\, dV$$

$$\bar{A} = \epsilon[\text{grad}\, \phi]\, \psi = \psi\, \epsilon\, \nabla\phi$$

COMBINING GIVES GREEN'S FIRST THEOREM:

$$\sum_{j=1}^{m} \oint \varepsilon\psi \frac{\delta\phi}{\delta n_j}\, dS_j + \sum_{p=1}^{q} \int_{S_p} \Big(\varepsilon'_p\psi'_p \frac{\delta\phi'_p}{\delta n'_p}$$

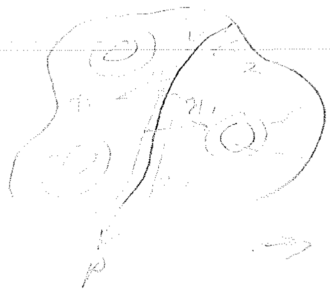
$$+ \varepsilon''_p\psi''_p \frac{\delta\phi''_p}{\delta n''_p}\Big)\, dS_p$$

$$= \int_V \epsilon\, \nabla\psi\ \nabla\phi\, dv$$

$$+ \int_V \psi\, \bar{\nabla}\cdot(\varepsilon\, \bar{\nabla}\phi)\, dv$$

RECALL GREEN'S RECIPROCATION THEOREM:

$$\sum_{j=1}^{m} Q_j V_j' = \sum_{j=1}^{m} Q_j' V_j$$



[illegible] DISCONTINUITIES

METAL — EQUIPOTENTIAL
$\Rightarrow$ FLUX IS SAME OUT [illegible] VOLUME.

$Q_1 \ldots Q_n$ DETERMINED BY $V(r) = \phi$
$Q_1' \ldots Q_n'$ " " $V'(r) = \phi'$

ON DIELECTRIC SURFACE

$V_{p1} = V_{p2}$; $V_{p1}' = V_{p2}'$

$D_{np_1} = D_{np_2}$; $D_{np_1}' = D_{np_2}'$

$$\Rightarrow \varepsilon_{1p} \frac{\delta V}{\delta n_{1p}} = \varepsilon_{2p} \frac{\delta V}{\delta n_{1p}} = -\varepsilon_{2p} \frac{\delta V}{\delta n_{2p}}$$

THUS, IF NO CHARGE ON [illegible] INTERFACE, [illegible] FURTHERMORE, THERE'S NO CHARGES & POISSON'S EQ. MUST BE SATISFIED $\Rightarrow \nabla \cdot \varepsilon \nabla V = 0$, THIS LEAVES

$$\underline{\sum_{j=1}^{m} \oint_{S_j} \varepsilon \left(\psi \frac{\delta \phi}{\delta n_j} - \phi \frac{\delta \psi}{\delta n_j} \right) ds_j}$$

OR

$$\sum_{j=1}^{m} \oint_{S_j} \left(V_j \varepsilon' \frac{\delta V'}{\delta n_j} - V_j' \varepsilon \frac{\delta V}{\delta n_j} \right) ds = 0$$

GIVES

$$\oint_S \left[\frac{1}{4\pi\epsilon r}\frac{\delta\phi}{\delta n} - \phi\frac{\delta}{\delta n}\frac{1}{4\pi\epsilon r}\right]ds$$

$$= \frac{1}{4\pi\epsilon}\oint_S \frac{1}{r}\frac{\delta}{\delta n}\left[\phi + \frac{1}{4\pi\epsilon r}\right]ds$$

$$= \frac{1}{4\pi\epsilon}\oint_S \frac{1}{r}\frac{\delta}{\delta n} G\,ds$$

$G = \phi + \frac{1}{4\pi\epsilon r}$ = GREEN FUNCTION

SECOND TERM IN SUM GIVES

= 0 SINCE ϕ_P IS CONST

$$\oint_{S'}\left[\cancel{\frac{1}{4\pi\epsilon a}\frac{\delta\phi_P}{\delta n}} - \phi_P\frac{\delta}{\delta n'}\frac{1}{4\pi\epsilon r}\right]ds'$$

$$= -\oint_{S'}\phi_P\frac{\delta}{\delta n'}\frac{1}{4\pi\epsilon r}ds'$$

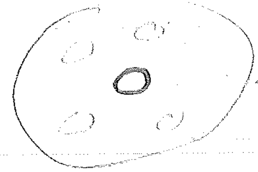
$$= -\phi_P\oint_{S'}\frac{\delta}{\delta n'}\frac{1}{4\pi\epsilon r}ds'$$? FIND

$$= -\frac{\phi_P}{\epsilon}\int_{V'}\rho\,dv' = -\frac{\phi_P}{\epsilon}$$

SUM OF THESE TWO = 0

$$\Rightarrow \frac{1}{4\pi\epsilon}\oint\frac{1}{r}\frac{\delta}{\delta n}G\,ds = \phi_P = \frac{1}{4\pi\epsilon}\int\frac{\sigma}{r}ds$$

INTRODUCING A METAL BALL INTO A SYSTEM REDUCES ENERGY



$$W = \frac{\epsilon}{2}\int_V E^2 dV$$

$$W - W' = \frac{\epsilon}{2}\int_V E^2 dV - \frac{\epsilon}{2}\int_{V'} E'^2 dV$$

ASSUME $Q_j = Q_j'$

$$W - W' = \frac{\epsilon}{2}\int_{V-V'} E^2 dV + \frac{\epsilon}{2}\int [(\vec{E} - \vec{E}')]^2 - 2\vec{E}'(\vec{E}'\vec{E})] dV$$

APPLY FIRST GREEN'S THEOREM WITH

$\psi = V'$, $\phi = V - V'$

SINCE $Q_j = Q_j'$ WE HAVE $\nabla^2 \phi = 0$

$\nabla^2 V - \nabla^2 V' = -\frac{\rho}{\epsilon} + \frac{\rho'}{\epsilon} = 0$ (SINCE $\rho = \rho'$)

THEN

$$\sum_{j=1}^{m} \int_{S_j} \psi \frac{\delta \phi}{\delta n_j} ds_v = \int_V [\bar{\nabla}\psi \cdot \nabla\phi - \cancel{\psi \nabla^2 \phi}^{\,0}] dV$$

$$= \int_{V'} \bar{\nabla} V' \cdot \bar{\nabla}(V - V') dV$$

$$= \int_{V'} \vec{E}' \cdot (E - E') dV \qquad (\text{FROM } \vec{E} = -\nabla V)$$

$$= \sum_{j=1}^{m} \int_{S_j'} V' \left[\frac{\delta V}{\delta n_j} - \frac{\delta V'}{\delta n_j} \right] dS_j$$

$$= \frac{1}{\epsilon} \sum_{j=1}^{m} V_j' \int_{S_j'} (\sigma - \sigma') dS_j \qquad \left(\text{FROM GAUSS FLUX THEM}\right)$$

$$= 0 = 2\bar{E}'(\bar{E}'\bar{E})$$

$\psi = \delta V \ ; \ \phi = V$
$\bar{\nabla} \cdot (\varepsilon \bar{\nabla} V) = -\rho$

SECOND THEOREM

$$\int_V \delta V \ \bar{\nabla} \cdot (\varepsilon \bar{\nabla} \phi) \, dv = -\int_V \delta V \rho \, dv$$

$$\sum_{j=1}^{m} \int_{S_j} \varepsilon \, \delta V \frac{\delta V}{\delta n_j} \, ds_j = \sum_{j=1}^{m} \delta V_j \int_{S_j} \sigma \, ds_j$$

$$\Rightarrow \int_V \varepsilon \bar{\nabla}(\delta V) \bar{\nabla} V \, dv = \sum_{j=1}^{m} \delta V_j Q_j + \int_V \delta V \rho \, dv$$

$$= 2\delta W \qquad \text{(pg. 59)}$$

$$\Rightarrow \delta W = -\frac{1}{2} \int_V \delta \epsilon (\nabla V)^2 dV$$

FOR $\delta \varepsilon > 0 \Longrightarrow \delta W < 0$

FEWER LINES OF FORCE ARE GENERATED

RECALL $W = (\vec{M} \cdot \vec{\nabla}) V$

$$dV = \frac{\sigma\, ds}{4\pi\epsilon r}$$

$$dV' = \frac{dV}{dn} = \frac{\sigma\, ds}{4\pi\epsilon} \frac{\delta}{\delta n}\left(\frac{1}{r}\right)$$

DIPOLE STRENGTH IS THEN $\sigma\, ds$

FOR A DOUBLE LAYER $\phi = \frac{M}{S}$ (pg. 14)

$$V_P = \frac{1}{4\pi\epsilon} \int_S \phi \frac{\delta}{\delta n}\left(\frac{1}{r}\right) ds$$

$$= \frac{1}{4\pi\epsilon} \int \phi \frac{\vec{n} \cdot \vec{r}}{r^3}\, ds$$

$$= \frac{1}{4\pi\epsilon} \int \phi\, d\Omega$$

~~GREEN'S~~ STRATUM

$\psi = r^{-1},\ \phi = V,\ \nabla^2 \phi = -\frac{\rho}{\epsilon}$

USE GREENS 2nd THEOREM
(WHAT DO THIS STUFF MEAN?)

$$\int_S \frac{1}{r} \frac{\delta V}{\delta n} ds - \int_S V \frac{\delta}{\delta n}\left(\frac{1}{r}\right) = -\frac{1}{\epsilon} \int \frac{\rho\, dv}{r} = -4\pi V_P$$

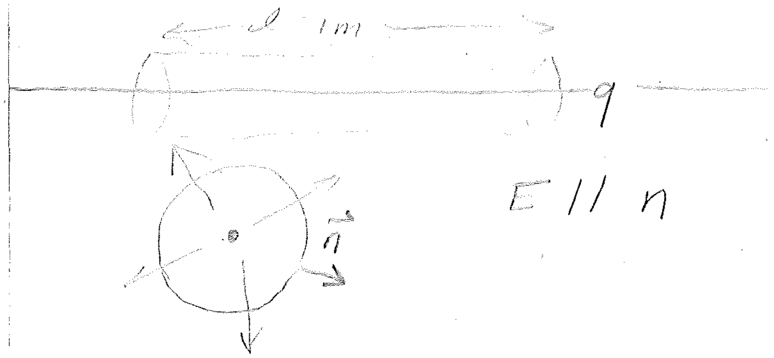
$$V_P = \underbrace{\frac{1}{4\pi\epsilon} \int \frac{\rho\, dv}{r}}_{\text{VOLUME}} = \underbrace{-\frac{1}{4\pi} \int_S \frac{1}{r} \frac{\delta V}{\delta n} ds + \frac{1}{4\pi} \int_S V \frac{\delta}{\delta n}\left(\frac{1}{r}\right)}_{\text{SURFACE}}$$

2-20-76 (THURS)

TWO DIMENSIONAL POTENTIAL

[ρ] HAS DIMENSIONS OF $\frac{C}{M}$

RECALL CYLYNDRICAL CAPACITOR



$q = \int_0^1 \rho \, dl$

$E \parallel n$

APPLY GAUSS' THEOREM:

$\oint \epsilon E ds = q \times 1 \text{ METER}$

$2\pi r \epsilon E = q \Rightarrow E = \frac{q}{2\pi \epsilon r}$

SIMILAR LAWS HOLD: $E = -\partial V / \partial r$

$\Rightarrow V = -\frac{q}{2\pi\epsilon} \ln r + C$

WE CHOOSE C TO FIT BOUNDRY CONDITION (LIKE V @ $r = \infty$ = 0)

$$\frac{\delta^2 \Theta}{\delta \theta^2} = -n^2 \Theta$$

$$\Rightarrow \Theta = A_n \cos n\theta + B_n \sin n\theta$$

$$r \frac{\delta R}{\delta r} + r^2 \frac{\delta^2 R}{\delta R^2} = n^2 \frac{R}{r^2}$$

$$\Rightarrow R = C_n r^n + D_n r^{-n}$$

FOR $n = 0$: $\Theta_0 = A\theta + B$
$R_0 = C \ln R + D$

THIS GIVES

$$V = \sum_n \Theta_n R_n$$

CONSIDER:

$$\sum_n A_n \cos n\theta + B_n \sin n\theta$$
$$= \sum_n (C_n \cos n\theta + D_n \sin n\theta)$$
$$\Rightarrow A_n = C_n \ ; \ B_n \sin n\theta$$

CONSIDER:

$$\Theta_1 = A_1 \cos\theta + B_1 \sin\theta + A_5 \cos 5\theta + B_5 \sin 5\theta$$
$$\Theta_2 = C_3 \cos 3\theta + D_3 \sin 3\theta + C_5 \cos 5\theta + D_5 \sin 5\theta$$

EQUATE: $\Theta_1 = \Theta$

$$\Rightarrow A_1 = B_1 = C_3 = D_3 = 0$$
$$A_5 = C_5 \ ; \ B_5 = D_5$$

BOUNDRY CONDITIONS

$$@\ r = b, \quad V_o = V_i$$

$$@\ r = b, \quad \epsilon_v \frac{\delta V_o}{\delta r} = \epsilon \frac{\delta V_i}{\delta r}$$

$$\xrightarrow{OR} \frac{\delta V_o}{\delta r} = K \frac{\delta V_i}{\delta r}$$

SPHERE (CONDUCTOR) HAS $V_i = 0$

$$\Rightarrow V_i(a) = 0$$

FOR $n \neq 1$ (pg. 66)

$$\Rightarrow A_n = B_n = C_n = 0$$

$$A_1 = -Eb^2 \frac{(K+1)a^2 + (K-1)b^2}{(K+1)b^2 + (K-1)a^2}$$

B_1, C_1 } IN BOOK (pg. 67)

POTENTIAL'S SHAPE:

$$V_o = \left(Er + \frac{A_1}{r}\right)\cos\theta$$

$$V_i = \left(B_1 r + \frac{C_1}{r}\right)\cos\theta$$

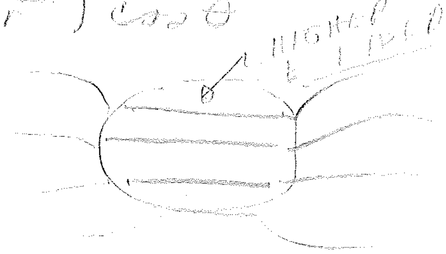
FOR JUST CONDUCTOR (SPHERE) LET $K = 1$

$$\Rightarrow V_o = E\left(r - \frac{a^2}{r}\right)\cos\theta$$

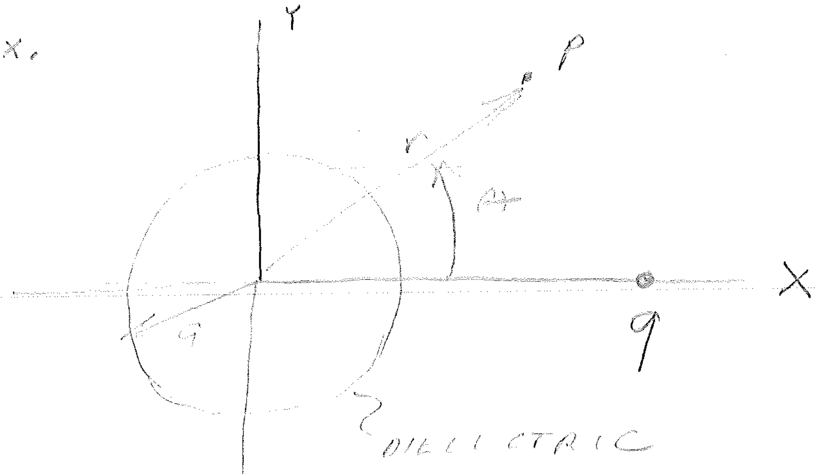
JUST DIELECTRIC $\Rightarrow$ LET $a = 0$

$$V_o = E\left(r - \frac{(K-1)}{(K+1)}\frac{b^3}{r^2}\right)\cos\theta$$

$$V_i = \frac{3E}{K+1}\, r\cos\theta$$

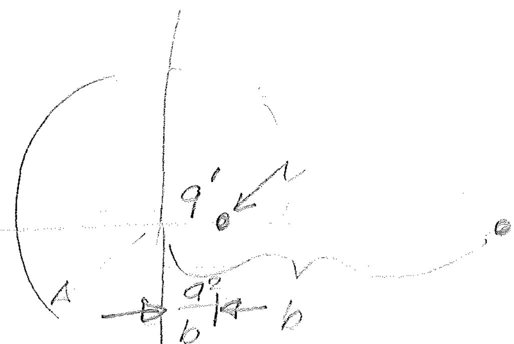


EX.



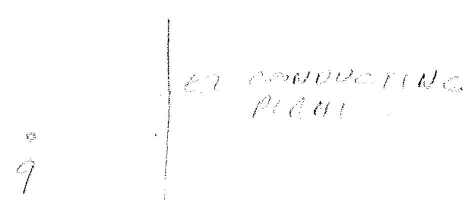
$$V_0 = \frac{q}{4\pi\epsilon}\left[\sum \frac{1}{n}\left(\frac{r}{b}\right)^n + \frac{A_n}{r_n}\cos n\theta - \ln b + C_i\right]$$

TURNS OUT POTENTIAL LOOKS LIKE



$$q' = \frac{1-K}{1+K}\, q \qquad \text{(BOTH LINE CHARGES)}$$

USE METHOD OF IMAGES:



Eq. 403

$$V_0 = E\left(r - \frac{K-1}{K+1}\frac{b^2}{r}\right)\cos\theta$$

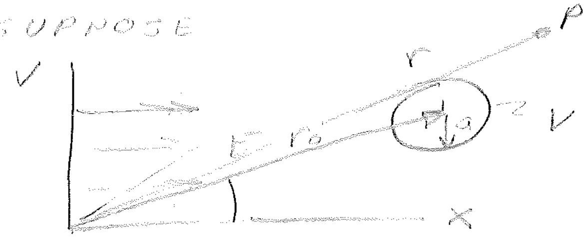
BEFORE $V = Er\cos\theta$

$$V_0 = Er\cos\theta - E\cos\theta\,\frac{K-1}{K+1}\frac{b^2}{r}$$

IS ~~IMPOSSIBLE~~ TO HAVE DIFFERENT POLARITY ON V & V_0.

CHANGE COORDINATE SYSTEM

SUPPOSE



$V_p = Er\cos\theta$ (BEFORE)

$$V_p = Er\cos\theta + A(r - r_0)^{-1}\cos\theta$$

BOUNDRY CONDITIONS: #1 (FOR $r = r_0 + a$) $V = 0$ (#2 $V = Er_0\cos\theta$)

$$E(r_0 + a) = -A(r_0 + a - r_0)^{-1}$$

$$\Rightarrow A = -Ea(r_0 + a)$$

$$\text{AND } V = E\left[r - \frac{a^2}{r - r_0} - \frac{a r_0}{r - r_0}\right]\cos\theta$$

FOR #2, $V = Er_0\cos\theta$

$$E(r_0 + a) = A(r_0 + a - r_0)^{-1} = E(r_0)$$

$$\Rightarrow A = -Ea^2$$

$$V = E\left[r - \frac{a^2}{r - r_0}\right]\cos\theta$$

$W = U + jV$

$W = \ln \frac{z+ja}{z-ja}$

$= \ln \frac{re^{j\theta}}{re^{-j\theta}}$ (r IS COMPLEX HERE)

$= \ln e^{j2\theta}$

$= j2\theta = j2\tan^{-1}\frac{a}{z} = j2\cot^{-1}\frac{z}{a}$

$\therefore \quad W = 2j\theta = j2\tan^{-1} a/z$

$z = a\cot\frac{U+jV}{j2} = a\cot\frac{\frac{U}{j} - V}{2}$

$= \frac{-a\sin(\frac{U}{j}) + a\sin V}{\cos(U/j) - \cos V}$

$= a\,\frac{\frac{j}{2}\left(e^{j(U/j)} - e^{-j(U/j)}\right) + \sin V}{\frac{1}{2}\left(e^{jU/j} + e^{-jU/j}\right) - \cos V}$

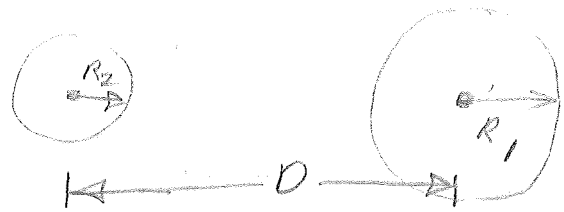
$= a\,\frac{j\sinh U + \sin V}{\cosh U - \cos V}$

$X = \frac{a\sin V}{\cosh U - \cos V}$; $Y = \frac{a\sinh U}{\cosh U - \cos V}$

$Z = X + iY$

FROM $q_1 = q_2 = q$, WE FOUND EQUIPOTENTIAL CYLINDERS $\Rightarrow f(x,y) =$ CONST U

NOW, WE WILL ASSUME A $f(x,y)$, AND FIND $a_1 = -a_2 \Rightarrow q_1 = -q_2$ AND THEN FIND CAPACITANCE



$R_1 = a / |\operatorname{csch} U_1|$

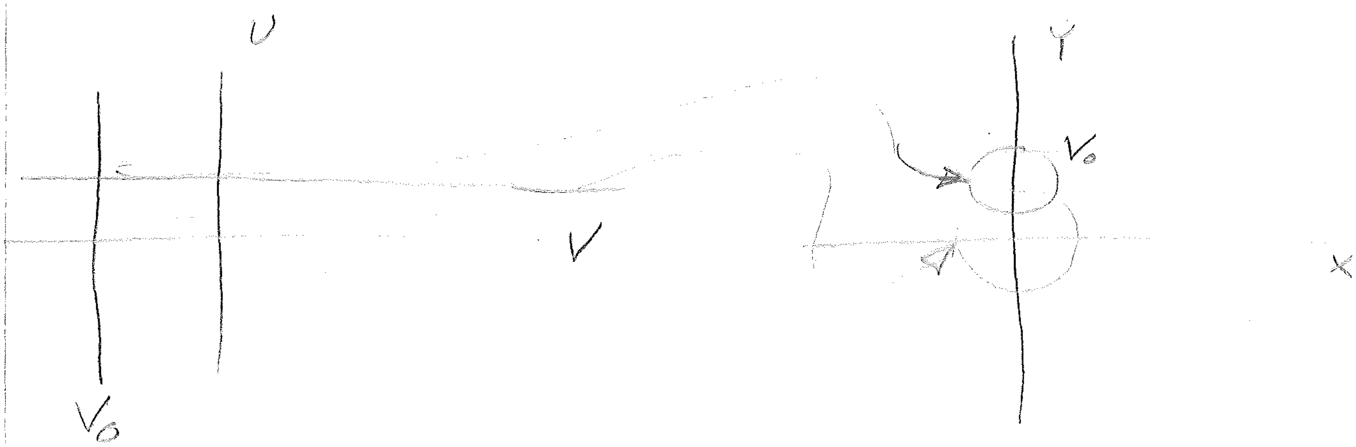
$R_2 = a / |\operatorname{csch} U_2|$

$D = a \left[|\coth U_1| \pm |\coth U_2| \right]$ ← "+" CYLINDERS SEPERATE, "−" INSIDE EACH OTHER

MAY FIND CAPACITANCE FROM $(U_2 - U_1)$

$$C = 2\pi\epsilon \left[\cosh^{-1}\left(\pm \frac{D^2 - R_1^2 - R_2^2}{2R_1R_2} \right) \right]^{-1}$$

(CAPACITANCE PER UNIT LENGTH)



TRASFORMATION USED WAS

$W = \ln \frac{z + ja}{z - ja}$ ← USE FOR POINT CHARGES

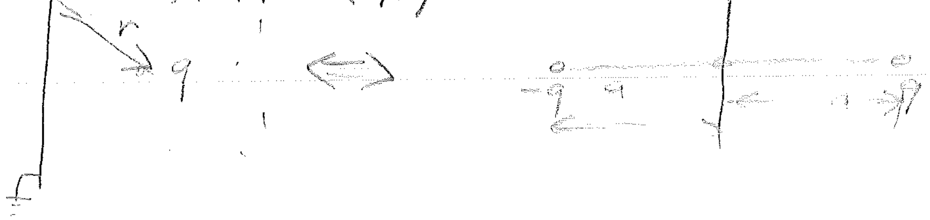
(POWERFUL WHEN USED WITH METHOD OF IMAGES)

3/2/76 (TUES)

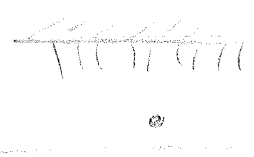
CHAPT. 5

5.04. METHOD OF IMAGES

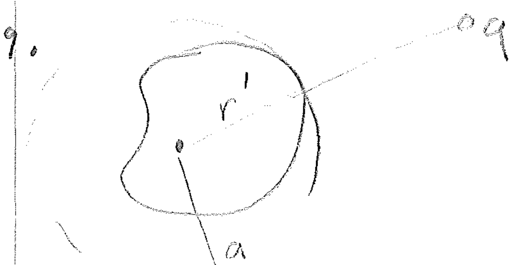
$\sigma = -q/2\pi r^2 \left(\frac{a}{r}\right)$



5.07. $q'' = \frac{2q}{K+1}$ $\qquad q = \frac{a}{b} q$



5.09.

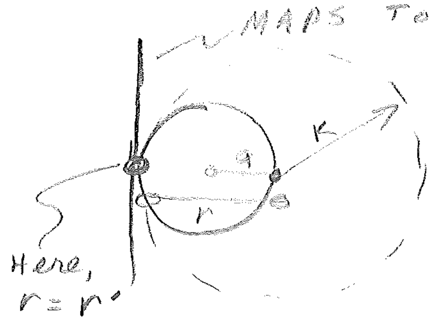


$r' = a^2/r$

$r' r = K^2$

$$F(r,\theta,\phi) \rightarrow F\left(\frac{K^2}{r},\theta,\phi\right)$$

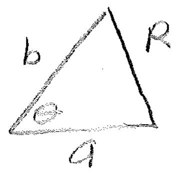
MAPS TO LINE



$K = 2a.$

$r' = \frac{K^2}{r} = \frac{4a^2}{r} = \frac{4a^2}{2a} = 2a$

3-9-76 (TUES)



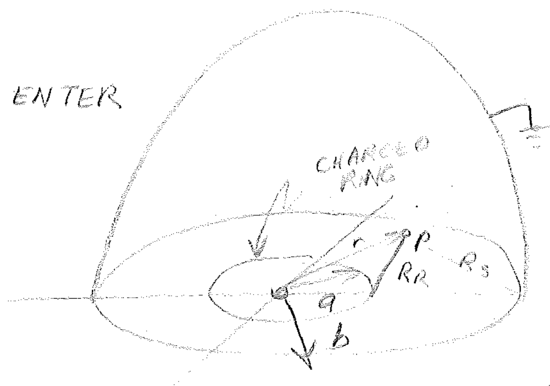
$$\frac{1}{R} = (a^2 + b^2 - 2ab\cos\theta)^{-\frac{1}{2}}$$

$\mu = \cos\theta$

FOR $b > a$, $\frac{1}{R} = \frac{1}{b}\left[1 + \mu\left(\frac{a}{b}\right) + \frac{3\mu^2}{2}\left(\frac{a}{b}\right)^2 + \dots\right]$

$$= \frac{1}{b}\left[P_0(\mu) + \frac{a}{b}P_1(\mu) + \left(\frac{a}{b}\right)^2 P_2(\mu) + \dots\right]$$

5.17. ENTER



WE CAN WRITE: $V = \sum_{n=1}^{\infty} \underbrace{\left(A_n r^n + B_n \frac{1}{r^{n+1}}\right)}_{R(r)} P_n(\mu)$

FOR RING

$$V_r = \frac{Q}{2\pi\epsilon a} \sum_{n=0}^{\infty} \left(\frac{a}{r}\right)^{2n+1}$$

$$\times P_{2n}(0) P_{2n}(\mu) \quad ; a < r$$

FOR SPHERE: $B_n = 0 \Rightarrow V_S = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$

$(r > a)$

$V = V_r + V_S$

$V(a) = 0$ TO FIND A_n

(ALL ODD VALUES DROP OUT)

$V_r(a) = -V_S(a)$

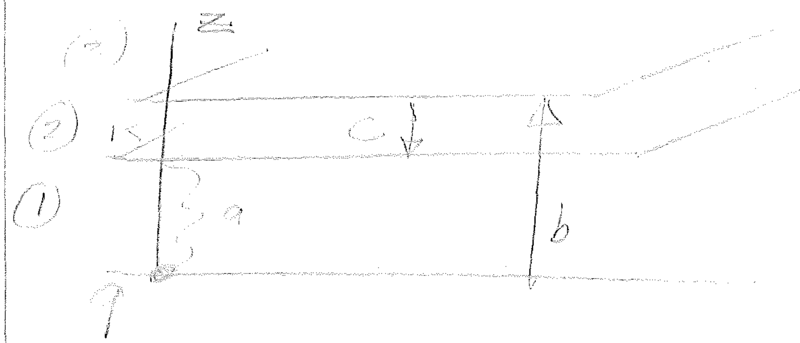
GIVES $A_{2n} b^{2n} = -\frac{Q}{4\pi\epsilon a}\left(\frac{a}{b}\right)^{2n+1} \cdot P_{2n}(0)$

$A_{2n+1} = 0$

RETURN

3 11 76 (THURS)
DO PROBS 60 & 61 IN CHAPT. 5

3-16-76 (TUES)
5.303



WITHOUT DIELECTRIC

$$V = \frac{q}{4\pi\epsilon_0 r} = \frac{q}{4\pi\epsilon_0}\int_0^\infty J_0(k\rho)\, e^{-k|z|}\, dk$$

FOR

$-\infty < z < a$ ← SO NO BLOW UP

WILL SOLVE LAPLACE'S EQN. ⇒ $V_1 = V + \frac{q}{4\pi\epsilon_0}\int_0^\infty \phi_1(k) J_0(k\rho)\, e^{+kz}\, dk$

FOR $a < z < b$

$$V_2 = \frac{q}{4\pi\epsilon}\int_0^\infty \psi(k) J_0(k\rho) \ldots \text{ETC.}$$

(TEST NXT. TUES

4-6-95

(1) $\nabla^2 W_1 = 0 \qquad \nabla^2 W_2 = 0$

$$B = \nabla \times A$$
$$= \nabla \times (\nabla \times W)$$
$$= \nabla \times [\nabla \times (\bar{U} W_1 + U \times \nabla W_2)]$$
$$= \nabla \times [\nabla \times \bar{U} W_1] + \nabla \times \nabla \times (U \times \nabla W^2)$$
$$\nabla \times (\nabla \times \bar{U} W_1) = -\nabla \times (\bar{r} \times \bar{\nabla} W_1)$$
$$= r \frac{\delta}{r} (\nabla W_1) + 2 \bar{\nabla} W_1$$
$$= \bar{U} \nabla (\nabla W_1)$$

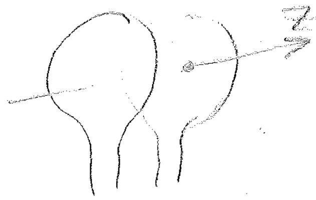
(2) $\bar{\nabla} \times (\nabla \times (\bar{U} \times \bar{\nabla} W_2))$
$$= \nabla \times [\bar{U} \nabla^2 W_2) - \bar{\nabla}(\bar{U} \cdot \bar{\nabla} W_2) - \bar{\nabla} W_2] = 0$$

ASSUME $\bar{W} = k W_1$ (IN z DIRECTION)
$$\nabla^2 W_1 = 0$$
FIND $\vec{A}$
$$\vec{A} = \bar{\nabla} \times \vec{W} = \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ 0 & 0 & W_1 \end{vmatrix}$$

$$= \hat{i} \frac{\delta}{\delta y} W_1 - \vec{j} \frac{\delta}{\delta x} W_1 + \vec{k}\, 0$$

THIS WILL BE USEFUL FOR BOUNDARY CONDITIONS. NOTE PERPENDICULAR COMPONENTS OF $\vec{A}$

7.07



SOL WAS (FROM LASTIAF)

$$A = \frac{\mu_0}{4\pi}\int_V \frac{J\,dV}{r} = \frac{\mu_0}{4\pi}\oint \frac{I\,dl}{r}$$

A WILL ONLY HAVE ϕ component

FOR $\nabla^2 W_1 = 0$

$$W_1 = \frac{1}{2\pi}\int_0^{2\pi}\phi(z + j\rho\sin\theta)\,d\theta$$

$$\nabla^2 W_1 = \frac{1}{\rho}\frac{\delta}{\delta\rho}\left(\rho\frac{\delta W_1}{\delta\rho}\right) + \frac{1}{\rho^2}\frac{\delta^2}{\delta\phi^2} + \frac{\delta^2}{\delta z^2}W_1 = 0$$

NOW $\phi(z)$ IS A REAL FUNCTION OF Z

$$W_1 = k W_2 \quad W_1/\delta\rho$$

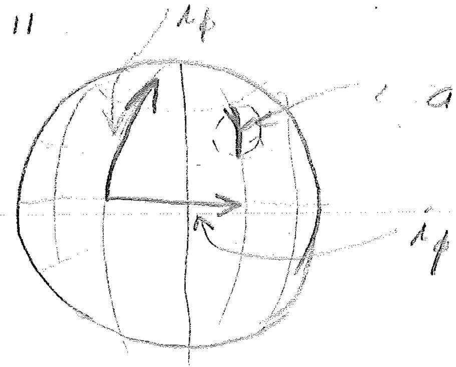
$$A_\phi = \frac{-1}{2\pi}\int_0^{2\pi}\psi(z + j\sin\theta)(\sin\theta)\,d\theta$$

$$A_\phi(\rho, z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+1)!}\frac{\delta^{2n+1}}{\delta z^{2n+1}}\left(\frac{\rho}{2}\right)^{2n+1}$$

$$B_z\big|_{\rho=0} = B_\rho(z) = \frac{\delta\psi(z)}{\delta z}$$

4-8-76 (THURS)

7.11



STREAM FUNCTION $\psi_n^m = S_n^m(\theta, \phi)$

FROM CHAPT 6: $i = \frac{1}{F}\left|\frac{dw}{dz}\right|$

IN SPHERICAL COORDINATES

$$i_\theta = \frac{1}{a \sin\theta}\frac{\delta\psi}{\delta\phi}$$

$$i_\phi = \frac{1}{a}\frac{\partial\psi}{\partial\theta}$$

APPLY AMPERE'S LAW (HOLD ϕ CONST)

$$\oint \bar{B}\, d\bar{s} = \mu I$$

$$I = a\Delta\theta\, i, \quad B_{\theta_{OUTSIDE}} - B_{\theta_{INSIDE}} = \vec{B}_\theta$$

$$\Rightarrow (B_{\theta_o} - B_{\theta_i})\, a\Delta\theta = \mu\, a\Delta\theta\, i$$

LET $W = rW_1$

GIVES $B_\theta = \frac{1}{r}\frac{\delta^2(rW_1)}{\delta r\,\delta\theta}$

$$\mu\frac{\delta\psi}{\delta\theta} = \frac{\delta^2(rW_{1o})}{\delta r\,\delta\theta} - \frac{\delta^2(rW_{1i})}{\delta r\,\delta\theta}$$

WHEN WE HOLD θ CONSTANT, WE GET

$$-\frac{\mu}{\cancel{\sin\theta}}\frac{\delta\psi}{\delta\phi} = -\frac{1}{\cancel{\sin\theta}}\frac{\delta^2(rW_{1o})}{\delta r\,\delta\phi} + \frac{1}{\cancel{\sin\theta}}\frac{\delta^2(rW_{1i})}{\delta r\,\delta\phi}$$

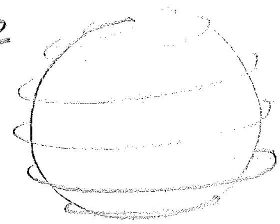
$$\Rightarrow \mu\psi = \frac{\delta}{\delta r}(rW_{1o} - rW_{1i})$$

BOUNDARY CONDITIONS : $r = a \Rightarrow W_{1o}(a) = W_{1i}(a)$

$$\Rightarrow \mu\psi = a\left(\frac{\delta\psi_{1o}}{\delta r} - \frac{\delta\psi_{1i}}{\delta r}\right)$$

$$\nabla^2 W_1 = 0$$

7.12



← PARALLEL WIRES

ANSWER NOT DEPEND ON ϕ.

WE HAVE

$$\psi(\theta) = \sum_{n=1}^{\infty} C_n P_n(\cos\theta)$$

WE KNOW $\vec{A} = [0, 0, A_\phi]$

SINCE, FROM AMPERE'S LAW, $\vec{i}$ IS

IS SAME DIRECTION AS $\vec{A}$.

$$i_\phi = \frac{1}{a}\frac{\delta \psi}{\delta \theta}$$

$$= -\sum \frac{C_n}{a} \frac{\delta P_n(\cos\theta)}{\delta(\cos\theta)} \frac{d(\cos\theta)}{\delta\theta}$$

$$= -\sum \frac{C_n}{a} \sin\theta \frac{\delta P_n(\cos\theta)}{\delta(\cos\theta)}$$

$$= \sum_n \frac{C_n}{a} P_n'(\cos\theta)$$

CHOOSE

$$W_{10} = \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n-1}) P_n(\cos\theta)$$

THIS GIVES, (ie)

$$W_{10} = \sum_n \frac{\mu}{2n+1} C_n \left(\frac{a}{r}\right)^{n+1} P_n(\cos\theta)$$

$$W_{1i} = \sum \frac{\mu}{2n+1} \left(\frac{r}{a}\right)^n P_n(\cos\theta)$$

NOW: $\bar{A} = [0, 0, A_\phi]$

$\bar{B} = [B_r, B_\theta, 0]$

$$B_{ri} = \frac{-\delta^2(rW_{1i})}{\delta r^2} = \frac{-\mu}{a} \sum_n \frac{n(n+1)}{2n+1} C_n \left(\frac{r}{a}\right)^{n-1} P_n(\mu)$$

$$B_{\theta i} = -\frac{1}{r}\frac{\delta}{\delta r}(rA_\phi) = \frac{\mu}{a} \sum_{n=1} \frac{n+1}{2n+1} \left[C_n \left(\frac{r}{a}\right)^{n-1}\right] P_n'(\mu)$$

(n+1)

$\mu=\cos\theta$

THUS, WE HAVE

$$C_n = \frac{(2n+1)}{2n(n+1)} \underbrace{I \sin\alpha \, P_n'(\cos\alpha)}$$

$$\int_{-1}^{1} (\ '' \ P_n'(\cos\alpha)\, \underbrace{k \sin\theta \, d\theta}_{= I}^{ds}$$

THUS,

$$A_{\phi_i} = \phi \frac{\mu I}{2} \sum \frac{\sin\alpha}{n(n+1)} \left(\frac{r}{a}\right)^n P_n'(\cos\alpha) P_n'(\cos\theta)$$

STEPS

DETERMINE B.C. WITH STREAM FUNCTIONS

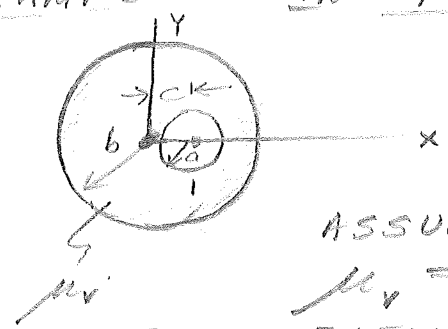
STREAM VIA SPHERICAL HARMONICS

CHOOSE W. REPRESENTATIVE OF SOLN.

EXAMPLES IN TWO DIMENSIONS

FIND $\vec{B}$ IN DA HOLE

7.16.



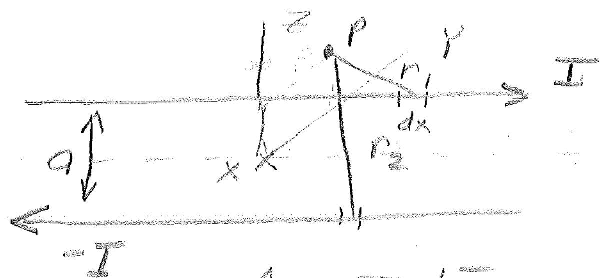
ASSUME UNIFORM CURRENT i

μ_v = CONSTANT

1. FIND FIELD FOR
2. " " "
3. SUBTRACT CURRENTS TO GIVE ZERO CURRENT IN ~~b~~

CONT →

EX.



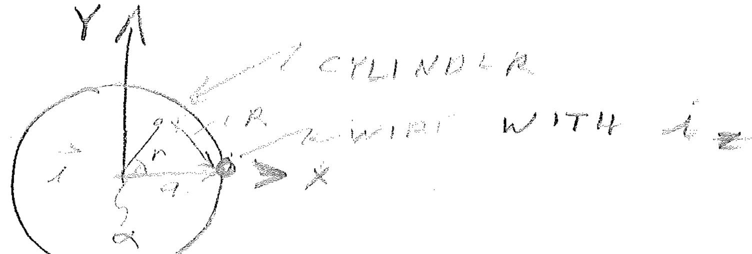
$$\bar{A} = \frac{\mu}{4\pi} \oint \frac{I\, d\bar{s}}{r}$$

$$A_x = \frac{\mu I}{4\pi}\left[\int_{\infty}^{-\infty} \frac{dx}{r_1} - \int_{-\infty}^{\infty} \frac{dx}{r_2}\right]$$

$$= \frac{\mu I}{2\pi} \int_0^{\infty} \frac{dx}{\sqrt{a^2+x^2}} - \int_{-\infty}^{\infty} \frac{dx}{(\quad)}$$

$$= \frac{\mu I}{2\pi} \ln a_2/a_1$$

TRY A COUPLE PROBLEMS FROM DA CHAPTER



ASSUME <u>ONLY</u> i_z COMPONENTS $[0, 0, i_z)$

FIND POTENTIAL @ POINT P

$$A_z = \frac{\mu I}{2\pi} \ln R = \mu i a \ln R$$

INTEGRATE AROUND a FOR MANY WIRES

$$A_z = \mu a \int i \ln R \, d\alpha$$

$$= \frac{\mu a}{2\pi} \int_0^{2\pi} i \ln R \, d\alpha$$

WE CAN EXPRESS i_z AS

$$i_z = \sum_{n=0}^{\infty} \left[C_n \cos n\alpha + D_n \sin n\alpha \right]$$

AND

$$\ln R = \ln a - \sum \frac{1}{m} \left(\frac{r}{a}\right)^m \left[\cos m\theta \cos m\alpha + \sin m\theta \sin m\alpha \right]$$

PUTTING IT TOGETHER:

$$A_z = \mu a C_0 \ln a + \frac{\mu a}{2} \sum \frac{1}{n} \left(\frac{r}{a}\right)^n \times \left[C_n \cos n\theta + D_n \sin n\theta \right]$$

IF WE HAVE UNIFORM i_z (IT DOES NOT DEPEND ON α), THEN $C_n = 0$

$A_z = \mu a i \ln a \Leftarrow$ CONSTANT <u>EVERYWHERE</u>

$\Rightarrow B = 0$

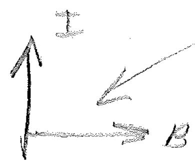
$$\vec{\nabla}(d\vec{l}\cdot\vec{A}) = \cancel{(\vec{A}\cdot\vec{\nabla})d\vec{l}}^{\,0} + \vec{A}\times\cancel{(\nabla\times d\vec{l})}^{\,0}$$
$$+ (d\vec{l}\cdot\nabla)\vec{A} + d\vec{l}\times(\nabla\times\vec{A})$$

THUS, WE CAN REWRITE THE PREVIOUS FORCE AS

$$F = I\oint\left[\cancel{(dl\cdot\nabla)\vec{A}}^{\,0} + dl\times(\nabla\times A)\right]$$

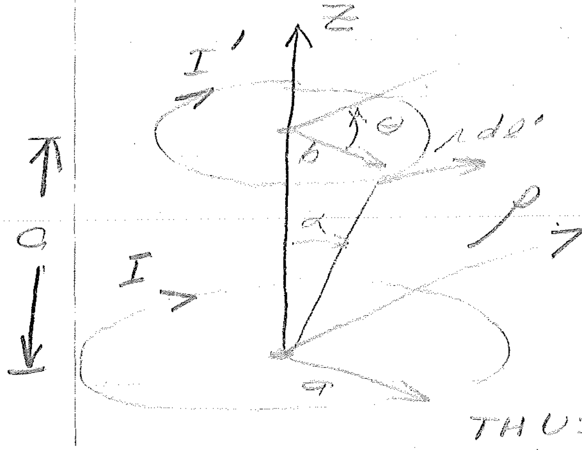
LEAVING

$$\underline{F = I\oint_C d\vec{l}\times\vec{B}}$$



FORCE $\perp$ TO THESE VECTORS

A SECOND SIMPLE CASE (7.19)



$$\vec{F} = \frac{\mu I I'}{2\pi a^2} \oint d\vec{l}' \times [\vec{l} \times \vec{a}]$$

$$\vec{F} = I' \oint d\vec{l} \times \vec{B}$$

DUE TO SYMMETRY, FORCE WILL ONLY BE IN Z DIRECTION, THUS, WE NEED ONLY TO LOOK @ B_ρ.

$$\Rightarrow F = I' \oint dl \, B_\rho(a, b, c)$$

$$= I' B_\rho(a, b, c) \int_0^{2\pi} b \, d\theta$$

$$= 2\pi I' B_\rho(a, b, c)$$

ELLIPTIC INTEGRAL

NOW

$$B_\rho = -\frac{\delta A_\phi}{\delta z} = \frac{\mu I}{2\pi} \frac{c}{\rho} \frac{\left[-K(m) + \frac{a^2+b^2+c^2}{(a-b)^2+c^2} E(m)\right]}{[(a^2+b)^2+c^2)]^{1/2}}$$

WHERE $m(a, b, c)$

4-15-76 (THURS)

$(\mu - \mu_v)\,\vec{i} = \mu_v(\nabla \times \vec{M})$

7-21. $M = B\left(\frac{1}{\mu_v} - \frac{1}{\mu}\right)$

$\oint B \cdot dl = \mu I$

$\oint \frac{\bar{B}}{\mu} \cdot d\bar{l} = I$

$$\vec{A} = \frac{\mu_v}{4\pi}\int_V \frac{\vec{i} + \nabla \times \vec{M}}{r}\,dv + \frac{\mu_v}{4\pi}\int_S \frac{\bar{M} \times \bar{n}}{r}\,ds$$

$$V = \frac{1}{4\pi\epsilon_v}\left[\int_V \frac{\rho}{r}\,dv + \int \frac{\sigma}{r}\,ds\right]$$

USE SAME BOUNDRY CONDITIONS AS ELEC

$V_i = V_o$

$\frac{\delta V_o}{\delta n} - \frac{\delta V_i}{\delta n} = \frac{\sigma}{\epsilon_v}$

SO FOR VECTOR $\vec{A}$

$\bar{A}_o = \bar{A}_i$

$\mu_v\,(\bar{M}\,n)$

$\frac{\delta \bar{A}_o}{\delta n} - \frac{\delta \bar{A}_i}{\delta n} = \mu_v(\bar{M} \times \bar{n})$

OR, EQUIVALENTLY,

SINCE $\bar{M} = \bar{B}\left(\frac{1}{\mu_v} - \frac{1}{\mu}\right)$

$\bar{B} = \nabla \times \vec{A}$

$\frac{\delta A_o}{\delta n} - \frac{\delta A_i}{\delta n} = \frac{(\mu - \mu_v)}{\mu}\left[\nabla \times \vec{A}_i \times \bar{n}\right]$

DUALS

$$\frac{dW}{dV}\Big|_B = \frac{B^2}{2\mu} \qquad \frac{\delta W}{\delta V}\Big|_E = \frac{D^2}{2\varepsilon}$$

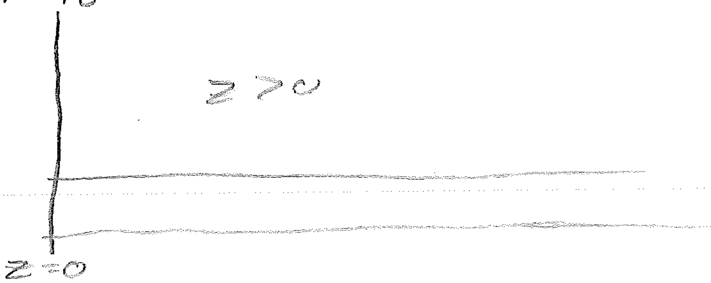
$$\nabla \cdot B = \nabla \cdot D = 0$$

$$F = -\frac{\delta W}{\delta \xi} \qquad F = +\frac{\delta W}{\delta \xi}$$

$$M = \left(\frac{1}{\mu_0} - \frac{1}{\mu}\right)\vec{B} \qquad w_B = \frac{1}{2}\int M \cdot B\, dV$$

4-29-76 (THURS)

10.16



$t = 0$

$A_1' = f_1(x, y, z) \qquad t < 0$

$A_2' = f_2(x, y, z) \qquad t > 0$

$A\big|_{t=0} = A_1' - A_2'$

FOR $t > 0$

$\dfrac{dA}{dt} = \dfrac{2s}{\mu v} \dfrac{\delta A}{\delta z}$ $\Rightarrow$ BOUNDRY CONDITION:

$A = f_1(x, y, -|z|)$ . . .

(Eq. 3. IN 10.10)

GREENS FUNCTIONS (FROM HAGLER LECTURE)

$\nabla^2 \psi(\vec{x}';t') - \frac{\delta^2}{\delta t^2}\psi(\vec{x}';t') = -4\pi f(\vec{x}',t) \Leftarrow$ DRIVEN WAVE EQ.

$\Box_{x'} = \nabla_{x'}^2 - \frac{\delta^2}{\delta t^2}$ = D'ALAMBERTIAN

$G = G(\vec{x}',t';\vec{x},t)$ = GREENS FUNCTION

$\Box_{x'} G = -4\pi\, \delta^3(\vec{x}-\vec{x}')\,\delta(t-t')$

$\begin{cases} \vec{x}',\ t' \Rightarrow \text{SOURCE COMPONENTS} \\ \vec{x},\ t \Rightarrow \text{OBSERVER COMPONENTS} \end{cases}$

IN A HOMOGENEOUS ISOTROPIC MEDIA:

$$G(\vec{x}',t';\vec{x},t) = G(\vec{x}'-\vec{x}\ ;\ t'-t)$$

CONSIDER:

$$\int dt' \int d^3x' \left[G\,\Box_{x'}\psi - \psi\,\Box_{x'} G\right]$$
$$= -4\pi \int dt' \int_V d^3x' G f + 4\pi \int dt' \int_V d^3x' \psi(\vec{x}';t')\delta^3(\vec{x}-\vec{x}')\delta(t-t')$$
$$= -4\pi \int dt' \int_V d^3x' G f + 4\pi\, \psi(\vec{x},t)$$

REARRANGE:

$$\psi(\vec{x},t) = \int dt' \int d^3x' G f + \frac{1}{4\pi}\int dt' \int_V d^3x' (G\Box_{x'}\psi - \psi\Box_{x'} G)$$
$$= \int dt' \int d^3x' G f$$
$$+ \frac{1}{4\pi}\int dt' \int_V d^3x' \left[G\nabla_{x'}^2\psi - \psi\nabla_{x'}^2 G + \frac{G}{c^2}\frac{\delta^2}{\delta t'^2}\psi - \frac{\psi}{c^2}\frac{\delta^2}{\delta t'^2}G\right]$$

BUT $\quad G\nabla_{x'}^2\psi - \psi\nabla_{x'}^2 G = \vec{\nabla}_{x'}\cdot(G\nabla_{x'}\psi - \psi\nabla_{x'} G)$

THRU THE DIVERGENCE THEOREM

$$\int_V \vec{\nabla}\cdot\vec{M}\, d^3x = \int_S \vec{n}\cdot\vec{M}\, dS$$

$$\Rightarrow \psi(\vec{x},t) = \int dt' \int d^3x' G f$$
$$+ \frac{1}{4\pi}\int dt' \int_S ds'\ \vec{n}\cdot\left[G\nabla_{x'}\psi - \psi\nabla_{x'} G\right]$$
$$+ \frac{1}{4\pi}\int_{V'} d^3x \left(-\frac{1}{c^2}\right)\left[G\frac{\delta}{\delta t}\psi - \psi\frac{\delta}{\delta t}G\right]_{t_1'}^{t_2'}$$

THESE TERMS PHYSICALLY CORRESPOND TO:

1) CONTRIBUTIONS FROM SOURCES WITHIN V'

2) CONTRIBUTIONS EXTERNAL TO V'

3) DUE TO FINITE DELAY

TO FIND G:

$$G = G(\vec{x}' - \vec{x}\,;\, t' - t) = G(\vec{\xi}\,;\, \tau)$$

$$\Box_{\vec{\xi}'} G = -4\pi\, \delta^3(\vec{\xi})\, \delta(\tau)$$

$$= \frac{-4\pi}{(2\pi)^4} \int d\omega \int d^3k\, e^{i(\vec{k}\cdot\vec{\xi} - \omega\tau)}$$

DEFINE $g(\vec{k}, \omega)$ AS THE FOURIER XFORM OF G:

$$G = \int d\omega \int d^3k\, g(\vec{k}, \omega)\, e^{i(\vec{k}\cdot\vec{\xi} - \omega\tau)}$$

THEN

$$\Box_{\xi'} G = \int d\omega \int d^3k\, g(\vec{k}, \omega) \left[\Box_{\vec{\xi}'}\, e^{i(\vec{k}\cdot\vec{\xi} - \omega\tau)}\right]$$

$$= \int d\omega \int d^3k\, g(\vec{k}, \omega) \left[\frac{\omega^2}{c^2} - k^2\right] e^{i(\vec{k}\cdot\vec{\xi} - \omega\tau)}$$

EQUATING WITH PREVIOUS EXPRESSION FOR $\Box_{\xi} G$ GIVES

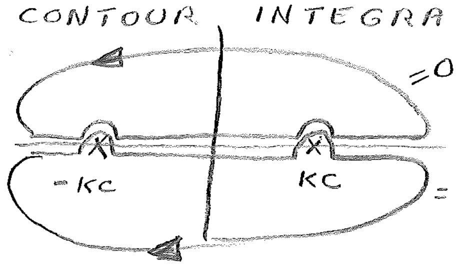
$$\int d\omega \int d^3k \left[\left\{\left(\frac{\omega}{c}\right)^2 - k^2\right\} g(\vec{k}, \omega) + \frac{1}{4\pi^3}\right] e^{i(\vec{k}\cdot\vec{\xi} - \omega\tau)} = 0$$

THUS: $\quad g(\vec{k}, \omega) = -\left[4\pi^3\left(\frac{\omega^2}{c^2} - k^2\right)\right]^{-1}$

REGAIN G BY AN INVERSE FOURIER XFORM:

$$G(\vec{\xi}, \tau) = \frac{-c^2}{4\pi^3} \int d^3k \int d\omega\, \frac{e^{i(\vec{k}\cdot\vec{\xi} - \omega\tau)}}{(\omega - kc)(\omega + kc)}$$

USE CONTOUR INTEGRATION:



$= 0$ ONLY FOR $\tau < 0$

$= 0$ ONLY FOR $\tau > 0$

POLES ARE DIVIED TO YIELD CAUSAL "RETARDED" GREENS FUNC.

GIVES, FOR $\tau > 0$:

$$G_R(\vec{\xi}, \tau) = \frac{-c^2}{4\pi^3} \int d^3k\, e^{i\vec{k}\cdot\vec{\xi}} (-i2\pi) \left[\left.\frac{e^{-i\omega\tau}}{\omega + kc}\right|_{\omega = kc} + \left.\frac{e^{-i\omega\tau}}{\omega - kc}\right|_{\omega = -kc} \right]$$

$$= \frac{c}{2\pi^2} \int d^3k\, e^{i\vec{k}\cdot\vec{\xi}}\, \frac{\sin ck\tau}{k} \quad ; \tau > 0$$

TO EVALUATE THIS, USE SPHERICAL COORDINATES IN k SPACE

$$G_R(\vec{\xi},\tau) = \frac{c}{2\pi^2}\int_0^\infty K^2 dK \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi\, e^{ik|\vec{\xi}|\cos\theta} \frac{\sin ck\tau}{K}$$

$$= \frac{c}{\pi}\int_0^\infty K\, dK\, e^{ik|\vec{\xi}|\cos\theta}\Big|_0^\pi \left(\frac{-1}{ik|\vec{\xi}|}\right)\sin ck\tau$$

$$= \frac{2c}{\pi|\vec{\xi}|}\int_0^\infty dK \sin k|\vec{\xi}| \sin ck\tau$$

SINCE THE INTEGRAND IS EVEN

$$G_R(\vec{\xi},\tau) = \frac{c}{\pi|\vec{\xi}|}\int_{-\infty}^{\infty} dK \sin k|\vec{\xi}| \sin ck\tau$$

USING TRIG. IDENTITY

$$G_R(\vec{\xi},\tau) = \frac{c}{2\pi|\vec{\xi}|}\int_{-\infty}^{\infty} dK \left[\cos\{k(|\vec{\xi}| - c\tau\} - \cos\{k|\vec{\xi}) + c\tau\}\right]$$

WE MAY ADD TO THE INTEGRAND $\sin$ FUNCTIONS, WHICH ARE EVEN AND WILL INTEGRATE TO 0. THIS GIVES

$$G_R(\vec{\xi},\tau) = \frac{c}{2\pi|\vec{\xi}|}\int_{-\infty}^{\infty} dK \left[e^{ik(|\vec{\xi}| - c\tau)} - e^{ik(|\vec{\xi}| + c\tau)}\right]$$

$$= \frac{c(2\pi)}{2\pi|\vec{\xi}|}\left[\delta\{|\vec{\xi}| - c\tau\} - \delta\{|\vec{\xi}| + c\tau\}\right]$$

SINCE $\tau > 0$, $\delta\{|\vec{\xi}| + c\tau\} = 0$ AND

$$G_R(\vec{\xi},\tau) = \frac{c}{|\vec{\xi}|}\,\delta\left[|\vec{\xi}| - c\tau\right] \quad \text{FOR } \underline{\text{ALL}}\ \tau$$

$$= \frac{1}{|\vec{\xi}|}\,\delta\left[\frac{|\vec{\xi}|}{c} - \tau\right]$$

$$= \frac{\delta\left[\frac{|\vec{x}' - \vec{x}|}{c} - (t - t')\right]}{|\vec{x}' - \vec{x}|} \Leftarrow \text{GREENS FUNCTION}$$

PLUG INTO FIRST TERM ON BOTTOM OF pg 1:

THIS IS THE STEADY STATE TERM:

$$\psi_s(\vec{x},t) = \int_{v'} d^3x' \int dt\, f(\vec{x}',t')\, |\vec{x} - \vec{x}'|^{-1}\, \delta\left[\frac{|\vec{x} - \vec{x}'|}{c} - (t - t')\right]$$

$$= \int_v d^3x' \frac{f(\vec{x}', t'_{RET})}{|\vec{x}' - \vec{x}|}$$

$$t'_{RET} = t - \frac{|\vec{x} - \vec{x}'|}{c} = \text{TIME FROM SOURCE TO OBSERVER}$$

ADVANCED FIELDS I TEST # 1 CRAM SHEET

I. BASIC IDEAS

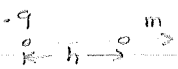
$\vec{F} = \frac{q_1 q_2}{4\pi\epsilon r^3}\vec{r}$ ⇐ COULOMB'S LAW

$\vec{E} = \frac{-1}{4\pi\epsilon}\sum_{i=1}^{n}\frac{q_i}{r_i^3}\vec{r}_i$ ⇐ ELECTRIC FIELD INTENSITY

$V_p = \frac{1}{4\pi\epsilon}\sum_{i=1}^{n}\frac{q_i}{r_i}$ ⇐ ELECTROSTATIC POTENTIAL

$\vec{E} = -\nabla V$

$V_p = \frac{1}{4\pi\epsilon}\int_V \frac{\rho\, dv}{r} + \frac{1}{4\pi\epsilon}\int_S \frac{\sigma\, ds}{r}$

$\vec{m} = q\,h$  ⇐ DIPOLE MOMENT

$V = \frac{\vec{m}\cdot\vec{r}}{4\pi\epsilon r^3}$ ⇐ ELECTROSTATIC POTENTIAL FROM DIPOLE MOMENT

$q = \int_S \epsilon\vec{E}\cdot\vec{n}\, ds$ ⇐ GAUSS' FLUX THEOREM

$\psi(E) = -\frac{\epsilon E^2}{2}$ ⇐ TENSION IN (HOMO/ISO) MEDIUM TWIXT FORCE LINES

$\phi(E) = \frac{\epsilon E^2}{2}$ ⇐ FORCE TWIXT FORCE LINES IN (HOMO/ISO) MEDIUM

ON A CONDUCTOR, $\sigma = \vec{D} = \epsilon\vec{E}$ = SURFACE CHARGE DENSITY

II. CAPACITORS, DIELECTRIC, SYSTEMS OF CONDUCTORS

$C = \frac{Q}{V}$ = CAPACITANCE

$= \frac{4\pi\epsilon ab}{b-a}$ FOR TWO CONCENTRIC SPHERES

$= 4\pi\epsilon a$ FOR A SINGLE SPHERE

$= \frac{2\pi\epsilon}{\ln(b/a)}$ FOR TWO CONCENTRIC CYLINDERS

$= \frac{\epsilon A}{a}$ FOR TWO PARALLEL PLATES

$W = \frac{1}{2}\sum q_i V_i$ ⇐ ENERGY IN CHARGED CAPACITOR $= \frac{1}{2}CV^2$

$\sum Q_i' V_i \neq \sum Q_i V_i'$ ⇐ GREEN'S RECIPROCATION THEM.

$Q_i \to V_i \Rightarrow Q_i + Q_i' \Rightarrow V_i + V_i' \Rightarrow$ FIELD SUPERPOSITION

$Q = -\frac{V_p'}{V'}q \Rightarrow$ CHARGE, Q, INDUCED BY PT. CHARGE q (2.14)

$\vec{V} = \overleftrightarrow{S}\vec{Q} \ni S_{sr}$ = POT. TO WHICH r IS RAISED/PLACING 1 COUL @ S = BLASTANCE

$\vec{Q} = \overleftrightarrow{C}\vec{V} \ni C_{sr}$ = CAPACITANCE

III. GENERAL THEOREMS

$$\oint_S \vec{A}\cdot\vec{n}\,ds = \int_V \vec{\nabla}\cdot\vec{A}\,dv \Leftarrow \text{GAUSS' THEM}$$

$$\oint_\ell \vec{F}\cdot d\vec{\ell} = \int_S \vec{n}\cdot\vec{\nabla}\times\vec{F} \Leftarrow \text{STOKES THEM}$$

$$\vec{\nabla}\cdot\epsilon\vec{\nabla}V = -\rho \Leftarrow \text{POISSONS Eq}$$

$$\vec{\nabla}\cdot\epsilon\vec{\nabla}V = 0 \Leftarrow \text{LAPLACE'S Eq.}$$

IV. TWO DIMENSIONAL POTENTIAL DISTRIBUTIONS

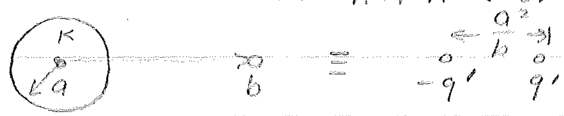
$$E = \frac{q}{2\pi\epsilon r} \qquad V = \frac{-q}{2\pi\epsilon}\ln r + c$$

$$V = R(\rho)\,\Phi(\phi)\,Z(z) = R(r)\,\Theta(\theta)$$

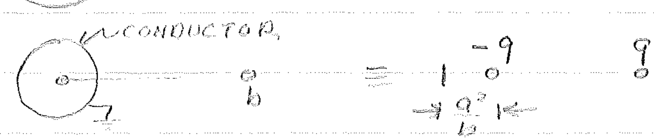
$$\Theta_n = \begin{cases} A\cos n\theta + B\sin n\theta & n\neq 0 \\ A\theta + B & n = 0 \end{cases} \Leftarrow \text{CIRCULAR HARMONICS}$$

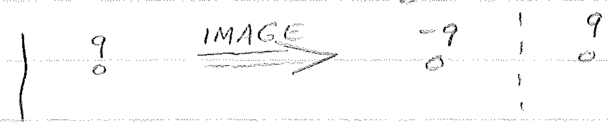
$$R_n = \begin{cases} Cr^n + Dr^{-n} & n\neq 0 \\ C\ln r + D & n = 0 \end{cases}$$

$$V_p = \begin{cases} \dfrac{q}{4\pi\epsilon}\sum_{n=1}\dfrac{1}{n}\left(\dfrac{r_0}{r}\right)^n(\cos n\theta_0\cos n\theta + \sin n\theta_0\sin n\theta) & r_0<r \\ \dfrac{q}{4\pi\epsilon}\sum_{n=1}\dfrac{1}{n}\left(\dfrac{r}{r_0}\right)^n(\cos n\theta_0\cos n\theta + \sin n\theta_0\sin n\theta) & r_0>r \end{cases}$$



$$q' = \frac{1-K}{1+K}\,q \qquad q'' = \frac{2}{1+K}\,q$$





$$W = U + jV$$

$-\frac{\delta W}{\delta z}$ GIVES X & Y COMPONENTS OF $\vec{E}$

GAUSS' FLUX THEM (V IS POTENTIAL)

$$Q = \text{FLUX} = -\epsilon\oint\vec{E}\cdot d\vec{n} = -\epsilon\int_{U_1}^{U_2}\frac{dV}{dn}\,ds = \epsilon\int_{U_1}^{U_2}\frac{dU}{ds}\,ds = \epsilon(U_2-U_1)$$

$$C = \frac{Q}{V_2-V_1} = \frac{\epsilon[U]}{V_2-V_1}$$

V. THREE DIMENSIONAL POTENTIAL DISTRIBUTIONS

$q' = \frac{-aq}{b}$

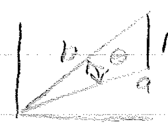
THREE DIMENSIONAL INVERSION

$rr' = K^2$ K = RADIUS OF INVERSION

$f(r, \theta, \phi) \Leftrightarrow f(\frac{K^2}{r}, \theta, \phi)$

$\frac{V'}{V} = \frac{q'}{q}\frac{r}{r'}$, $\frac{\sigma'}{\sigma} = K^3/r'^3$; q_i's @ P_i's $\Leftrightarrow q_i'$'s @ P_i''s

LEGENDRE POLY. EXPANSION OF $\frac{1}{R}$



$b > a \Rightarrow \frac{1}{R} = \frac{1}{b}\sum_{n=0}^{\infty}\left(\frac{a}{b}\right)^n P_n(\mu)$

$\mu = \cos\theta$

ADVANCED FIELDS I, TEST # 2 CRAM SHEET.

VI. ELECTRIC CURRENT

$I = \frac{dQ}{dt}$ $\quad i = \frac{dI}{dS}$

$i = \gamma \Delta E = \frac{1}{\tau} \nabla E \Leftarrow$ OHM'S LAW

$P = I^2 R \Leftarrow$ JOULE'S LAW

$RC = \gamma \epsilon$

VII. MAGNETIC INTERACTION OF CIRCUITS.

$\nabla \cdot B = 0 \quad \nabla \times B = \mu i$

$A = \frac{\mu}{4\pi} \int_S I\,ds / r \quad B = \nabla \times A$

$B = \mu I / 2\pi a \Leftarrow$ BIOT AND SAVART'S LAW

$F = I \oint d\bar{s} \times \bar{B}$

$M = \left(\frac{1}{\mu_v} - \frac{1}{\mu}\right) \vec{B} = K \vec{B} =$ MAGNETIZATION

VIII. ELECTROMAGNETIC INDUCTION

$\nabla \times E = \frac{-\delta B}{\delta t} \Leftrightarrow E = \frac{-\delta A}{\delta t} \Rightarrow$ FARADAY'S LAW

$W = \frac{B^2}{2\mu} \Rightarrow$ ENERGY DENSITY IN B FIELD

$M_{12} = \frac{\mu}{4\pi} \oint \oint d\bar{s}_1 d\bar{s}_2 / r =$ MUTUAL INDUCTANCE

$L_{11} = N_{11} / I_1 \Rightarrow$ SELF INDUCTANCE

IX. MAGNETISM

$M = KH$, $K =$ SUSCEPTABILITY

X. EDDY CURRENTS

$\nabla \times \bar{B} = \mu i \Leftarrow$ AMPERE'S LAW

XI. PLANE ELECTROMAGNETIC WAVES

MAXWELL'S EQ.

LORENTZ & COULOMB GAUGES

HERTZ VECTOR

POYNTING VECTOR

I. BASIC IDEAS OF ELECROSTATICS

1.05. ELECTRIC FIELD INTENSITY

1.06. ELECTROSTATIC POTENTIAL

1.07. ELECTRIC DIPOLES

1.071. INTERACTION OF DIPOLES

1.08. LINES OF FORCE

1.09. EQUAPOTENTIAL SURFACES

1.10. GAUSS' FLUX THEM.

1.12. POTENTIAL OF A DOUBLE LAYER

1.13. TUBES OF FORCE

1.14. STRESSES & TENSION

1.15. GAUSS' LAW FOR NON-HOMO ISOTROPIC MEDIA

1.16. BOUNDRY CONDITIONS & STRESSES ON CONDUCTOR SURFACE

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CURRENT.

I. BASIC IDEAS OF ELECTROSTATICS

1.05 ELECTRIC FIELD INTENSITY

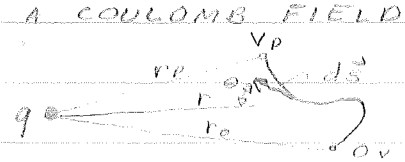
$$\vec{E}_p = \frac{-1}{4\pi\epsilon}\sum_{i=1}^{n}\frac{q_i}{r^3}\vec{r}_i$$

ϵ = CAPACITIVITY OF HOMOGENEOUS MATERIAL

1.06 ELECTROSTATIC POTENTIAL

$$dV = -\vec{E}\cdot d\vec{s}$$

- IN A COULOMB FIELD:



$$dV = \frac{-q\cos\theta}{4\pi\epsilon r^2}\,ds$$

$$V_p = \int_0^{V_p} dV$$

$$= \frac{-q}{4\pi\epsilon}\int_{r_0}^{r_p}\frac{dr}{r^2}$$

$$= \frac{q}{4\pi\epsilon}\left(\frac{1}{r_p} - \frac{1}{r_0}\right) = \frac{q}{4\pi\epsilon r_p} \quad (r_0 \to \infty)$$

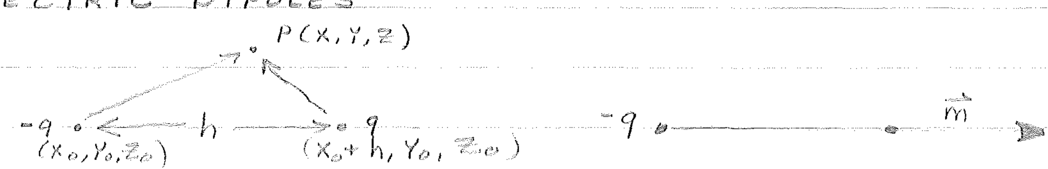
- FOR A NUMBER OF CHARGES, $V_p = \frac{1}{4\pi\epsilon}\sum_{i=1}^{n}\frac{q_i}{r_i}$

$$\vec{E} = -\vec{\nabla} V$$

- FOR CHARGE DENSITIES:

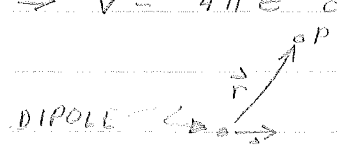
$$V = \frac{1}{4\pi\epsilon}\left[\int_S \frac{\sigma}{r}\,dS + \int_V \frac{\rho}{r}\,dv\right]$$

1.07 ELECTRIC DIPOLES



LET $q \to \infty$ AND $\vec{h} \to 0 \ni qh = |\vec{m}|$ REMAINS CONSTANT

$$\Rightarrow V = \frac{-M}{4\pi\epsilon}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\cos\theta = \frac{\vec{M}\cdot\vec{r}}{4\pi\epsilon r^3}$$



- TRANSLATIONAL FORCE = $\Sigma\vec{F}$ ACTING ON CHARGES

$$\vec{F} = (\vec{M}\cdot\vec{\nabla})\vec{E} \quad (=0 \text{ IN UNIFORM FIELD})$$

- TORQUE ON DIPOLE:

$$\vec{T} = \vec{M}\times\vec{E}$$

1.071 • INTERACTION OF DIPOLES

POTENTIAL ENERGY OF DIPOLE = WORK TO BRING CHARGES INTO PLAC

V_2 V_1 ; $-q$ q ; S ; P_1P_2

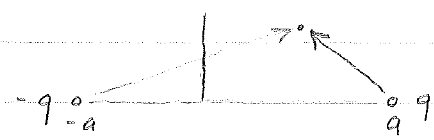
$$\Rightarrow W = q(V_1 - V_2) = q\,\overline{P_1P_2}\,\frac{\delta V}{\delta S} = |\vec{M}|\,\frac{\delta V}{\delta S} = (\vec{M}\cdot\vec{\nabla})V$$

1.08 • LINES OF FORCE

$$d\vec{s} = \lambda\vec{E} \Rightarrow dX_x = \lambda E_x\ ;\ dY = \lambda E_Y\ ;\ dz = \lambda E_z$$

$$\therefore\ \frac{dx}{E_x} = \frac{dY}{E_Y} = \frac{dz}{E_z}$$

EXAMPLE:



$$4\pi\epsilon E_x = \frac{q(x-a)}{[Y^2+(x-a)^2]^{3/2}} - \frac{q(x+a)}{[Y^2+(x+a)^2]^{3/2}}$$

$$= \frac{qV}{Y^2(1+V^2)^{3/2}} - \frac{qU}{Y^2(1+U^2)^{3/2}}\ ;\ U = \frac{x+a}{Y},\ V = \frac{x-a}{Y}$$

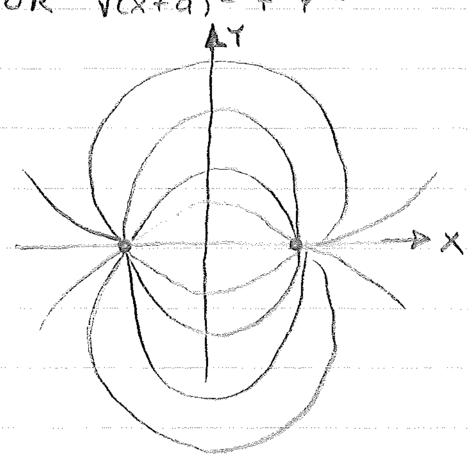
$$4\pi\epsilon E_Y = \frac{q}{Y^2(1+V^2)^{3/2}} - \frac{q}{Y^2(1+U^2)^{3/2}}$$

$$\Rightarrow \frac{dY}{dx} = \frac{E_Y}{E_x} = \frac{(1+V^2)^{3/2} - (1+U^2)^{3/2}}{U(1+V^2)^{3/2} - V(1+U^2)^{3/2}}$$

$$= \frac{dV - dU}{U\,dV - V\,dU}$$

$$\Rightarrow \frac{dU}{dV} = \left(\frac{1+U^2}{1+V^2}\right)^{3/2} \Rightarrow \frac{U}{\sqrt{1+U^2}} - \frac{V}{\sqrt{1+V^2}} = C$$

$$\text{OR}\quad \frac{x+a}{\sqrt{(x+a)^2+Y^2}} - \frac{x-a}{\sqrt{(x-a)^2+Y^2}} = C$$



1.09 EQUAPOTENTIAL SURFACES

→ V = CONST (PERPUNDICULAR TO $\vec{F}$ LINES)

IF $\vec{F}$ LINES ARE FROM $F(x,y) = C$,

AND V LINES FROM $V(x,y) = C'$,

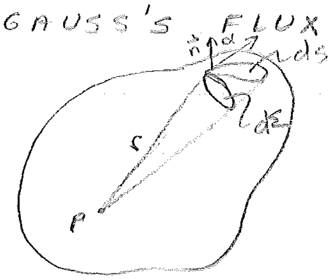
THEN $F(x,y) = |\vec{\nabla} V(x,y)|$

→ EQUALIBRIUM POINTS (LINES)

PLACE AT WHICH AN EQUAPOTENTIAL SURFACE CROSSES ITSELF AT LEAST TWICE.

∇V (AND THUS $\vec{E}$ & $\vec{F}$) VANISH THERE

1.10 GAUSS'S FLUX THEOREM



$$d\Sigma = dS \cos\alpha$$

$$E_n = \frac{q(\vec{r}\cdot\vec{n})}{4\pi\epsilon r^3} = \frac{q\cos\alpha}{4\pi\epsilon r^2}$$

$$dN = \epsilon E_n dS = \frac{q\cos\alpha\, dS}{4\pi r^2} = \frac{q\, d\Sigma}{4\pi r^2}$$

$$d\Omega = \frac{d\Sigma}{r^2} \Leftarrow \text{SOLID ANGLE}$$

$$\Rightarrow dN = \frac{q\, d\Omega}{4\pi} \Rightarrow 4\pi \int_S dN = q\int_0^{4\pi} d\Omega$$

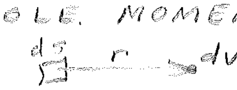
OR: $N = q$

$$\Rightarrow \epsilon \int_S E\cdot n\, ds = q$$

1.12 POTENTIAL OF ELECTRIC DOUBLE LAYER

→ ONE MAY OBTAIN THE POTENTIAL OF A DIPOLE MOMENT BY DIFFERENTIATING THE POTENTIAL OF A SINGLE CHARGE IN THE DIRECTION OF THE DIPOLE MOMENT

→ $dV = \frac{\sigma}{4\pi\epsilon r} dS$ = POTENTIAL DUE TO dS



$\therefore \frac{\sigma\, ds}{4\pi\epsilon_1} \frac{d}{dn}\left(\frac{1}{r}\right)$ = EQUIVALENT DIPOLE POTENTIAL

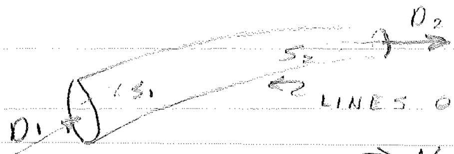
$$\Rightarrow V = \frac{1}{4\pi\epsilon}\int_S \Phi \left(\frac{d}{dn}\frac{1}{r}\right) ds = \frac{1}{4\pi\epsilon}\int_S \Phi\, \frac{\vec{n}\cdot\vec{r}}{r^3}\, ds$$

IS POTENTIAL DUE TO DIPOLE DOUBLE LAYER W/ STRENGTH Φ

NOW $d\Omega = \frac{\vec{n}\cdot\vec{r}}{r^3} ds \Rightarrow V = \frac{1}{4\pi\epsilon}\int_S \Phi\, d\Omega$

IF Φ IS UNIFORM, WITH STRENGTH ψ;

1.13● TUBES OF FORCE



$\Rightarrow N =$ FLUX LINES $= DS_1 = DS_2$

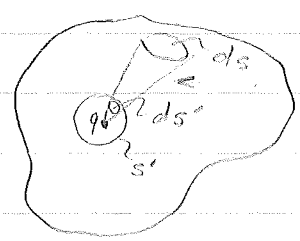
1.14● STRESSES AND TENSION (TWIXT LINES IN $\vec{E}$ FIELD)

$\Phi(E) =$ TENSION $= \dfrac{\epsilon E^2}{2}$

$\Psi(E) =$ FORCE $= \dfrac{-\epsilon E^2}{2}$

SINCE Φ & Ψ DEPEND ONLY ON ϵ & E, THE ORIGIN OR SHAPE OF THE FIELD IS IMMATERIAL

1.15● GAUSS'S LAW FOR NON-HOMOGENEOUS ISOTROPIC MEDIA



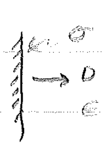
$\epsilon' \vec{E}' \cdot \vec{n}' ds' = \epsilon \vec{E} \cdot \vec{n}\, ds$

$\epsilon \int_{s'} \vec{E}' \cdot \vec{n}' ds = \int_s \epsilon \vec{E} \cdot n\, ds$

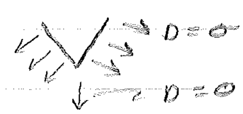
$\Rightarrow \int_s \epsilon \vec{E} \cdot \vec{n}\, ds = q$

1.16● BOUNDRY CONDITIONS & STRESSES ON SURFACE OF CONDUCTORS

$\rightarrow$ ON AND IN A CONDUCTOR, THE POTENTIAL IS CONSTANT



$\vec{D} = \epsilon \vec{E} = \sigma$



$\rightarrow$ TENSION (FORCE) $= \dfrac{D^2}{2\epsilon} = \dfrac{\sigma^2}{2\epsilon}$

(INDEPENDENT OF CHARGE POLARITY)

II. CAPACITOR'S, DIELECTRICS, & SYSTEMS OF CONDUCTORS

2.00 UNIQUENESS THEOREM

ONLY ONE DISTRIBUTION OF CHARGE WILL GIVE 1) A SPECIFIED POTENTIAL TO EVERY CONDUCTOR IN $\vec{E}$ FIELD

2) A SPECIFIED TOTAL CHARGE TO EACH CONDUCTOR IN AN $\vec{E}$ FIELD

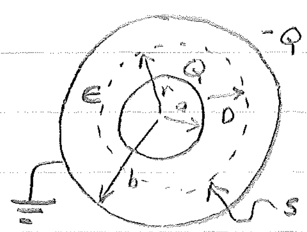
2.01 CAPACITANCE

C = CAPACITANCE (FARADS)

S = ELASTANCE (DARAFS)

$$Q = CV \quad \text{AND} \quad V = SQ$$

2.03 SPHERICAL CAPACITORS



APPLY GAUSS'S FLUX THEOREM TO SPHERE OF RADIUS r:

$$\int_S \epsilon \vec{E} \cdot \vec{n}\, ds = (4\pi r^2)\epsilon E = Q$$

$$\Rightarrow E = -\frac{\delta V}{\delta r} = \frac{Q}{4\pi \epsilon r^2}$$

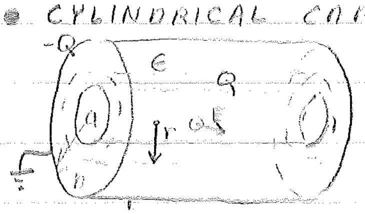
THUS $\Delta V = V_a - V_b = \frac{-Q}{4\pi\epsilon}\int_a^b \frac{dr}{r^2} = \frac{(b-a)Q}{4\pi\epsilon ab}$

$$\Rightarrow C = \frac{Q}{\Delta V} = \frac{4\pi\epsilon ab}{b-a}$$

WE CAN LET $b \to \infty$ & GET C FOR A SINGLE HOLLOW SPHERE: $C = 4\pi\epsilon a$

NOTE: IF b WAS NOT GROUNDED, THIS XTRA CAPACITANCE WOULD BE ADDED

2.04 CYLINDRICAL CAPACITORS



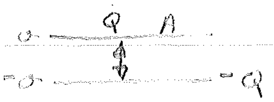
$$Q = \int_S \epsilon \vec{E} \cdot n\, ds = (2\pi r)\epsilon E$$

$$\Rightarrow E = -\frac{\delta V}{\delta r} = \frac{Q}{2\pi\epsilon r}$$

$$\Delta V = V_a - V_b = \frac{-Q}{2\pi\epsilon}\int_a^b \frac{dr}{r} = \frac{-Q}{2\pi\epsilon}\ln\left(\frac{a}{b}\right)$$

$$\Rightarrow C = \frac{2\pi\epsilon}{\ln(b/a)}$$

2.05 ● PARALLEL PLATE CAPACITOR



$$D = {}^{+}\epsilon E = \sigma = {}^{-}\epsilon \frac{\delta V}{\delta r}$$

$$\Delta V = \frac{\sigma}{\epsilon}\int_0^a dx = \frac{\sigma a}{\epsilon} \Rightarrow C = \frac{\epsilon A}{a}$$

2.07 ● ENERGY OF A CHARGED CAPACITOR

CONSIDER A FIELD OF n POINT CHARGES.

THE WORK REQUIRED TO PUT THE j^{TH} CHARGE IN PLACE IS

$$W_j = q_j V_j = \frac{q_j}{4\pi\epsilon}\sum_{i=1}^{n}\frac{q_i}{r_{ij}} \quad ; i \neq j$$

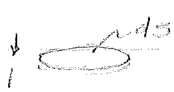
$$\Rightarrow W = \frac{1}{2}\Sigma q_i V_i$$

IF CHARGES ARE ALL ON SAME CONDUCTOR, $W = \frac{1}{2} Q_a V_a$

ON A CONDUCTOR WITH CAPACITANCE C: $W = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2$

ON A CAPACITOR @ Q & -Q : $W = \frac{1}{2}Q(V_1 - V_2)$

2.08 ● ENERGY IN AN $\vec{E}$ FIELD



$$ds\left(\frac{\delta V}{\delta s}\right) = -E\,ds$$

$$\vec{D}\cdot n\,ds = \frac{D\cdot E}{E}\,ds \Rightarrow \frac{dW}{dV} = \frac{D\cdot E}{2}$$

2.12 ● GREEN'S RECIPROCATION THEOREM

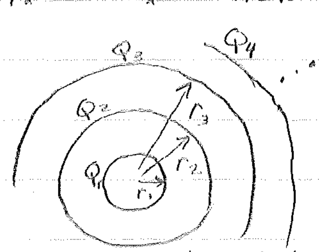
$$\sum_{i=1}^{n} Q_i V_i' = \sum_{i=1}^{n} Q_i' V_i$$

Q_i & V_i ARE THE CHARGE & POTENTIAL OF THE i^{TH} CONDUCTOR

2.13 ● FIELD SUPERPOSITION

$Q_i \rightarrow V_i$ THEN $Q_i' + Q \rightarrow V_i + V_i'$

EX.



FIND VOLTAGE ON S^{TH} SPHERE

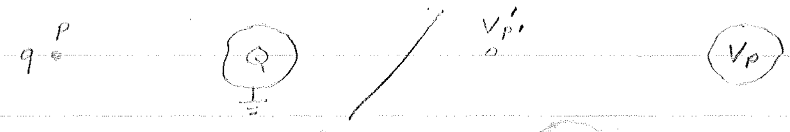
1. VOLTAGE INSIDE A SHELL IS

$$V_{IN} = \frac{Q}{4\pi\epsilon a}$$

2. OUTSIDE: $V_o = \frac{Q}{4\pi\epsilon r}$

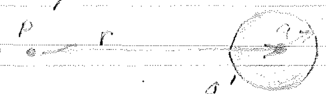
$$\Rightarrow V_S = \frac{1}{4\pi\epsilon}(Q_1 + Q_2 + \ldots Q_S)\,r_S^{-1} + \frac{Q_{S+1}}{4\pi\epsilon r_{S+1}} + \cdots + \frac{Q_n}{4\pi\epsilon r_n}$$

2.14 INDUCED CHARGES ON EARTHED CONDUCTORS



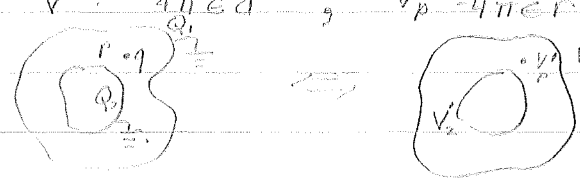
$\Rightarrow Q = \frac{-V_p'}{V'} q$

EX, CONSIDER



$V' = \frac{q'}{4\pi\epsilon a}$, $V_p' = \frac{q'}{4\pi\epsilon r}$ $\Rightarrow Q = -\frac{aq}{r}$

EX.



$Q_1 V_1' + Q_2 V_2' + q V_p' = 0$; $\frac{Q_1}{q} = \frac{V_2' - V_p'}{V_1' - V_2'}$; $\frac{Q_2}{q} = \frac{V_1' - V_p'}{V_2' - V_1'}$

- CYLINDERS: $Q_1 = -\frac{\ln(r_2/r)}{\ln(r_2/r_1)} q$; $Q_2 = -\frac{\ln(r_1/r)}{\ln(r_1/r_2)} q$
- SPHERES: $Q_1 = -\frac{r_1(r_2 - r)}{r(r_2 - r_1)} q$; $Q_2 = -\frac{r_2(r - r_1)}{r(r_2 - r_1)} q$
- PLATES: $Q_1 = \frac{-bq}{a+b}$; $Q_2 = -\frac{aq}{a+b}$

2.15 SELF & MUTUAL ELASTANCE

$$\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1N} \\ S_{21} & S_{22} & & \\ \vdots & & & \\ S_{n1} & \cdots & \cdots & S_{nn} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_n \end{bmatrix} \qquad S_{rn} = S_{nr} > 0$$

S_{sr} = POTENTIAL TO WHICH r IS RAISED WHEN A UNIT CHARGE IS PLACED ON S = MUTUAL ELASTANCE

S_{rr} = SELF ELASTANCE

2.16 MUTUAL AND SELF CAPACITANCE

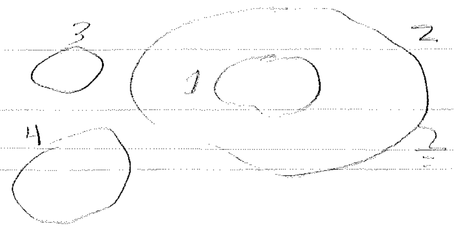
$\vec{Q} = \overleftrightarrow{C} \vec{V}$

$$C_{11} = \frac{1}{\Delta} \begin{vmatrix} S_{22} & S_{23} & \cdots \\ S_{32} & & \\ \vdots & & \end{vmatrix} \qquad C_{12} = C_{21} = -\frac{1}{\Delta} \begin{vmatrix} S_{21} & S_{31} & \cdots & S_{n1} \\ S_{23} & S_{33} & & \\ \vdots & & & \\ S_{2n} & S_{3n} & \cdots & S_{nn} \end{vmatrix}$$

C_{rr} = SELF CAPACITANCE > 0

$C_{rs} = C_{sr}$ = MUTUAL CAPACITANCE ≤ 0

2.17 ● ELECTRIC SCREENING



IF $V_2 = 0$, Q_1 DEPENDS ONLY ON ITS POTENTIAL

$\Rightarrow C_{31} = C_{41} = \ldots = C_{n1} = 0$

2.19 ● ENERGY IN A CHARGED SYSTEM

$$W = \frac{1}{2} \int_v \vec{D} \cdot \vec{E} \, dv$$

$$W_V = \frac{1}{2} \left(c_{11} V_1^2 + 2 c_{12} V_1 V_2 + c_{22} V_2^2 + \ldots \right)$$

$$W_Q = \frac{1}{2} \left(s_{11} Q_1^2 + 2 s_{12} Q_1 Q_2 + \ldots \right)$$

2.20 ● FORCES & TORQUES ON CHARGED CONDUCTORS

$$\vec{F} \cdot \vec{p} = \int_S \frac{1}{2} \vec{D} \cdot \vec{E} \quad \vec{p} \cdot \vec{n} \, dS$$

III. GENERAL THEOREMS

3.00 • GAUSS'S THEOREM

$$\oint_S \vec{A}\cdot\vec{n}\,ds = \int_V \vec{\nabla}\cdot\vec{A}\,dv$$

3.01 • STOKE'S THEOREM

$$\oint_\ell \vec{F}\cdot d\vec{\ell} = \int_S n\cdot\vec{\nabla}\times\vec{F}\,ds$$

3.02 • POISSON & LAPLACE EQUATIONS

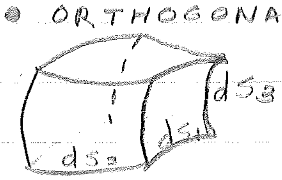
FROM GAUSS: $\int_V \nabla\cdot\dot{D}\,dv = q = \int_V \rho\,dv$

$\Rightarrow \bar{\nabla}\cdot\bar{D} = \frac{dq}{dV} = \rho$

$\vec{D} = \epsilon\vec{E} = -\epsilon\vec{\nabla}V \Rightarrow \bar{\nabla}\cdot\epsilon\bar{\nabla}V = -\rho \Leftarrow$ POISSON'S EQ

FOR NO CHARGE $\Rightarrow \bar{\nabla}\cdot\epsilon\bar{\nabla}V = 0 \Leftarrow$ LAPLACE'S EQ

3.03 • ORTHOGONAL CURVILINEAR COORDINATES

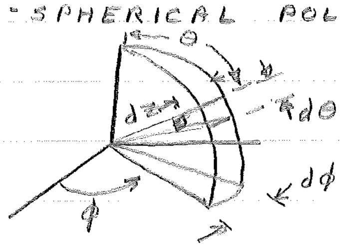


(u_1, u_2, u_3)

$$dS_i = h_i\,du_i$$
$$\frac{\delta V}{\delta S_i} = \frac{\delta V}{h_i\,\delta u_i}$$

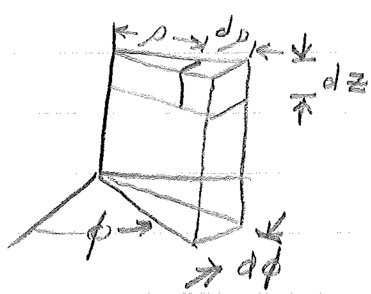
3.05 • LAPLACE'S EQ. IN VARIOUS COORDINATE SYSTEMS

- SPHERICAL POLAR COORDINATES



$$\frac{1}{r^2}\frac{\delta}{\delta r}\left(\epsilon r^2\frac{\delta V}{\delta r}\right) + \frac{1}{r^2\sin\theta}\frac{\delta}{\delta\theta}\left(\epsilon\sin\theta\frac{\delta V}{\delta\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\delta}{\delta\phi}\left(\frac{\epsilon\,\delta V}{\delta\phi}\right) = 0$$

- CYLINDRICAL COORDINATES



$$\frac{1}{\rho}\frac{\delta}{\delta\rho}\left(\epsilon\rho\frac{\delta V}{\delta\rho}\right) + \frac{1}{\rho^2}\frac{\delta}{\delta\phi}\left(\epsilon\frac{\delta V}{\delta\phi}\right) + \frac{\delta}{\delta z}\left(\epsilon\frac{\delta V}{\delta z}\right) = 0$$

IV. TWO DIMENSIONAL POTENTIAL DISTRIBUTIONS

4.00 FIELD & POTENTIAL IN TWO DIMENSIONS

$$E = \frac{q}{2\pi\epsilon r}$$

$$V = \frac{-q}{2\pi\epsilon} \ln r + C$$

4.01 CIRCULAR HARMONICS

$$V = R(\rho)\,\Phi(\phi)\,Z(z) = R(r)\,\Theta(\theta) \leftarrow \text{IN 2 DIMENSIONS}$$

LAPLACE'S EQN. BECOMES: $\frac{1}{R}\left(r\frac{\delta R}{\delta r} + r^2 \frac{\delta^2 R}{\delta r^2}\right) + \frac{1}{\Theta}\frac{\delta^2 \Theta}{\delta \theta^2} = C$

LET $\frac{\delta^2 \Theta}{\delta \theta^2} = -n^2 \Theta$; $\frac{\delta^2 R}{\delta r^2} + \frac{1}{r}\frac{dR}{dr} = n^2 R/r^2$

GIVES
$$\Theta_n = \begin{cases} A\cos n\theta + B\sin n\theta & ; n \neq 0 \\ A\theta + B & ; n = 0 \end{cases}$$

$$R_n = \begin{cases} Cr^n + Dr^{-n-1} & ; n \neq 0 \\ C\ln r + D & ; n = 0 \end{cases}$$

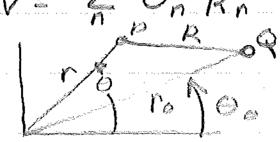
n IS "DEGREE OF THE HARMONIC"

$$\begin{cases} n = 0 \Rightarrow V = (A\theta + B)(C\ln r + D) \\ n \neq 0 \Rightarrow V = (A\cos n\theta + B\sin n\theta)(Cr^n + Dr^{-n}) \end{cases}$$

SOLUTIONS TO LAPLACE'S EQ.

$$V = \sum_n \Theta_n R_n \quad \text{OR} \quad V = \int f(n)\,\Theta_n R_n\,dn$$

4.02

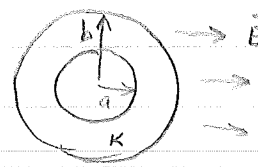


$$4\pi V = -2q\ln R = -q\ln\left[r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0)\right]$$
$$= -2q\ln r - q\ln\left[1 - \frac{r_0}{r}e^{j(\theta-\theta_0)}\right]\left[1 - \frac{r_0}{r}e^{-j(\theta-\theta_0)}\right]$$

IT TURNS OUT THAT

$$V = \begin{cases} \frac{q}{4\pi\epsilon}\left[\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r_0}{r}\right)^n(\cos n\theta_0\cos n\theta + \sin n\theta_0\sin n\theta) - \ln r\right] & r_0 < r \\ \frac{q}{4\pi\epsilon}\left[\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{r_0}\right)^n(\cos n\theta_0\cos n\theta + \sin n\theta_0\sin n\theta) - \ln r_0\right] & r_0 > r \end{cases}$$

4.03 CONDUCTING OR DIELECTRIC CYLINDER IN UNIFORM FIELD



INITIALLY : $V = Ex = Er\cos\theta$

THE FIELD @ ∞ MUST BE ZERO:

$$V_o = Er\cos\theta + \sum_{n=1}^{\infty} A_n r^{-n}\cos n\theta$$

INSIDE DIELECTRIC: $V_i = \sum_{n=1}^{\infty} (B_n r^n + C_n r^{-n})\cos n\theta$

BOUNDRY CONDITIONS @ $r = b$ ARE

$$\frac{\delta V_o}{\delta r} = K\frac{\delta V_i}{\delta r} \quad \text{AND} \quad V_o = V_i$$

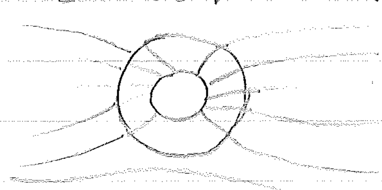
THIS REQUIRES $A_n = B_n = 0 \;\; \forall \; n \neq 1$

$$A_1 = -Eb^2 \frac{(K+1)a^2 + (K-1)b^2}{(K+1)b^2 + (K-1)a^2}$$

$$B_1 = 2Eb^2 \frac{1}{(K+1)b^2 + (K-1)a^2}$$

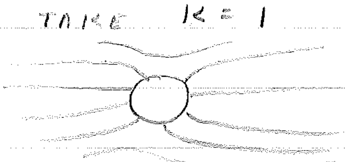
$$C_1 = -2Ea^2b^2 \frac{1}{(K+1)b^2 + (K-1)a^2}$$

$$\Rightarrow V_o = \left(Er + \frac{A_1}{r}\right)\cos\theta \quad ; \quad V_i = \left(B_1 r + \frac{C_1}{r}\right)\cos\theta$$



$\vec{D}$

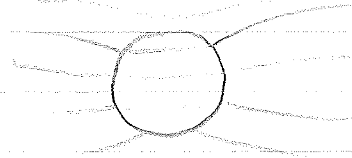
TAKE $K = 1$



$$V_o = E\left(r - \frac{a^2}{r}\right)\cos\theta$$

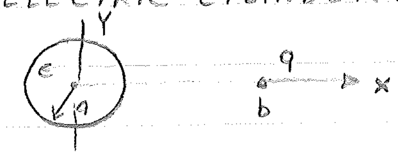
LET $a \to 0 \Rightarrow$ DIELECTRIC CILYINDER

$$V_o = E\left(r - \frac{K-1}{K+1}\frac{b^2}{r}\right)\cos\theta \qquad V_i = \frac{2E}{K+1} r\cos\theta$$



$\vec{D}$

4.04 DIELECTRIC CYLINDER. METHOD OF IMAGES



WITHOUT DIELECTRIC: $V = \frac{q}{4\pi\epsilon}\left[\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{r}{b}\right)^n \{\cos n\theta\} - \ln b\right]$ ← FROM 4.02

MUST ADD EFFECT OF DIELECTRIC POLARIZATION.

FIELD SYMMETRIC WITH X AXIS:

$$V_o = \frac{q}{4\pi\epsilon}\left[\sum_{n=1}^{\infty}\left\{\frac{1}{n}\left(\frac{r}{b}\right)^n + \frac{A_n}{r^n}\right\}\cos n\theta - \ln b + c_1\right]$$

FIELD INSIDE DIELECTRIC MUST BE FINITE

$$V_i = \frac{q}{2\pi\epsilon_v}\left(\sum_{n=1}^{\infty} B_n r^n \cos n\theta + C_2\right) \Leftarrow \text{FROM SYMMETRY}$$

1. $V_o = V_i$ AT $r = a$

$$\Rightarrow \frac{1}{n}\left(\frac{a}{b}\right)^n + \frac{A_n}{a^n} = a^n B_n \quad ; \quad C_2 = -\ln b + C_1$$

2. $\epsilon_v \frac{\delta V_o}{\delta r} = \epsilon \frac{\delta V_o}{\delta r} \Rightarrow K \frac{\delta V_i}{\delta r} = \frac{\delta V_o}{\delta r}$ @ $r = a$

$$\Rightarrow \frac{a^{n-1}}{b^n} - \frac{n}{a^{n+1}} A_n = n K a^{n-1} B_n$$

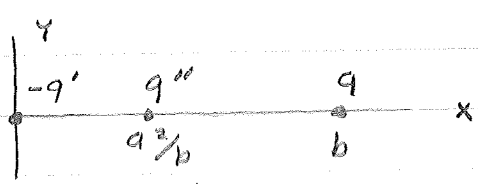
COMBINING GIVES: $A_n = \frac{1-K}{1+K}\frac{a^{2n}}{n b^n}$; $B_n = \frac{2}{(1+K) n b^n}$

$$\therefore V_o = \frac{q}{2\pi\epsilon_v}\left[\sum_{n=1}^{\infty}\frac{1}{n}\left\{\left(\frac{r}{b}\right)^n + \frac{1-K}{1+K}\left(\frac{a^2}{b}\right)^n\frac{1}{r^n}\right\}\cos n\theta - \ln b + c_1\right]$$

$$V_i = \frac{q}{\pi\epsilon_v(1+K)}\left[\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{b}\right)^n \cos n\theta - \frac{q}{2\pi\epsilon_v}(\ln b - c_1)\right]$$

LET $C_1 = 0 = -\frac{1-K}{1+K}\ln r + \frac{1-K}{1+K}\ln r$

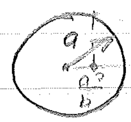
THE POTENTIAL OUTSIDE (V_o) MAY BE FOUND VIA



$$q' = \frac{1-K}{1+K} q \quad ; \quad q'' = \frac{2}{1+K} q$$

4.05 • IMAGE IN CONDUCTING CYLINDER

MAY GET IMAGE, LET $K \to \infty$



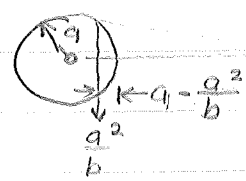
$\overset{b}{\circ}\ q \quad \equiv \quad \underset{a^2/b}{-q} \qquad \underset{b}{q}$

4.06 • IMAGE IN PLANE FACE OF DIELECTRIC OR CONDUCTOR

(INTERSECTING CONDUCTING PLANES)

LET RADIUS (a) OF CYLINDERS $\to \infty$ KEEPING

$d = b - a$ CONSTANT.

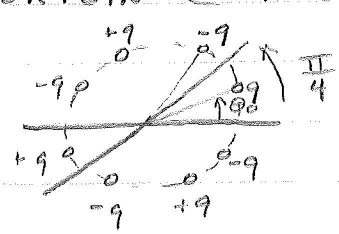


$$a - \frac{a^2}{b} = \frac{a}{b}(b-a) \to d$$

FOR A CONDUCTOR: $\underset{-b}{-q} \quad \underset{0}{} \quad \underset{b}{+q}$

FOR A DIELECTRIC: $\underset{-b}{+q''} \quad \underset{0}{-q''} \quad \underset{b}{+q}$

- FOR CONDUCTING PLANES INTERSECTING @ ORIGIN @ ANGLE $\frac{\pi}{m}$



$+q$'s @

$\theta,\ \frac{2\pi}{m} + \theta_0,\ \frac{4\pi}{m} + \theta_0, \ldots,\ \frac{2(M-1)\pi}{m} + \theta_0$

$-q$'s @

$\frac{2\pi}{m} - \theta_0,\ \frac{4\pi}{m} - \theta_0, \ldots,\ 2\pi - \theta_0$

4.09 • COMPLEX QUANTITIES

$$\frac{\delta^2 U}{\delta X^2} + \frac{\delta^2 U}{\delta Y^2} = 0 \Leftarrow \text{LAPLACE'S EQ. IN TWO DIMENSIONS}$$

$$U = \Phi(x+jY) + \psi(x-jY) \Leftarrow \text{A (REAL) SOLUTION}$$

$$\Rightarrow \Phi(x+jy) = u + jv \qquad \psi(x-jY) = u - jv$$

$$U = 2u \;;\; V = 2v, \quad W = U + jV$$

$W = U + jV$ SATISFIES LAPLACE'S EQ. $(W = f(z))$

U AND V ARE "CONJUGATE FUNCTIONS"

4.10 • THE STREAM FUNCTION

$$\frac{\delta V}{\delta X} = \frac{-\delta U}{\delta Y} \quad \text{AND} \quad \frac{\delta V}{\delta Y} = \frac{\delta U}{\delta X}$$

$\Rightarrow$ V = CONST & U = CONST. INTERSECT ORTHOGONALLY

4.11 • ELECTRIC FIELD INTENSITY / ELECTRIC FLUX

$$\frac{\delta W}{\delta z} = \frac{\delta V}{\delta Y} + j\frac{\delta V}{\delta X} = \frac{\delta U}{\delta X} - j\frac{\delta U}{\delta Y}$$

• IF V IS POTENTIAL, THEN

a. $-\frac{\delta W}{\delta z}$ GIVES X AND Y COMPONENTS OF $\vec{E}$ FIELD

b. $\left|\frac{\delta W}{\delta z}\right|$ GIVES $|\vec{E}|$

• $\left|\frac{\delta W}{\delta z}\right| = \frac{\delta U}{\delta n} = \frac{\delta V}{\delta s}$ (FOR POTENTIAL U)

$\left|\frac{\delta W}{\delta z}\right| = \frac{\delta V}{\delta n} = \frac{-\delta U}{\delta s}$ (FOR POTENTIAL V)

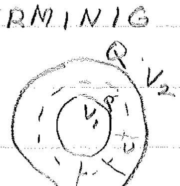
CONSIDER GAUSS' FLUX THEM: (HERE, V IS POTENTIAL)

$$\text{FLUX} = -\epsilon \int_s \vec{E}\cdot d\vec{n} = -\epsilon \int_{U_1}^{U_2} \frac{\delta V}{\delta n}\, ds = \epsilon \int_{U_1}^{U_2} \frac{\delta U}{\delta s}\, ds$$

$$= \epsilon(U_2 - U_1)$$

IF OUR SURFACE WAS CLOSED, $\epsilon(U_2 - U_1) = Q$

DETERMININIG CAPACITANCE



Q = FLUX PER UNIT LENGTH

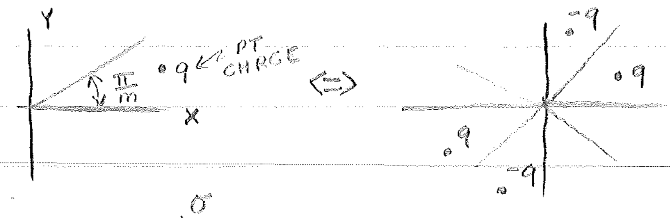
$$C = \frac{[Q]}{[V_2 - V_1]} = \frac{\epsilon[U]}{[V_2 - V_1]}$$

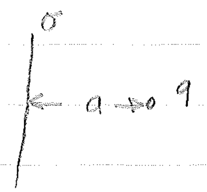
$$\text{FIELD ENERGY} = \tfrac{1}{2} C |V_2 - V_1|^2$$

$$= \tfrac{1}{2}\epsilon |U_2 - U_1| \, |V_2 - V_1|$$

V. THREE DIMENSIONAL POTENTIAL DISTRIBUTIONS

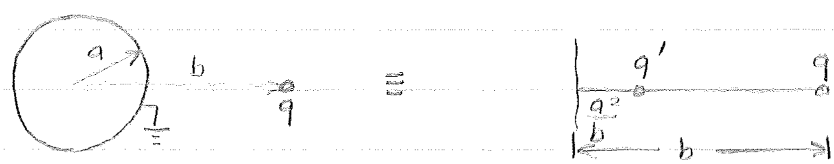
5.04. METHOD OF IMAGES. CONDUCTING PLANES.





$$\sigma = \frac{-aq}{2\pi r^3}$$

5.06. IMAGE IN SPHERICAL CONDUCTOR



$$q' = \frac{-aq}{b}$$

IT THE SPHERE IS UNGROUNDED (AT POTENTIAL V),
WE MAY ADD A CHARGE $q = 4\pi\epsilon a V$ @ THE ORIGIN.

5.09. INVERSION IN THREE DIMENSIONS, GEOMETRICAL PROPERTIES

$rr' = K^2$; K = RADIUS OF INVERSION

PLANES & SPHERES ⟺ SPHERES & PLANES

$$f(r, \theta, \phi) \Leftrightarrow f(K^2/r, \theta, \phi)$$

5.10. INVERSE OF POTENTIAL AND IMAGING SYSTEMS.

$$rr' = K^2$$

$$\frac{V'}{V} = \frac{q'r}{qr'} \qquad \frac{\sigma'}{\sigma} = \frac{K^3}{r'^3}$$

$$\frac{q'}{q} = \frac{V'r'}{Vr}$$

q_i's @ P_i's ⟺ q_i''s @ P_i''s

● 5.12. SPHERICAL HARMONICS

IN SPHERICAL CO-ORDINATES, LAPLACE'S EQN IS

$$\frac{\delta}{\delta r}\left(r^2 \frac{\delta V}{\delta r}\right) + \frac{1}{\sin\theta}\frac{\delta}{\delta\theta}\left(\sin\theta \frac{\delta V}{\delta\theta}\right) + \frac{1}{\sin^2\theta}\frac{\delta^2 V}{\delta\phi^2} = 0$$

ASSUME SOLUTION: $V = R\Theta\Phi = RS$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = K \Rightarrow R = Ar^n + Br^{-n-1}$$

AND $V = (Ar^n + Br^{-n-1})S_n$

● 5.14. DIFFERENTIAL EQS. FOR SURFACE HARMONICS

SURFACE HARMONIC DIFF. EQ. $(S = \Theta\Phi)$

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} + n(n+1)\sin^2\theta = 0$$

LET: $\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -K_1 = -m^2 \Rightarrow \Phi(\phi) = C\cos m\phi + D\sin m\phi \quad (m \neq 0)$

$= M\phi + N \quad (m = 0)$

$$\Rightarrow \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0$$

● 5.15. SURFACE ZONAL HARMONICS. LEGENDRE'S EQ.

FOR $\mu = \cos\theta$, ABOVE EQ. BECOMES

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Theta_n}{d\mu}\right] + n(n+1)\Theta_n = 0$$

● 5.151 SERIES SOLUTION OF LEGENDRE'S EQ:

$$\Theta_n = A_n p_n + B_n q_n$$

$$n q'_{n-1} + (n+1) q'_{n+1} = (2n+1) p_n$$

$$p_{n-1} - p_{n+1} = (2n+1)\mu q_n$$

$$(n+1)p'_{n-1} + n p'_{n+1} = -n(n+1)(2n+1) q_n$$

● 5.152. LEGENDRE POLYNOMIALS. RODRIGUES FORMULA.

$$P_n(\mu) = \frac{(-1)^{n/2}\, n!}{2^n (n/2!)^2} p_n \quad (n \text{ ODD})$$

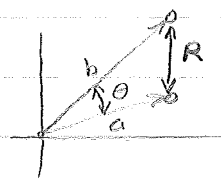
$$= \frac{(-1)^{\frac{n-1}{2}}\, n!}{2^{n-1}[\frac{1}{2}(n-1)!]^2} q_n$$

$$= \sum_{s=0}^{m} (-1)^s \frac{(2n-2s)!}{2^n s!(n-s)!(n-2s)!}\mu^{n-2s}$$

$$m = \frac{n}{2} \text{ OR } \frac{1}{2}(n-1)$$

$$= \frac{1}{2^n n!}\frac{d^n}{d\mu^n}(\mu^2-1)^n \Leftarrow \text{RODRIGUES'S FORMULA}$$

● 5.153. LEGENDRE COEFFICIENTS. INVERSE DISTANCE



$b > a \qquad \mu = \cos\theta$

$$\frac{1}{R} = \frac{1}{b}\left[P_0(\mu) + \left(\frac{a}{b}\right)P_1(\mu) + \left(\frac{a}{b}\right)^2 P_2(\mu) + \dots\right]$$
$$= \frac{1}{b}\sum_{n=0}^{\infty}\left(\frac{a}{b}\right)^n P_n(\mu)$$

● 5.154. RECURRANCE FORMULAS FOR LEGENDRE POLYNOMIALS

$$n P_{n-1} + (n+1) P_{n+1} = (2n+1)\mu P_n$$
$$P'_{n+1} - P'_{n-1} = (2n+1) P_n$$
$$P'_{n+1} = \mu P'_n + (n+1) P_n \qquad P_n' = \mu P'_{n-1} + n P_{n-1}$$
$$P_n' = \frac{-n(n+1)}{1-\mu^2}\int P_n(\mu)\,d\mu \qquad \int P_n(\mu) = \frac{P_{n+1} - P_{n-1}}{2n+1}$$

● 5.155. INTEGRAL OF PRODUCT OF LEGENDRE POLYNOMIALS

$$\int_{-1}^{1} P_n(\mu) P_m(\mu)\,d\mu = \frac{2}{2n+1}\,\delta_{nm} \;;\; \mu = \cos\theta$$

● 5.156. EXPANSION OF FUNCTION IN LEGENDRE POLYNOMIALS

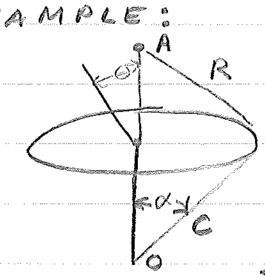
$$f(\mu) \in [-1, 1]$$
$$f(\mu) = \sum_{n=0}^{\infty} a_n P_n(\mu)$$
$$a_m = \tfrac{1}{2}(2m+1)\int_{-1}^{1} f(\mu) P_m(\mu)\,du$$
$$= \frac{2m+1}{2^{m+1} m!}\int_{-1}^{1} \frac{d^m f(\mu)}{d\mu^m}(1-\mu^2)^m\,d\mu \leftarrow \text{VIA RODRIGOUS}$$

● 5.157 TABLE OF LEGENDRE POLYNOMAILS (pp. 147-8)

● 5.16. POTENTIAL OF A CHARGED RING

– IF V IS SYMMETRICAL ABOUT THE X AXIS, AND IF THIS POTENTIAL CAN BE EXPRESSED IN A SERIES OF X^m ($m \in$ INTEGER), THEN, REPLACING X^m BY $r^m P_n(\cos\theta)$ WILL GIVE V EVERYWHERE.

EXAMPLE: RING WITH CHARGE Q



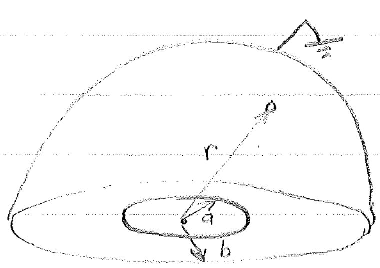
$$V_A = \frac{Q}{4\pi\epsilon R} = \frac{Q}{4\pi\epsilon\sqrt{c^2+x^2-2cx\cos\alpha}}$$

$$= \begin{cases} \frac{Q}{4\pi\epsilon c} \sum_{n=0}^{\infty} \left(\frac{c}{x}\right)^{n+1} P_n(\cos\alpha)\ ; x \geq c \\ \frac{Q}{4\pi\epsilon c} \sum_{n=0}^{\infty} \left(\frac{x}{c}\right)^{n} P_n(\cos\alpha)\ ; x \leq c \end{cases}$$

THUS $\Rightarrow$

$$V_P = \begin{cases} \frac{Q}{4\pi\epsilon c} \sum_{n=0}^{\infty} \left(\frac{c}{r}\right)^{n+1} P_n(\cos\alpha) P_n(\cos\theta)\ ; r > c \\ \frac{Q}{4\pi\epsilon c} \sum_{n=0}^{\infty} \left(\frac{r}{c}\right)^{n} P_n(\cos\alpha) P_n(\cos\theta)\ ; r < c \end{cases}$$

● 5.17. CHARGED RING IN CONDUCTING SPHERE



FROM ABOVE, POTENTIAL DUE TO RING ($\alpha = \frac{\pi}{2}$) IS $V_r = \frac{Q}{2\pi\epsilon a} \sum_{n=0}^{\infty} \left(\frac{a}{r}\right)^{2n+1} P_{2n}(0) P_{2n}(\mu)$

(THIS IS GOOD FOR $a < r$)

FOR SPHERE:

$$V_s = \sum_n R(r) P_n(\mu) = \sum_n A_n r^n P_n(\mu)$$

THE TOTAL POTENTIAL IS THE SUM OF THESE

$$V = V_r + V_s$$

WE REQUIRE THAT $V(b) = 0 \Rightarrow V_r(b) = -V_s(b)$

$$\frac{Q}{2\pi\epsilon a} \sum_n \left(\frac{a}{b}\right)^{2n+1} P_{2n}(0) P_{2n}(\mu) = \sum_n A_n b^n P_n(\mu)$$

$$\Rightarrow A_{2n} = -(-1)^n \frac{(2n-1)!!}{2n!!} \frac{1}{b^{2n}} \left(\frac{a}{b}\right)^{2n+1}$$

QED.

● 5.29. LAPLACE'S EQ. IN CYLINDRICAL COORDINATES

$$\frac{d^2 V}{d\rho^2} + \frac{1}{\rho}\frac{\delta V}{\delta \rho} + \frac{1}{\rho^2}\frac{\delta^2 V}{\delta \phi^2} + \frac{\delta^2 V}{\delta z^2} = 0$$

$x = \rho \cos\phi \qquad y = \rho \sin\phi$

LET $V = R\Phi Z$

$$\Rightarrow \frac{\rho}{R}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \frac{1}{\Phi}\frac{\delta^2 \Phi}{d\phi^2} + \frac{\rho^2}{Z}\frac{d^2 Z}{dz^2} = 0$$

● 5.291. BESSEL'S EQ. AND BESSEL FUNCTIONS

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -n^2, \quad \frac{1}{Z}\frac{d^2 Z}{dz^2} = k^2$$

$$\Phi = A\cos n\phi + B\sin n\phi$$

$$Z = C\cosh kz + D\sinh kz$$

THEN, FOR $V = k\rho$, R MUST SATISFY

$$\frac{d^2R}{dV^2} + \frac{1}{V}\frac{dR}{dV} + \left(1 - \frac{n^2}{V^2}\right)R = 0 \Leftarrow \text{BESSEL'S EQ.}$$

$$\Rightarrow V = R_n(k\rho)\,\Phi(n\phi)\,Z(kz)$$

NOTE, FOR $k=0$

$$V = \begin{cases} (M\rho^n + N\rho^{-n})(Cz + D)(A\cos n\phi + B\sin n\phi) \\ (M\cos(n\ln\rho) + N\sin(n\ln\rho))(zC + D) \times (A\cosh n\phi + B\sinh n\phi) \end{cases}$$

FOR $n = k = 0$

$$V_{00} = (M\ln\rho + N)(Cz + D)(A\phi + B)$$

5.293. SOLUTION OF BESSEL'S EQ

$$J_n(v) = \sum_{r=0}^{\infty} (-1)^r \frac{(v/2)^{n+2r}}{r!\,\Gamma(n+r+1)}$$

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \; ; \; x > 0, \; \Gamma(n+1) = n!$$

FOR $n \neq$ INTEGER, ANOTHER INDEPENDENT SOL. IS

$$Y_\nu(v) = \frac{1}{\sin \pi\nu}\left[J_\nu(v) \cos \nu\pi - J_{-\nu}(v) \right]$$

FOR $n =$ INTEGER, THIS BECOMES

$$Y_n(v) = \frac{2}{\pi} J_n(v) \ln(\alpha v) - \sum_{r=0}^{\infty} (v/2)^{2r-n} (n-r-1)! \Big/ \pi r!$$

$$- \sum_{r=0}^{\infty} \frac{(-1)^r (v/2)^{n+2r}}{\pi r! (n+r)!} \left(\sum_{m=1}^{r} \frac{1}{m} + \sum_{m=1}^{n+r} \frac{1}{m} \right)$$

THUS, FOR $n =$ INTEGER, A COMPLETE SOLN' OF BESSEL IS

$$R_n(v) = A\, J_n(v) + B\, Y_n(v)$$

5.294. RECURRENCE FORMULAS FOR BESSEL EQ'S

$$\frac{d}{dv} v^n J_n(v) = v^n J_{n-1}(v)$$

$$J_n' = J_{n-1} - \frac{n}{v} J_n$$

$$= -J_{n+1} + \frac{n}{v} J_n$$

$$= \frac{1}{2}(J_{n-1} - J_{n+1})$$

$$\frac{2n}{v} J_n = J_{n-1} + J_{n+1}$$

$$Y_n' = Y_{n-1} - \frac{n}{v} Y_n$$

$$= -Y_{n+1} + \frac{n}{v} Y_n$$

$$\frac{2n}{v} Y_n = Y_{n-1} + Y_{n+1}$$

$$\left\{ \begin{aligned} \int v^{-n} J_{n-1}(v)\, dv &= -v^{-n} J_n(v) \\ \int v^{-n} Y_{n+1}(v)\, dv &= -v^{-n} Y_n(v) \end{aligned} \right\}$$

$$\left\{ \begin{aligned} \int v_n J_{n-1}(v)\, dv &= v^n J_n(v) \\ \int v^n Y_{n-1}(v)\, dv &= v^n Y_n(v) \end{aligned} \right\}$$

• 5.295. VALUES OF BESSEL FUNCTIONS @ ∞

$$J_n(v) \xrightarrow[v\to\infty]{} \left(\frac{2}{\pi v}\right)^{1/2} \cos\left(v - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

$$Y_n(v) \xrightarrow[v\to\infty]{} \left(\frac{2}{\pi v}\right)^{1/2} \sin\left(v - \frac{\pi n}{2} - \frac{\pi}{4}\right)$$

• 5.297. EXPANSIONS IN SERIES OF BESSEL FUNCTIONS

$v \in [0, a]$, LET $f(v) = \sum_{r=1}^{\infty} A_r J_n(\mu_r v)$

POSSIBLE BOUNDRY CONDITIONS

(a) $f(a) = 0 \Rightarrow f(a)$ IS POTENTIAL, AND a IS GROUNDED

(b) $f'(a) = 0 \Rightarrow$ BOUNDRY IS A LINE OF FORCE

(c) $a f'(a) + B f(a) = 0 \Rightarrow$ (a) FOR $B = \infty$, (b) IF $B = 0$

FOR THESE RESPECTIVE BOUNDRY CONDITIONS, CHOOSE

(a) $J_n(\mu_r a) = 0$ (b) $J_n'(\mu_r a) = 0$ (c) $\mu_r a J_n'(a\mu_r) + B J_n(\mu_r a) = 0$

COEFFICIENTS GIVEN BY

$$A_s = \frac{\int_0^a v f(v) J_n(\mu_s v)\,dv}{\int_0^a v [J_n(\mu_s v)]^2\,dv}$$

$$\int_0^a v [J_n(\mu_s v)]^2 dv = \frac{a^2}{2}\left[J_n^2(\mu_s a) + J_{n\pm1}^2(\mu_s a) - \frac{na}{\mu_s} J_n(\mu_s a) J_{n\pm1}(\mu_s a)\right]$$

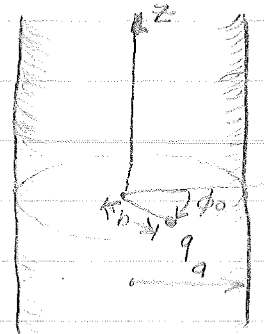
FOR THE GIVEN BOUNDRY CONDITIONS, WE HAVE

(a) $A_s = \frac{2}{a^2 J_{n\pm1}^2(\mu_s a)} \int_0^a v f(v) J_n(\mu_s v)\,dv$

(b) $A_s = \frac{2}{(a^2 - n^2/\mu^2) J_n^2(\mu_s a)} \int_0^a v f(v) J_n(\mu_s v)\,dv$

(c) $A_s = \frac{2}{[a^2 + (B^2 - n^2)/\mu_s^2] J_n^2(\mu_s a)} \int_0^a v f(v) J_n(\mu_s v)\,dv$

● 5.298. GREEN'S FUNCTION FOR A CYLINDER. INVERSE DISTANCE



- V VANISHES @ $z = \infty$
- V IS SYMMETRICAL ABOUT $\phi = \phi_0$
- $V = 0$ FOR $\rho = a$
- V IS BOUNDED (AS IN $\vec{E}$)

$$V = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} A_{rs} e^{-\mu_r z} J_s(\mu_r \rho) \cos s(\phi - \phi_0)$$

THE PLANE $z=0$ HAS ONLY LINES OF FORCE

THIS GIVES, WITH SOME CRANKING, THE GREEN'S FUNCTION FOR A CIRCULAR CYLINDER:

$$V = \frac{q}{2\pi\epsilon a^2} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} (2 - \delta_{s,0}) e^{-\mu_r |z|} \frac{J_s(\mu_r b) J_s(\mu_r \rho)}{\mu_r J_{s+1}^2(\mu_r a)} \cos s(\phi - \phi_0)$$

VI. ELECTRIC CURRENT

6.00 ● ELECTRIC CURRENT DENSITY. EQUATION OF CONTINUITY.

$I = \frac{dQ}{dt}$ = CURRENT

$i = \frac{dI}{dS}$

$\nabla \cdot i = 0 \Rightarrow$ CONTINUITY EQ. (NO SOURCES OR SINKS)

6.01 ● ELECTROMOTANCE

$\mathcal{E} = \oint E \cdot ds =$ ELECTROMOTANCE

$E = E' + E''$

$E' =$ DUE TO ELECTROSTATIC $= -\nabla V$

$E'' =$ DUE TO ELECTROMOTANCE

6.02 ● OHM'S LAW. RESISTIVITY.

$R_{AB} = V_A - V_B / I_{AB} = \mathcal{E}_{AB} / I_{AB}$

$i = \Delta\mathcal{E}/\tau = \gamma \nabla \mathcal{E}$ $\qquad \oint i \cdot ds = \gamma \mathcal{E}$

τ = RESISTIVITY $\qquad \gamma$ = CONDUCTIVITY

6.03 ● HEATING EFFECT OF ELECTRIC CURRENT

$P = I^2 R \leftarrow$ JOULES LAW

6.04 ● STEADY CURRENTS IN EXTENDED MEDIUMS

$\nabla \cdot (\frac{1}{\tau} \nabla V) = 0$

$\nabla^2 V = 0$ IN HOMOGENEOUS MEDIA

γ PLAYS SAME ROLE AS ϵ IN ELECTROSTATIC CASE.

BOUNDRY CONDITIONS (AND DUALS)

$\bar{i} = \frac{1}{\tau} \nabla \mathcal{E} = -\frac{1}{\tau} \frac{dV}{dn}$ $\qquad \bar{D} = \epsilon \bar{E} = -\epsilon \frac{dV}{dn}$

$\frac{1}{\tau'} \frac{dV'}{dn} = \frac{1}{\tau''} \frac{dV''}{dn}$ $\qquad \epsilon' \frac{dV'}{dn} = \epsilon'' \frac{dV''}{dn}$

$V' = V''$ $\qquad V' = V''$

$\frac{1}{\tau} = \gamma$ $\qquad \epsilon$

i $\qquad D$

ALSO $\frac{\sigma}{\epsilon_v} = K_1 \frac{\delta V_1}{\delta n} - K_2 \frac{\delta V_2}{\delta n} = -(K_1\tau_1 - K_2\tau_2)\, i_n$

6.05 GENERAL THEOREMS

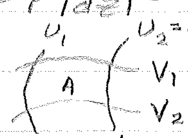
1. GIVEN — V OVER ALL CONDUCTOR BOUNDRIES
   LOCATION OF SOURCES & SINCS INSIDE
   SPECIFIES — V INSIDE THE BOUNDRY
2. GIVEN — $i_n = \frac{1}{\gamma}\frac{dV}{dn}$ OVER CONDUCTOR BOUNDRIES
   LOCATION OF SOURCES AND SINCS INSIDE
   SPECIFIED — $V_b - V_a$ = POTENTIAL DIFFERENCE
3. AS γ INCREASES, THE TOTAL RESISTANCE INCREASES OR STAYS THE SAME
4. AS γ DECREASES, THE TOTAL RESISTANCE DECREASES OR STAYS THE SAME
5. THE CURRENT DENSITY IS DISTRIBUTED SUCH THAT MINIMUM POWER IS LOST

6.06 CURRENT FLOW IN TWO DIMENSIONS (INFINITE SHEET)

$W = U + jV = f(x + iy)$ = POTENTIAL / STREAM FUNCTION

$$i = \frac{1}{\gamma}\left|\frac{dW}{dz}\right| = \frac{1}{\gamma}\frac{\delta V}{\delta n} = -\frac{1}{\gamma}\frac{\delta U}{\delta s} \quad \text{FOR } V \text{ IS POTENTIAL}$$

$$i = \frac{1}{\gamma}\left|\frac{dW}{dz}\right| = \frac{1}{\gamma}\frac{\delta U}{\delta n} = \frac{1}{\gamma}\frac{\delta V}{\delta s} \quad \text{FOR } U \text{ IS POTENTIAL}$$

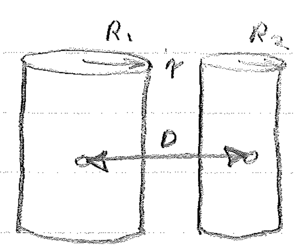


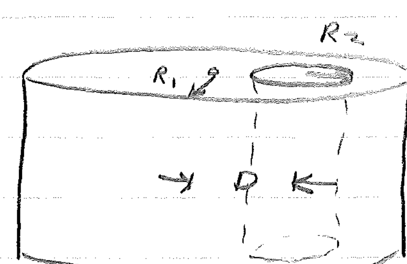
CURRENT FLOWING THRU A IS

$$I = \frac{V_2 - V_1}{\gamma}$$

$$R = \frac{U_2 - U_1}{I} = \gamma\frac{|U_2 - U_1|}{|V_2 - V_1|} \iff C = \frac{\epsilon[V]}{U_2 - U_1}$$

$$RC = \gamma\epsilon$$





$$R = \frac{\gamma}{2\pi}\cosh\frac{D^2 - R_1^2 - R_2^2}{2R_1R_2}$$

$$R = \frac{\gamma}{2\pi}\cosh - \frac{D^2 - R_1^2 - R_2^2}{2R_1R_2}$$

(CONT)

6.06. CURRENT FLOW IN TWO DIMENSIONS: $W = U + jV$

$$i\left(\frac{\text{AMP}}{\text{METER}}\right) = \frac{1}{\gamma}\left|\frac{dW}{dz}\right|$$

$$= \frac{1}{\gamma}\left|\frac{\delta V}{\delta n}\right| = -\frac{1}{\gamma}\frac{\delta U}{\delta s} \quad \text{FOR } V \text{ THE POTENTIAL}$$

$$= \frac{1}{\gamma}\left|\frac{\delta U}{\delta n}\right| = +\frac{1}{\gamma}\frac{\delta V}{\delta s} \quad \text{FOR } U \text{ THE POTENTIAL}$$

FOR A CONDUCTOR BOUND BY POTENTIALS U_1 & U_2

AND BY LINES OF FORCE V_1 & V_2:

$$I = \int_{V_1}^{V_2} i\,ds = \frac{1}{\gamma}\int_{V_1}^{V_2}\frac{\delta U}{\delta n}ds = \frac{1}{\gamma}\int_{V_1}^{V_2}\frac{\delta V}{\delta s}ds = \frac{V_2 - V_1}{\gamma}$$

$$R = \text{CONDUCTORS' RESISTANCE} = \gamma\frac{|U_2 - U_1|}{|V_2 - V_1|}$$

IF U_1 & U_2 ARE CLOSED CURVES, THEN

$$C = \frac{\epsilon[V]}{|U_2 - U_1|} \ni V \text{ IS INTEGRAL AROUND } U_2 \text{ OR } U_1$$

$$\Rightarrow R = \gamma\epsilon/C$$

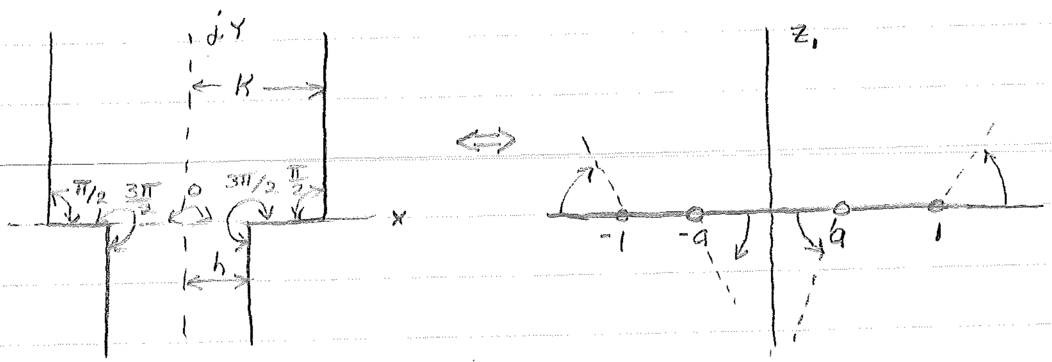
∴ RESISTANCE TWIXT TWO ELECTRODES IS

$$R = \frac{\gamma}{2\pi}\cosh^{-1}\pm\left(\frac{D^2 - R_1^2 - R_2^2}{2R_1R_2}\right)$$

$\ni R_1$ & R_2 ARE RADII, D IS SEPARATION

"+" ⇒ OUTSIDE EACH OTHER. "−" ⇒ ONE IS IN THE OTHER

6.07. LONG STRIP WITH ABRUPT CHANGE IN WIDTH



IN THE Z_1 PLANE, WE ARE BENDING THE REAL AXIS AS SHOWN. ORIGIN OF Z_1 GOES TO $Z = -i\infty$.

RECALL THE SCWARTZ XFORM (4-18 (7)):

$$\frac{dz}{dz_1} = C_1 (z_1 - U_1)^{\frac{\alpha_1}{\pi} - 1} (z_1 - U_2)^{\frac{\alpha_2}{\pi} - 1} \ldots$$

HERE, $U_1 = -1$, $U_2 = -a$, $U_3 = 0$, $U_4 = a$, $U_5 = 1$

$\alpha_1 = \frac{\pi}{2}$, $\alpha_2 = \frac{3\pi}{2}$, $\alpha_3 = 0$, $\alpha_4 = \frac{3\pi}{2}$, $\alpha_5 = \frac{\pi}{2}$

$$\Rightarrow \frac{dz}{dz_1} = C \frac{(z_1^2 - a^2)^{1/2}}{z_1 (z_1^2 - 1)^{1/2}} = \frac{cz}{[(z_1^2 - 1)(z_1^2 - a^2)]^{1/2}} - \frac{ca^2}{z_1[(z_1^2 - 1)(z_1^2 - a^2)]^{1/2}}$$

WE MUST NOW EVALUATE THE C'S. SET $r_1 = \text{CONST}$

SO THAT $dz_1 = j r_1 e^{j\theta_1} = j z_1 \, d\theta_1$

$$\Rightarrow dz = jC \frac{(z_1^2 - a^2)^{1/2}}{(z_1^2 - 1)^{1/2}} d\theta_1 = jc\left[(r_1^2 e^{2j\theta} - a^2)/(r_1^2 e^{2j\theta} - 1)^2\right]^{1/2} d\theta$$

- FOR $r_1 \ll 1$, AS $0 < \theta_1 < \pi$, $Y = -i\infty$, $+h > x > -h$

$$\int_h^{-h} dz = jc\left[\int_0^{\pi}\left[\frac{r_1^2 e^{j2\theta_1} - a^2}{r_1^2 e^{j2\theta_1} - 1}\right]^{1/2} d\theta \right|_{r_1 \to 0} = \pm jc \int_0^{\pi} a \, d\theta$$

- FOR $r_1 \gg 1$, AS $0 < \theta_1 < \pi$, $+k > x > -k$

$$\int_k^{-k} dz = \pm jc \int_0^{\pi} d\theta_1$$

$$\Rightarrow a = h/k \quad ; \quad c = \pm \frac{jkz}{\pi}$$

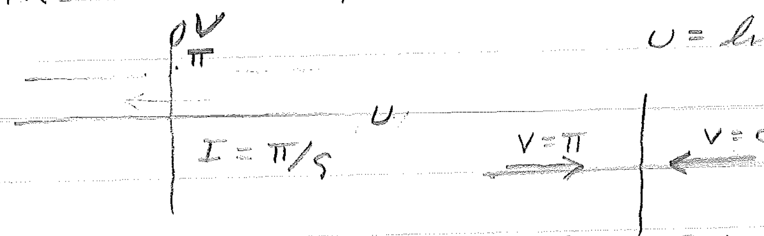
GIVES

$$z = \frac{2}{\pi}\left\{ k \tan^{-1}\left[\frac{(z_1^2 - a^2)}{(1 - z_1^2)}\right]^{\frac{1}{2}} + h \tan^{-1}\left[a\sqrt{\frac{(1 - z_1^2)}{z_1^2 - a^2}}\right]\right\}$$

6.07 (CONT)

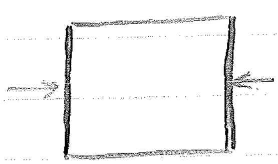
TAKE $W = \ln z_1 \quad (z_1 = e^W) \quad W = U + jV$

$U = \ln r_1 \quad V = \theta$

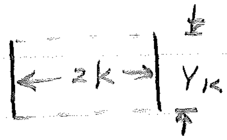


THUS $Z = \frac{2}{\pi}\left[k \tan^{-1}\left\{\frac{e^{2W} - a^2}{1 - e^{2W}}\right) + h \tan^{-1} a\left(\frac{1 - e^{2W}}{e^{2W} - a^2}\right)^{\frac{1}{2}}\right]$

CONSIDER A ONE METER CUT:



ρ IS RESISTANCE TWIXT OPPOSITE SIDES



$R_K = \frac{\rho / Y_K}{2K}$

6.08 CURRENT FLOW IN THREE DIMENSIONS

ELECTRODES (UNIFORM ISOTROPIC CONDUCTING MEDIA) $\Rightarrow$ SOLVE $\nabla^2 V = 0$

MAY SOLVE BY FINDING CAPACITANCE

- ELECTROSTATIC BOUNDRY CONDITIONS (ON ELECTRODE)

$V = V_a \qquad Q_a = -\int_S \epsilon \frac{\delta V}{\delta n} dS_a$

- CORRESPONDING CURRENT BOUNDRY CONDITIONS

$V = V_a \qquad I_a = -\int_S \frac{1}{\tau} \frac{\delta V}{\delta n} dS_a$

THE EQUAPOTENTIAL SURFACES IN BOTH OF THESE CASES CORRESPOND EXACTLY.

$RC = \epsilon_v \tau$: C IS CAPAC. IN VACUUM.

6.09 SYSTEMS OF ELECTRODES. TWO SPHERES. DISTANT ELEC.

FOR n PERFECTLY CONDUCTING ELECTRODES IN A HOMO ISOTROPIC MEDIUM

- WRITE I_s INSTEAD OF Q_s
- MULTIPLY CAPACITANCES BY $\frac{1}{\tau \epsilon_v}$

EX. TWO SPHERES (INTERNAL OR EXTERNAL) SEPARATED BY A DISTANCE c

$V_1 - V_2 = \tau \epsilon_v (S_{11} - 2S_{12} + S_{22}) I_1$

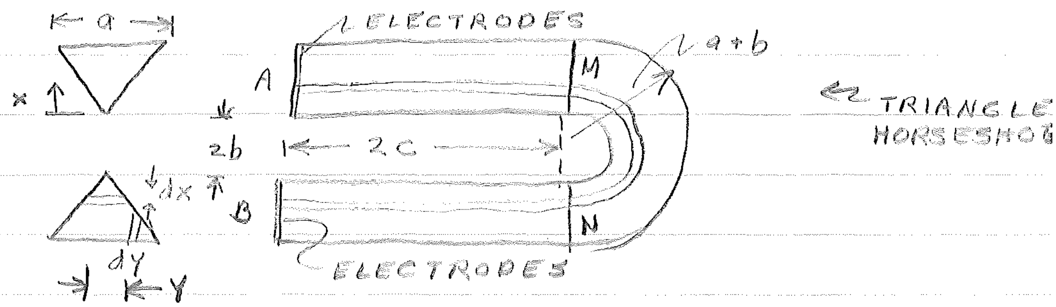
$\Rightarrow R = \left|\frac{V_1 - V_2}{I}\right| = \tau \epsilon_v (S_{11} - 2S_{12} + S_{22})$

● 6.16. LIMITS OF RESISTANCE

1) LOWER LIMIT: "INSERT" THIN SHEETS OF CONDUCTOR TO COINCIDE WITH EQUAPOTENTIAL LINES

2) UPPER LIMIT: "INSERT" THIN INSULATING SHEETS ALONG THE LINES OF FLOW.

EXAMPLE:



UPPER LIMIT: MAKE INSULATING SURFACE LENGTH $2c+\pi(b+x)$ & AREA $x\,dx$

$$dR_u = \frac{\gamma l}{A} = \frac{\gamma[2c+\pi(b+x)]}{x\,dx}$$

EVERYTHING IS IN PARALLEL, SO THAT

$$R_u = \left(\int \frac{1}{dR_u}\right)^{-1} = \left(\frac{1}{\gamma}\int_0^a \frac{x\,dx}{2c+\pi b+\pi x}\right)^{-1}$$

$$= \pi^2\gamma\left[\pi a - (2c+\pi b)\ln\frac{2c+\pi(a+b)}{2c+\pi b}\right]^{-1}$$

LOWER LIMIT: PUT PERFECT CONDUCTORS @ M & N

$\Rightarrow$ CURRENT IS UNIFORMLY DISTRIBUTED

IN EACH LEG, $R = \frac{4c\gamma}{a^2}$

IN SEMICIRCAL PART, $W = \ln z$

$U = \ln r \qquad V = \theta \qquad$ (V IS POTENTIAL)

THE RESISTANCE IN dy IS

$$dR = \gamma\left|\frac{U_2-U_1}{V_2-V_1}\right| = \gamma\pi\left(\ln\frac{a+b}{b+2y}\,dy\right)^{-1}$$

$$R = \left(\int dR\right)^{-1} = \frac{-\pi\gamma}{b\ln(a+b) - b\ln b - a}$$

AND THE LOWER LIMIT RESISTANCE IS

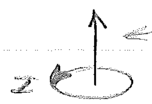
$$R_l = \gamma\left[\frac{4c}{a^2} - \frac{\pi\gamma}{b\ln(a+b) - b\ln b - a}\right]$$

VII. MAGNETIC INTERACTION OF CURRENTS

7.00 ● DEFINITION OF THE AMPERE IN TERMS OF THE MAGNETIC MOMENT

$$\mu_v = 4\pi \times 10^{-7}$$

7.01 ● MAGNETIC INDUCTION AND PERMEABILITY



MAGNETIC MOMENT (RIGHT HAND RULE)

$\nabla \cdot B = 0$; B = MAGNETIC INDUCTION OR FLUX DENSITY

$\nabla \times B = \mu i$

7.02 ● MAGNETIC VECTOR POTENTIAL. UNIFORM FIELD.

LET $B = \nabla \times A \Rightarrow \nabla \cdot A = 0$

A = VECTOR POTENTIAL

$$\Rightarrow \nabla^2 A = -\mu \vec{i}$$

$$A_x = \frac{\mu}{4\pi}\int_V \frac{i_x\, dv}{r} \qquad A_y = \frac{\mu}{4\pi}\int_V \frac{i_y\, dv}{r} \qquad A_z = \frac{\mu}{4\pi}\int_V \frac{i_z\, dv}{r}$$

$$\Rightarrow A = \frac{\mu}{4\pi}\int_V \vec{i}\, dv / r$$

OR $A = \frac{\mu}{4\pi}\int_S I\, ds / r$

IF $\vec{B}$ IS IN X DIRECTION ONLY

$$A_y = -\alpha Z B \qquad A_z = (1-\alpha) Y B$$

7.03 ● UNIQUENESS THEOREMS FOR MAGNETOSTATICS

7.04 ● ORTHOGONAL EXPANSIONS FOR VECTOR POTENTIAL

LET $A = \nabla \times W$

$\vec{W} = \vec{U} W_1 + \vec{U} \times \nabla W_2$

$\vec{U}$ IS AN ARBITRARY VECTOR

NOTE: COMPONENTS OF $\vec{W}$ ARE PERPENDICULAR

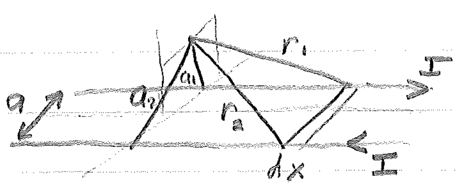
(IN MAGNETOSTATICS, W_2 OFFERS NOTHING TO B)

7.05 ● VECTOR POTENTIAL IN CYLINDRICAL COORDINATES

7.06 ● VECTOR POTENTIAL IN SPHERICAL COORDINATES

7.07 ● VECTOR POTENTIAL IN TERMS OF MAGNETIC INDUCTION ON AXIS → MAGNETIC LENSES

7.09 ● VECTOR POTENTIAL AND FIELD OF BIFILAR CIRCUIT



$\bar{A}$ IS IN DIRECTION OF $\bar{i}$

FROM 7.02 $\quad A = \frac{\mu}{4\pi} \oint \frac{I\,ds}{r}$

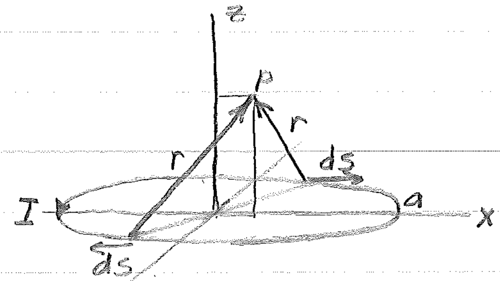
$$\Rightarrow A_x = \frac{\mu I}{4\pi} \int_{-\infty}^{\infty} \left(\frac{1}{r_1} - \frac{1}{r_2}\right) dx = \frac{\mu I}{2\pi} \ln a_2/a_1$$

$$B_x = 0$$

$$B_z = \frac{-\delta A_x}{\delta Y} = -\frac{\mu I}{2\pi}\left[\frac{Y + \frac{1}{2}a}{a_2{}^2} - \frac{Y - \frac{1}{2}a}{a_1{}^2}\right]$$

$$B_Y = \frac{\delta A_x}{\delta Z} = \frac{\mu I Z}{2\pi}\left(\frac{1}{a_2{}^2} - \frac{1}{a_1{}^2}\right)$$

7.10 VECTOR POTENTIAL & FIELD OF CIRCULAR LOOP



A WILL BE ϕ INDEPENDENT
$\therefore$ LET P BE ON XZ PLANE
(ie, $\phi = 0$

IN GENERAL (FROM 7.02):

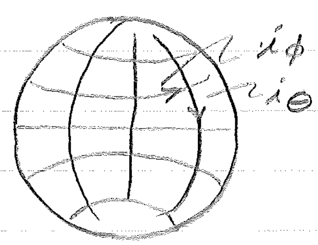
$$A = \frac{\mu}{4\pi}\int_S \frac{I\,ds}{r}$$

HERE: $A_\phi = \frac{\mu I}{4\pi}\oint \frac{ds_\phi}{r} = a^2 \mu I \sin\theta / 4r^2$

ON AXIS, $B_\rho = B_\phi = 0$, $B_z = \frac{1}{2}\mu a^2 / (a^2+z^2)^{3/2}$

7.11 FIELD OF CURRENTS IN SPHERICAL SHELL

7.12 ZONAL CURRENTS IN SPHERICAL SHELL



ψ = STREAM FUNCTION

IF CURRENT FLOWS IN "LATITUDE"

$$\psi = \sum_{n=1}^{\infty} C_n P_n(\cos\theta)$$

$$i_\phi = -\sum_n \frac{1}{a} C_n P_n'(\mu)$$

FOR $r > a$ $\quad A = \vec{\phi}\,\mu \sum_n \frac{-C_n}{(2n+1)}\left(\frac{a}{r}\right)^{n+1} P_n'(\mu)$

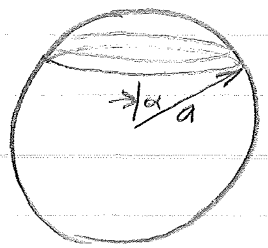
$$B_r = -\frac{\mu}{a}\sum_n \frac{n(n+1)}{2n+1} C_n \left(\frac{a}{r}\right)^{n+2} P_n(\mu)$$

$$B_\theta = -\frac{\mu}{a}\sum_n \frac{n\,C_n}{2n+1}\left(\frac{a}{r}\right)^{n+2} P_n'(\mu)$$

IN ORDER TO MAKE FIELD UNIFORM INSIDE:

$$N = 2\pi r \sin\theta \times A_\phi$$

7.13 • FIELD OF CIRCULAR LOOP IN SPHERICAL HARMONICS



CURRENT DENSITY NON-ZERO

ONLY WHEN $\theta = \alpha$

$$i = -\frac{1}{a}\sum_n C_n P_n^1(\mu)$$

$$C_n = -\frac{(n-1)!}{(n+1)!}\,\frac{2n+1}{2}\int_{-1}^{1} a\, i\, P_n^1(\mu)\, d\mu$$

$$= -\frac{(2n+1)\, I \sin\alpha}{2n(n+1)}\, P_n^1(\cos\alpha)$$

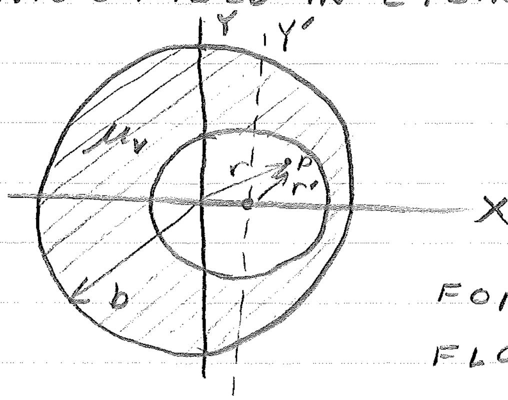
$$\Rightarrow A = \vec{\phi}\,\frac{\mu I}{2}\sum_n \frac{\sin\alpha}{n(n+1)}\left(\frac{r}{a}\right)^n P_n^1(\cos\alpha)\, P_n^1(\cos\theta)$$

B_r & B_θ FOLLOW

7.14 • BIOT AND SAVART'S LAW. FIELD OF STRAIGHT WIRE

$B @ p = \mu I / 2\pi a$

7.15 • FIELD IN CYLINDRICAL HOLE IN CONDUCTING ROD



i_z IS UNIFORM

WITHOUT "HOLE"

$$\mu \pi r^2 i = \oint B' \cdot ds = 2\pi r B'_\theta$$

$$\Rightarrow B' = \tfrac{1}{2}\mu_v\, \vec{i} \times \vec{r}$$

FOR A CURRENT DENSITY, $\vec{i}$, FLOWING ONLY IN INNER CYLINDER:

$$\bar{B} = \tfrac{1}{2}\mu_v\, \vec{i} \times (\bar{r} - \bar{r}^{\circ}) = \vec{j}\, \tfrac{1}{2}\mu_v\, i\, C$$

THE TOTAL CURRENT IS $I = \pi(b^2 - a^2)\,\vec{i}$

SO, SUBTRACTING THE $\vec{B}$'S GIVES

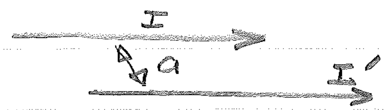
$$B_Y = \mu_v C I / 2\pi(b^2 - a^2)$$

7.18 ● FORCE ON ELECTRIC CIRCUIT IN MAGNETIC FIELD

$$\vec{F} = I \oint d\vec{s} \times \vec{B} \qquad (F = \vec{v}q \times B = \text{LORENTZ FORCE})$$

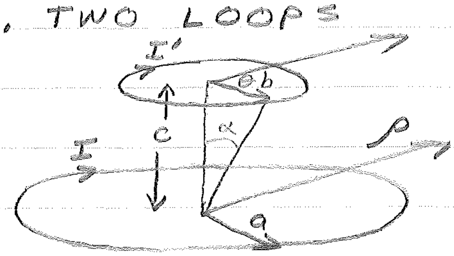
7.19 ● EXAMPLES OF FORCES TWIXT ELECTRIC CIRCUITS

1. TWO INFINATE PARALLEL WIRES



$$F = -\mu I I'/2\pi a$$

2. TWO LOOPS



ONLY B_ρ COMPONENT

$$F = -\pi\mu I I' \sin\alpha \sum_n \left(\frac{a^2}{a^2+c^2}\right)^{\frac{n}{2}} P_n^1(\cos\alpha) P_n(\cdots$$

7.20 ● VECTOR POTENTIAL & MAGNETIZATION

USE MAGNETIZATION, $\vec{M}$, WHEN μ IS NOT UNIFORM

$$\vec{M} = \left(\frac{1}{\mu_v} - \frac{1}{\mu}\right)\vec{B}$$

$$I = \oint B \cdot ds/\mu = I$$

7.21 ● MAGNETIC BOUNDRY CONDITIONS

$$\frac{\delta V_o}{\delta n} - \frac{\delta V_i}{\delta n} = -\sigma/\epsilon_v \quad ; V_o = V_i$$

$$A' = A'' \quad , \quad \frac{\delta A'}{\delta n} - \frac{\delta A''}{\delta n} = \mu_v[(M' - M'') \times \vec{n}\,]$$

[EQUIVALENT TO $n \times \left[\frac{B'}{\mu'} - \frac{B''}{\mu''}\right] = 0$]

(PLUS OTHER GIVEN CONDITIONS)

7.28 MAGNETOMOTANCE AND MAGNETIC INTENSITY

SOME DUALITIES

$\nabla \cdot \bar{i} = 0$ $\qquad\qquad$ $\nabla \cdot B = 0$

$\oint \frac{i \cdot ds}{\gamma} = \mathcal{E}$ $\qquad\qquad$ $\oint \frac{B \cdot ds}{\mu} = I$

$i = \gamma \nabla \mathcal{E}$ $\qquad\qquad$ $B = -\mu \nabla \Omega$

[Ω IS MAGNETOMOTANCE $= nI$ (IN AMP-TURNS)]

$D = \epsilon E$ $\qquad\qquad$ $B = \mu H$

(BOUNDRY CONDITIONS GIVEN FOR $\bar{H}$)

H = MAGNETIC FIELD INTENSITY

VIII ELECTROMAGNETIC INDUCTION

8.00 ● FARADAY'S LAW OF INDUCTION

$$\nabla \times E = -\,\delta B/dt$$

OR $E = -dA/dt$

8.01 ● MUTUAL ENERGY OF TWO CIRCUITS

BRING TWO LOOPS, CARRYING CURRENTS I & I', TOGETHER. THE CURRENTS ARE KEPT CONSTANT BY APPROPRIATELY CHANGING THE $\mathcal{E}$ OF THEIR BATTERIES. HALF OF THE ENERGY SUPPLIED BY THE BATTERIES IS USED DOING THE MECHANICAL WORK. HALF GOES INTO THE MAGNETIC FIELD.

8.02 ● ENERGY IN A MAGNETIC FIELD

$$W_B = \frac{1}{2\mu}\int_V \bar{B}^2\, dV$$

$$W_B/V = B^2/2\mu = \text{ENERGY DENSITY}$$

8.03 ● MUTUAL INDUCTANCE

DEF: FLUX, N_{12} THROUGH CIRCUIT 1 PRODUCED BY 2

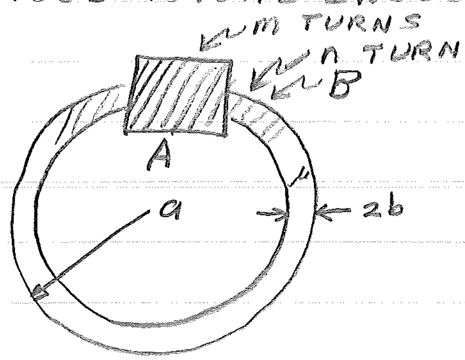
$$M_{12} = \int_{S_1} B_2 \cdot n\, dS_1 = M_{21}$$

$$= \oint A_2 \cdot dS_1$$ ⇐ FOLLOWS FROM STOKES THEM.

$$= \frac{\mu}{4\pi}\oint\oint \frac{d\bar{S}_1\, d\bar{S}_2}{r}$$

TORQUE: $F = I_1 I_2\, \delta M_{12}/\delta\Theta$

8.05 ● MUTUAL INDUCTANCE OF SIMPLE CIRCUITS



$$M_{12} = \mu\, n m \left[a - (a^2 - b^2)^{\frac{1}{2}} \right]$$

IF $a >> b$, $M_{12} = \mu m n b^2 / 2a$

8.06 ● MUTUAL INDUCTANCE OF TWO COAXIAL LOOPS

8.07 ● VARIABLE MUTUAL INDUCTANCE

8.08 ● SELF-INDUCTANCE

$$W_B = \frac{1}{2\mu} \int_v B^2 dv = \frac{1}{2} L_{11} I_1^2 = \frac{1}{2} \int i \cdot A\, dv$$

$$L_{11} = N_{11} / I_1$$

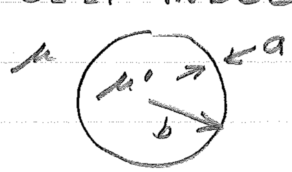
$$\mathcal{E}_1 = -L_{11}\, dI_1/dt$$

8.09 ● SELF INDUCTANCE OF A THIN WIRE

$L_{11}' =$ SELF INDUCTANCE PER UNIT LENGTH

$= \mu' / 8\pi$

8.10 ● SELF INDUCTANCE OF CIRCULAR LOOP



$$L_{11} \simeq b \left[\mu \left(\ln \frac{8b}{a} - 2 \right) + \frac{1}{4} \mu' \right]$$

IX. MAGNETISM

9.00 • PARAMAGNETISM AND DIAMAGNETISM

$\mu > 1 \Rightarrow$ PARAMAGNETIC

$\mu < 1 \Rightarrow$ DIAMAGNETIC (TEMP. INDEPENDENT)

$\mu >> 1 \Rightarrow$ FERROMAGNETIC

IN STRONG PARAMAGNETIC (BUT NOT DIAMAGNETIC)

$$\mu = \mu_v + \mu_v C / T + \Theta \Leftarrow \text{CURIE'S LAW}$$

9.01 • MAGNETIC SUSCEPTABILITY

$$\vec{M} = \kappa \vec{H} = \left(\frac{1}{\mu_v} - \frac{1}{\mu}\right)\bar{B}$$

κ = MAGNTETIC SUSCEPTABILITY

$$= \left(\frac{1}{\mu_v} - \frac{1}{\mu}\right) = K_m - 1 \quad = \frac{C}{T + \Theta}$$

ENERGY INCREASE (DENSITY)

$$\Delta W/_V = -\frac{\mu_v}{2} \kappa H^2$$

9.02 • MAGNETIC PROPERTIES OF CRYSTALS

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}$$

$\mu_{12} = \mu_{21}$ ETC

BY SUITABLY ORIENTING AXES:

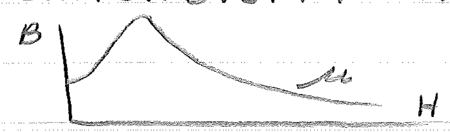
$$B_x = \mu_1 H_x \qquad B_y = \mu_2 H_y \qquad B_z = \mu_3 H_z$$

ON AXIS, ALSO

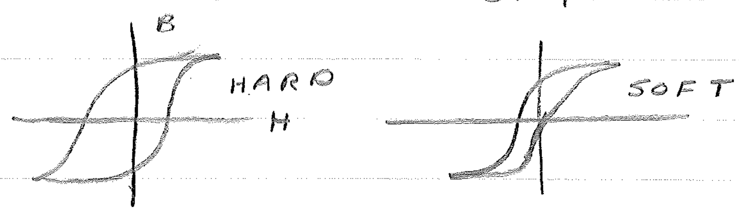
$$M_x = \kappa_1 H_x \qquad M_y = \kappa_2 H_y \qquad M_z = \kappa_3 H_z$$

9.04 • FERROMAGNETISM

PERMEABILITY VARIES WITH B



9.05 • HYSTERISIS. PERMANENT MAGNETISM



9.06 • THE NATURE OF PERMANENT MAGNETISM

9.07 • UNIFORM MAGNETISM. EQUIVALENT SHELL CURRENT

THE VECTOR POTENTIAL DUE TO M IS

$$\vec{A}_M = \frac{\mu_0}{4\pi} \int_S \frac{\bar{M} \times \bar{n}}{r} \, dS \quad \text{(FOR UNIFORM M)}$$

$$= \frac{\mu_0}{4\pi} \iint \frac{M \, d \times}{r} \, d\vec{S}$$

$\hat{i} \propto M$ FOR SUCH A CASE

X. EDDY CURRENTS

10.00 INDUCED CURRENTS IN EXTENDED CONDUCTORS

$$\left.\begin{aligned}\nabla \times \bar{E} &= -\frac{dB}{dt} \\ E &= -\frac{dA}{dt}\end{aligned}\right\} \Leftarrow \text{FARADAY'S LAW}$$

$$\left.\begin{aligned}\gamma(\nabla \times \vec{i}) &= -\frac{dB}{dt} \\ \gamma \vec{i} &= -\frac{dA}{dt}\end{aligned}\right\} \Leftarrow \text{OHM'S LAW}$$

$$\left.\begin{aligned}\nabla \times B &= \mu \vec{i} \\ \nabla^2 A &= -\mu \vec{i}\end{aligned}\right\} \Leftarrow \text{AMPERES LAW}$$

MAY COMBINE THESE TO GIVE:

$$\frac{\mu}{\gamma}\frac{dA}{dt} = \nabla^2 A \qquad \frac{\mu}{\gamma}\frac{dB}{dt} = \nabla^2 B$$

AVERAGE POWER OVER A CYCLE $d\bar{P} = \frac{1}{2}\gamma \hat{i}\check{i}\,dv$

$\check{i} = \hat{i}$ CONJUGATED

10.01 SOLUTION FOR VECTOR POTENTIAL FOR EDDY CURRENTS

$$A = \nabla \times (\vec{U} W_1 + \bar{U} \times \nabla W_2)$$

GIVES $\nabla^2 W = \frac{\mu}{\gamma}\frac{dw}{dt}$

$$\vec{\phi} = \bar{i}, \bar{j}, \bar{k} \Rightarrow \vec{B} = -\frac{\mu}{\gamma}\frac{d}{dt}[\bar{\phi} \times \nabla W_2 + \vec{U} W_1] + \bar{\phi} \cdot \nabla(\nabla W_1$$

$$\phi = \vec{r} \Rightarrow \vec{B} = -\frac{\mu}{\gamma}\frac{d}{dt}[\vec{r} \times \nabla W_2 + \vec{r} W_1] + \bar{r} \cdot \nabla(\nabla W_1) + 2\bar{\nabla} W_1$$

10.02 STEADY STATE SKIN EFFECT

$\hat{i}_x$ IS UNIFORM

$\bar{P}$ = POWER ABSORBED PER METER $= \frac{1}{2}\omega\mu\delta I_e^2$

$\delta = (\frac{1}{2}\omega\mu\gamma)^{-\frac{1}{2}}$ = SKIN DEPTH

$$B_0 = \sqrt{2}\,\mu_v I_e$$

$$L_i = \frac{\mu}{\omega\delta} = \frac{1}{\omega\delta\gamma} = \frac{R'}{\omega} \quad (\text{L \& R PER SQUARE METER})$$

10.03 SKIN EFFECT ON TUBULAR CONDUCTOR

10.04 SKIN EFFECT ON SOLID CYLINDRICAL CONDUCTOR

10.09 ● EDDY CURRENTS IN PLANE SHEETS

THIN SHEET. UNIFORM CURRENT DENSITY $\vec{i}$

FLUCTUATING MAGNETIC FIELD $A'(x,y,z,t)$.

RESULTING MAGNETIC FIELD $A(x,y,z,t)$

A_S = TANGENTAL COMPONENTS OF A

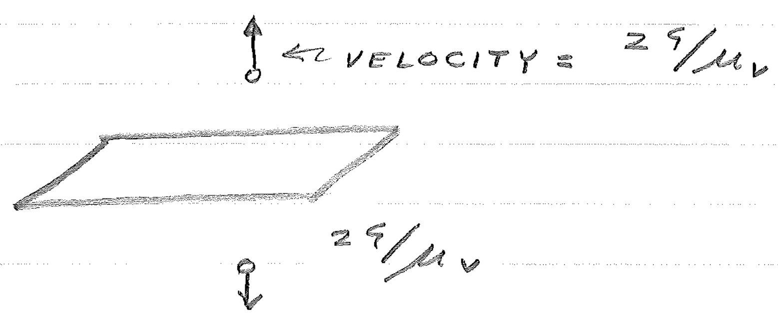
$$\vec{i} = \frac{2}{\mu_v} \frac{\delta A_s}{\delta z}$$

10.10 ● EDDY CURRENTS IN PLANE SHEET BY IMAGE METHOD

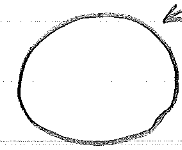
A FROM $\frac{dA}{dt} = \frac{2\rho}{\mu_v} \frac{dA}{dz}$

$f_1 = A_1' \Rightarrow t < 0$ $\qquad$ $f_2 = A_2' \Rightarrow t > 0$

$$A = f_1(x, y, -|z| - 2\rho\mu_v^{-1} t) - f_2(x, y, -|z| - 2\rho\mu_v^{-1} t)$$



10.16 • ZONAL EDDY CURRENTS IN SPHERICAL SHELL



THIN CONDUCTING SHELL

ALL EDDY CURRENTS FLOW IN COAXIAL SHELLS

$A' + A$ = TOTAL VECTOR POTENTIAL

A = VECTOR POTENTIAL DUE TO EDDY CURRENTS

LET $\epsilon_n \Rightarrow \tan \epsilon_n = -(2n+1)\rho / \mu_v a \omega$

FOR SMALL ϵ_n WE HAVE GOOD SHEILDING

10.18 • GENERAL EDDY CURRENTS IN SPHERICAL SHELL

10.19 • TORQUE ON SPINNING SPHERICAL SHELL TWIXT MAGNETIC POLES

10.20 • EDDY CURRENTS IN THIN CYLINDRICAL SHELLS

PLANE ELECTROMAGNETIC WAVES

11.00 ● MAXWELL'S FIELD EQUATIONS

AMPERE'S LAW: $\nabla \times H = \nabla \times \frac{B}{\mu} = \bar{i} + \frac{\delta D}{\delta t} + \rho V$

FARADAY'S LAW: $\nabla \times E = -\delta B/\delta t$

$$\nabla \cdot D = \rho$$

$$\nabla \cdot B = 0$$

WE ALSO HAVE:

CHARGE CONSERVATION: $\nabla \cdot \bar{i} = -\frac{\delta \rho}{\delta t}$

OHM'S LAW: $E = r\bar{i}$ $\quad (\bar{i} = \gamma \bar{E})$

11.01 ● PROPAGATION EQ., DYNAMIC POTENTIALS. GAUGES. HERTZ VECTOR.

IN UNIFORM MEDIUM WITH NO CHARGES

$$\nabla^2 \bar{B} = \upsilon\gamma \frac{\delta B}{dt} + \mu\epsilon \frac{\delta^2 B}{\delta t^2}$$

$$\nabla^2 \bar{E} = \upsilon\gamma \frac{dE}{dt} + \mu\epsilon \frac{\delta^2 E}{\delta t^2}$$

$$\nabla^2 A - \mu V \frac{\delta A}{\delta t} - \mu\epsilon \frac{\delta^2 A}{\delta t^2} = -\mu_s i_s$$

- COULOMB GAUGE $\quad \nabla^2 \psi = \frac{\rho}{\epsilon}$
- LORENTZ GAUGE $\quad \nabla^2 \psi = \delta \nabla \cdot A/\delta t$
- HERTZ VECTOR: $\quad \psi = -\bar{\nabla} \cdot \mathbf{Z}$

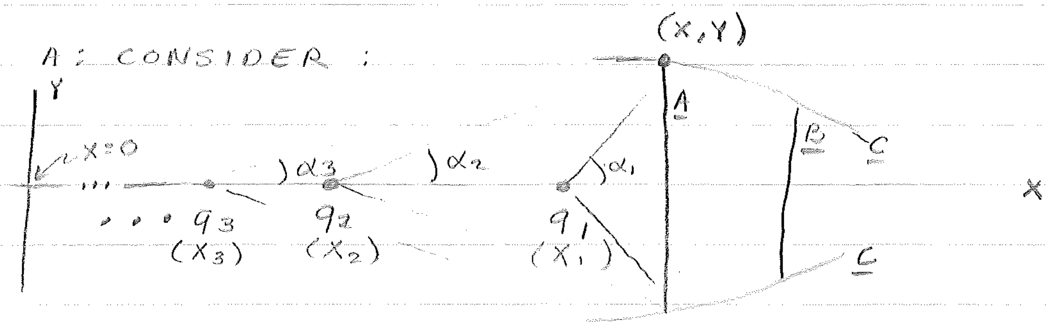
11.02 ● POYNTING VECTOR

$$\Pi = E \times B/\mu$$

BOB MARKS
EE 5341
ADV. FIELDS I
DUE 2/24/76

1. P: FOR A GIVEN SET OF COLINEAR CHARGES, FIND THE RELATIONSHIP DESCRIBING THE EQUAL LINES OF FORCE.

A: CONSIDER :



WE HYPOTHESIZE THE "LINES" OF FORCE C. ALL FLUX PASSING THROUGH SURFACE A MUST PASS THRU B (SINCE NONE GOES THROUGH C). ASSUME THE POINT CHARGE q_i WILL GIVE OUT q_i FLUX LINES (AS IN TUBES OF FORCE). THE NUMBER OF FLUX LINES, N_i, DUE TO q_i WHICH PASS THRU A ARE:

$$N_i = \frac{\Omega_i}{4\pi} q_i$$

WHERE Ω_i IS THE SOLID ANGLE WHICH A SUBTENDS AT X_i. IT FOLLOWS THAT

$\Omega = \frac{2\pi R h}{R^2} = 2\pi(1-\cos\alpha)$

$$N = \sum_{i=1}^{n} \frac{1}{4\pi} \Omega_i q_i$$
$$= \sum_{i=1}^{n} \frac{1}{4\pi} \left(2\pi \sin^2 \frac{\alpha_i}{2}\right) q_i$$
$$= \sum_{i=1}^{n} \frac{1}{2} (1 - \cos\alpha_i) q_i$$
$$= \frac{1}{2} Q - \frac{1}{2} \sum_{i=1}^{n} \cos\alpha_i \; q_i$$

WHERE THE CONSTANT Q IS

$$Q = \sum_{i=1}^{n} q_i$$

WE WISH TO SET $N = C$ = CONSTANT. IT FOLLOWS THAT

$$C = \frac{1}{2}Q - N = \frac{1}{2}\sum_{i=1}^{n} q_i \cos\alpha_i$$

$$= \frac{1}{2}\sum_{i=1}^{n} \frac{q_i(x - x_i)}{\sqrt{Y^2 + (x - x_i)^2}}$$

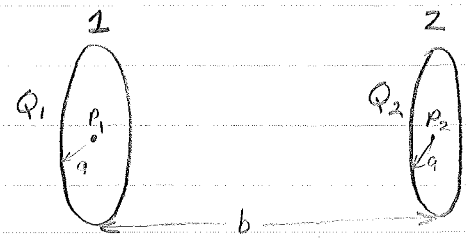
SINCE Φ (TENSION) AND ψ (FORCE) ARE FUNCTIONS OF ϵ & E ONLY, THE ORIGIN OR SHAPE OF THE FIELD IS IMMATERIAL. WE HAVE (IN HOMO/ISOTROPIC MEDIA)

$$\psi(E) = -\epsilon E^2/2$$
$$\Phi(E) = \epsilon E^2/2$$

FOR OUR PROBLEM:

$$\vec{E} = \frac{-1}{4\pi\epsilon}\sum_{i=1}^{n} \frac{q_i}{r_i^3}\vec{r}_i$$

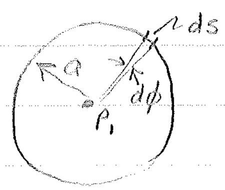
1.1



FIRST, FIND THE POTENTIAL AT P_1 BY SUPERPOSITION:

$$V_{P_1} = V_{P_1}^{(1)} + V_{P_1}^{(2)} \quad (V_{P_1}^{(i)} \text{ DUE TO RING } i,\ i = 1, 2)$$

FIRST, FIND $V_{P_1}^{(1)}$:

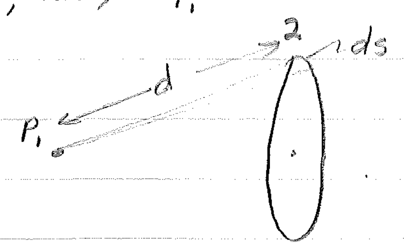


$$dV_{P_1}^{(1)} = \frac{\rho_1 ds}{4\pi\epsilon a} \quad ; \rho_1 = \frac{Q_1}{2\pi a}$$

$$= \frac{\rho_1 (a\, d\phi)}{4\pi\epsilon a} = \frac{\rho_1 d\phi}{4\pi\epsilon}$$

$$V_{P_1}^{(1)} = \frac{\rho_1}{4\pi\epsilon} \int_0^{2\pi} d\phi = \frac{\rho_1}{2\epsilon} = \frac{Q_1}{4\pi\epsilon a}$$

FIND, NOW, $V_{P_1}^{(2)}$



$$dV_{P_1}^{(2)} = \frac{\rho_2 ds}{4\pi\epsilon d}$$

$$= \frac{\rho_2 a\, d\phi}{4\pi\epsilon d}$$

$$\Rightarrow V_{P_1}^{(2)} = \frac{\rho_2 a}{2\epsilon d} = \frac{Q_2}{4\pi\epsilon d}$$

WHERE, AS BEFORE, $\rho_2 = \frac{Q_2}{2\pi a}$

ALSO $d = \sqrt{a^2 + b^2}$

THIS GIVES:

$$V_{P_1} = \frac{1}{4\pi\epsilon}\left[\frac{Q_2}{d} + \frac{Q_1}{a}\right]$$

DUE TO SYMMETRY, WE MAY ALSO WRITE:

$$V_{P_2} = \frac{1}{4\pi\epsilon}\left[\frac{Q_1}{d} + \frac{Q_2}{a}\right]$$

THE WORK, W_i, REQUIRED TO BRING A TEST CHARGE q_i (FROM ∞ WHERE $V = 0$) TO P_i IS:

$$W_i = q\, V_{P_i}$$

(CONT ⇒)

THUS

$$W_1 = \frac{q}{4\pi\epsilon}\left[\frac{Q_2}{d} + \frac{Q_1}{a}\right] \Rightarrow Q_2 = \frac{4\pi\epsilon W_1 d}{q} - \frac{Q_1 d}{a}$$

$$W_2 = \frac{q}{4\pi\epsilon}\left[\frac{Q_1}{d} + \frac{Q_2}{a}\right] \Rightarrow Q_2 = \frac{4\pi\epsilon W_2 a}{q} - \frac{Q_1 a}{d}$$

EQUATING GIVES:

$$\frac{4\pi\epsilon W_1 d}{q} - \frac{Q_1 d}{a} = \frac{4\pi\epsilon W_2 a}{q} - \frac{Q_1 a}{d}$$

SOLVING FOR Q_1:

$$Q_1\left[\frac{a}{d} - \frac{d}{a}\right] = \frac{4\pi\epsilon}{q}\left[a W_2 - d W_1\right]$$

$$Q_1 \frac{d^2 - a^2}{a d} = \frac{4\pi\epsilon}{q}\left[d W_1 - a W_2\right]$$

$$\Rightarrow Q_1 = \frac{4\pi\epsilon a d}{q(d^2 - a^2)}\left[d W_1 - a W_2\right]$$

AGAIN: $d = \sqrt{a^2 + b^2} \Rightarrow d^2 - a^2 = b^2$

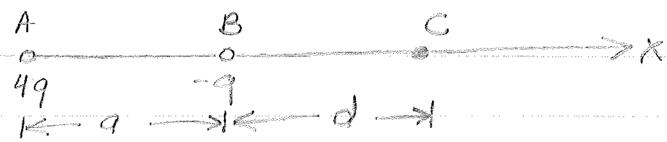
AND

$$Q_1 = \frac{4\pi\epsilon a\sqrt{a^2+b^2}}{q b^2}\left[\sqrt{a^2+b^2}\, W_1 - a W_2\right]$$

DUE TO THE PROBLEM'S SYMMETRY, WE MAY IMMEDIATELY WRITE:

$$Q_{1,2} = \frac{4\pi\epsilon a\sqrt{a^2+b^2}}{q b^2}\left[\sqrt{a^2+b^2}\, W_{1,2} - a W_{2,1}\right]$$

1.106C.



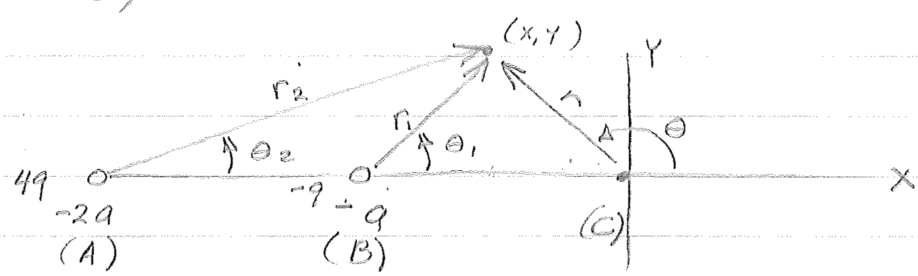
OBVIOUSLY, C MUST LIE TO THE RIGHT OF B, SINCE AT C (EQUALIBRIUM), WE REQUIRE $\vec{E} = -\nabla V = 0$. TO FIND THIS POINT EXPLICITLY:

$$\vec{E} = \frac{1}{4\pi\epsilon}\left[\frac{4q}{(a+d)^2} - \frac{q}{d^2}\right] = 0$$

OR

$$\frac{4}{(a+d)^2} = \frac{1}{d^2} \Rightarrow d = a \qquad \text{what with } d = -\tfrac{1}{3}a \;?$$

THUS, WE HAVE



THE EQ. FOR LINES OF EQUAL FORCE (FROM 1.101 (1)) ARE GIVEN BY

$$\frac{1}{q}C = \frac{4\,(x+2a)}{\sqrt{(x+2a)^2+y^2}} - \frac{(x+a)}{\sqrt{(x+a)^2+y^2}} \qquad (1)$$

$$= 4\cos\theta_2 - \cos\theta_1$$

THE LINE OF FORCE PASSING THRU (0,0)@C HAS A C GIVEN BY

$$\frac{C_0}{q} = \frac{8a}{2a} - \frac{a}{a} = 3$$

THE LINE IS THUS DESCRIBED BY

$$3 = 4\cos\theta_2 - \cos\theta_1 \quad \Leftarrow \text{PASS THRU C} \qquad (2)$$

[THE CORRESPONDING POTENTIAL TO Eq. 1 IS

$$\frac{1}{\sqrt{(x+a)^2+y^2}} - \frac{4}{\sqrt{(x+2a)^2+y^2}} = C'$$

C' FOR $x=y=0$ IS $\frac{1}{a} - \frac{1}{2a} = C_0' = \frac{1}{a}$]

ALL LINES OF FORCE INTERSECT POINTS A & B (ala FIG 1.08).

DUE TO THE PROBLEM'S SYMMETRY, THE LINES OF FORCE WILL BE SYMMETRIC ABOUT THE X AXIS. THERE WILL BE NO DISCONTINUITY OF THE LINE PASSING THRU C. IT STANDS TO REASON THAT THE LINE PASSES THRU C AT 90°. TO DETERMINE THIS ANALYTICALLY, ONE MAY SOLVE (OR APPROXIMATE) Eq. 2 IN THE FORM

$$Y = f(x)$$

(I WASN'T ABLE TO TO THIS). IF $\left.\frac{dy}{dx}\right|_{x=0} = \infty$, THEN THE 90° CONJECTURE IS PROVED. SIMILARLY, $\left.\frac{dy}{dx}\right|_{x=-2a} = \tan 60° = \sqrt{3}$ WOULD SUFFICE FOR SHOWING Eq. 2 PASSES THRU A AT AN

$\frac{dy}{d\theta} = 0$

$\frac{\partial f}{\partial y} = 0$

$\frac{dy}{dx} = \frac{\partial f/\partial x}{\partial f/\partial y}$

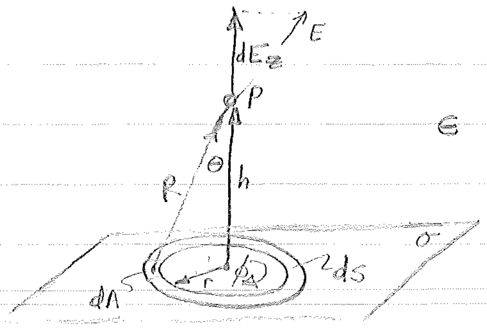
$f(x,y) = C$

$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$

COULD IN PRINCIPLE BE APPLIED TO
THE SECOND HALF OF THE
PROBLEM. AGAIN, I COULDN'T GET
A HANDLE ON IT.

Contact Dale Wilson
to see how he did it

1.18 C.



ϵ

$h = \frac{1}{2}''$

WE ARE GIVEN THAT THE TOTAL E FIELD (WHICH BY SYMMETRY, WILL POINT "UP") HAS AN INTENSITY $= \frac{\sigma}{2\epsilon}$ (WILL SHOW THIS).

FOR A POINT CHARGE, $\vec{E} = \frac{q\,\vec{r}}{4\pi\epsilon r^3}$

THE Z COMPONENT OF $\vec{E}$ DUE TO THE AREA dA IS

$$dE_{z_{(A)}} = \frac{\sigma\, dA}{4\pi\epsilon\, R^2} \cdot \cos\theta$$

BUT $dA = r\,dr\,d\phi$

$$\Rightarrow dE_{z_{(A)}} = \frac{\sigma\, r\,dr\,d\phi}{4\pi\epsilon\, R^2}$$

THE E FIELD INTENSITY DUE TO dS IS THEN

$$dE_{z_S} = \oint_{dS} dE_{z(A)}$$

$$= \cos\theta \frac{\sigma\, r\,dr}{4\pi\epsilon R^2} \int_0^{2\pi} d\phi = \frac{r\,\sigma\,dr}{2\epsilon_0 R^2} \cdot \cos\theta$$

BUT $R = \frac{h}{\cos\theta}$ $\quad r = h\tan\theta \Rightarrow dr = h\sec^2\theta\, d\theta$

THUS

$$dE_{z_s} = \frac{\sigma\, h\, \cos^2\theta\, d\theta\, \sin\theta \cdot h \cdot \cos\theta}{2\epsilon \cos^2\theta\; h^2\; \cancel{\cos\theta}\, \cos\theta} = \frac{\sigma}{2\epsilon} \sin\theta\, d\theta$$

THE TOTAL INTENSITY, AS PREVIOUSLY MENTIONED, IS

$$E = \int_{PLANE} dE_{z_s} = \frac{\sigma}{2\epsilon} \int_0^{\pi/2} \sin\theta\, d\theta$$

E_z (?) but $E_z \stackrel{?}{=} E$ $\quad = \frac{\sigma}{2\epsilon}$

~~...~~ applying Gauss theorem for infinite plane (CONT →)
it could be done "in one line".

WE WISH TO CONSIDER THE CASE WHERE $R = 2h$ (OR MORE SPECIFICALLY, $h = \frac{1}{2}''$ AND $R = 1''$). THE $\vec{E}$ FIELD RESULTING FROM WITHIN THE CORRESPONDING CIRCLE (RADIUS $= \frac{\sqrt{3}}{2}''$) IS THUS

$$E_{z\,(\text{CIRCLE})} = \frac{\sigma}{2\epsilon} \int_0^{\cos^{-1}\frac{h}{R} = 60^\circ = \frac{\pi}{3}} \sin\theta \, d\theta$$

$$= \frac{\sigma}{2\epsilon} \; -\cos\theta \Big|_0^{\pi/3}$$

$$= \frac{\sigma}{4\epsilon} = \frac{1}{2} E_z$$

THUS, HALF THE TOTAL E FIELD RESULTS FROM THE CHARGE DISTRIBUTION WITHIN THE CIRCLE. THE OTHER HALF OBVIOUSLY COMES FROM THE CHARGE EXTERNAL TO THE SURFACE.

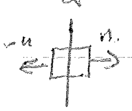
2●P. FIND $\vec{D}$ ON SURFACE OF TWO CONDUCTORS

#1 : PLANE CONDUCTOR: THICKNESS = 0

#2 : SLAB OF THICKNESS d

BOTH HAVE SAME DIMENSIONS (SAY $a \times a = A$)

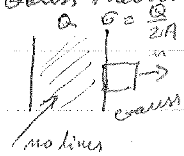
WHERE $a \gg d$.



① CASE 1: BOTH CONDUCTORS HAVE SAME CHARGE Q:

#1 WILL HAVE A SURFACE DENSITY OF $\sigma_1 = \dfrac{Q}{A} = D_1$ ($\vec{D}$ NORMAL TO PLANE)

Gauss theorem gives $E\cdot\varepsilon = D = \dfrac{\sigma}{2} = \dfrac{Q}{2A}$

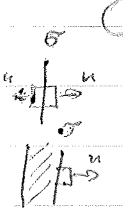


#2 WILL HAVE A SURFACE DENSITY OF $D_2 = \sigma_2 = Q\left[\dfrac{1}{2A + 2ad}\right] \simeq \dfrac{Q}{2A}$

Gauss theorem gives $E\cdot\varepsilon = D = \sigma = \dfrac{Q}{2A}$

($\vec{D}$ NORMAL TO SURFACE EVERYWHERE)

$\Rightarrow \frac{1}{2}$ THE DISPLACEMENT OF #1



② CASE #2: BOTH WILL HAVE SAME σ AND THUS HAVE SAME NORMAL $\vec{D}$ COMPONENT (IN MAGNETUDE) AT EACH POINT. (EXCEPT @ #2'S CORNERS $\Rightarrow \vec{D} = 0$)

$D = \dfrac{\sigma}{2}$

$D = \sigma$

<u>No</u>!! see how CHIN S. did it

20P2. ARE $\vec{D}$ AND $\vec{E}$ ALWAYS PARALLEL?
IF NO, GIVE EXAMPLE.

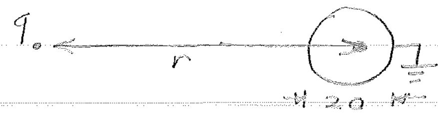
$\vec{D}$ AND $\vec{E}$ ARE NOT ALWAYS PAR. TO EACH OTHER (THEY ARE IN A HOMOGENEOUS ISOTROPIC MEDIA). what in nonhomogeneous but isotropic media?

A PLACE WHERE $\vec{D}$ & $\vec{E}$ ARE NOT PARALLEL IS IN A CRYSTALINE DIELECTRIC. HERE, THE TERM ϵ IN THE DEFINITION $\vec{D} = \epsilon\vec{E}$ TAKES ON A TENSOR NATURE:

$$\begin{bmatrix} D_x \\ D_Y \\ D_Z \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{21} & \epsilon_{31} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{32} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_Y \\ E_Z \end{bmatrix} \qquad \text{(FROM 1.19)}$$

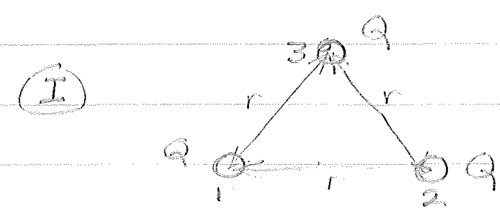
HERE, $\vec{D}$ IS OBVIOUSLY NOT PARALLEL TO $\vec{E}$.

204. WE KNOW THAT A POINT CHARGE q INDUCES A CHARGE $\frac{-qa}{r}$ ON A GROUNDED SPHERE;



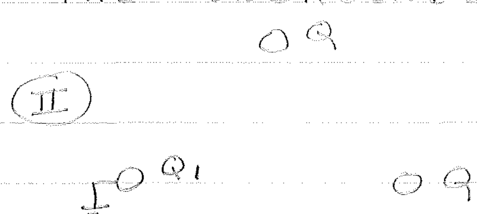
CONSIDER THEN, THE PROBLEM AT HAND.

INITIALLY, WE HAVE THE THREE CHARGED SPHERES EACH OF RADIUS a

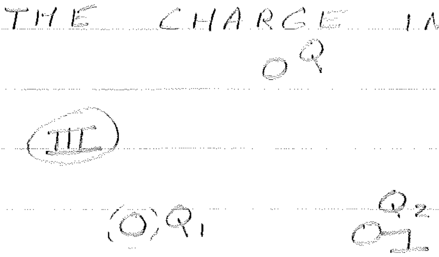


GROUND SPHERE # 1. THE INDUCED CHARGE BY THE UNGROUNDED SPHERES IS

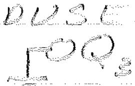
$$Q_1 = \frac{-2Qa}{r}$$



INSULATE SPHERE # 1 SO THAT ITS CHARGE DOESN'T CHANGE & GROUND SPHERE # 2. THE CHARGE INDUCED IS THEN $Q_2 = \frac{-a(Q+Q_1)}{r}$



INSULATE & UNGROUND Q_2. GROUND Q_3. THE INDUSED CHARGE IS $Q_3 = \frac{-a(Q_2+Q_1)}{r}$



TO FIND THE Q_i'S, SIMPLY WORK BACKWARDS:

$$Q_1 = \frac{-2Qa}{r}$$

$$Q_2 = \frac{-a}{r}(Q + Q_1)$$

$$= \frac{-a}{r}\left(Q - \frac{2Qa}{r}\right)$$

$$= \frac{-aQ}{r}\left(1 - \frac{2a}{r}\right)$$

$$Q_3 = \frac{-a}{r}\left[Q_2 + Q_1\right]$$

$$= \frac{-a}{r}\left[\frac{-aQ}{r}\left(1 - \frac{2a}{r}\right) - \frac{2Qa}{r}\right]$$

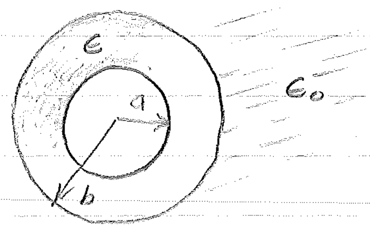
$$= \frac{a}{r}\left[\frac{aQ}{r}\left(1 - \frac{2a}{r}\right) + \frac{2Qa}{r}\right]$$

$$= \frac{a^2Q}{r^2}\left[\left(1 - \frac{2a}{r} + 2\right)\right]$$

$$= \frac{a^2Q}{r^3}\left(3 - \frac{2a}{r}\right)$$

r^2

2.13.



$K = \frac{\epsilon}{\epsilon_0}$

WE MAY LOOK AT THE PROBLEM AS TWO CAPACITORS IN SERIES. C_1 IS THE CAPACITANCE FROM THE CONDUCTOR TO THE OUTER SURFACE OF THE DIELECTRIC.

$$\Rightarrow C_1 = \frac{4\pi\epsilon ab}{b-a} = \frac{4\pi K\epsilon_0 ab}{b-a} \qquad \text{(Eq. 2.03 (1))}$$

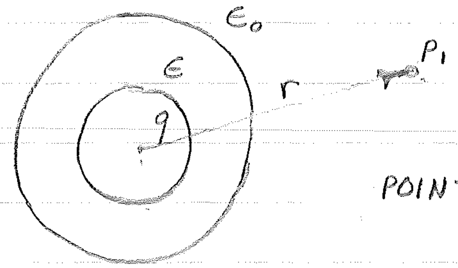
C_2 IS THE CAPACITANCE FROM THE OUTER SURFACE OF THE DIELECTRIC TO ∞:

$$C_2 = 4\pi\epsilon_0 b \qquad \text{(Eq. 2.03(2))}$$

THE TOTAL CAPACITANCE, C, IS THEN

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$
$$= \frac{b-a}{4\pi K\epsilon_0 ab} + \frac{1}{4\pi\epsilon_0 b}$$
$$= \frac{1}{4\pi\epsilon_0 b}\left[\frac{b-a}{Ka} + 1\right]$$
$$= \frac{1}{4\pi\epsilon_0 b}\,\frac{b-a+Ka}{Ka}$$
$$\Rightarrow C = \frac{4\pi Kab\epsilon_0}{b-a+Ka}$$

2.14. TO SOLVE THIS, USE GREEN'S RECIPROCATION THEM:

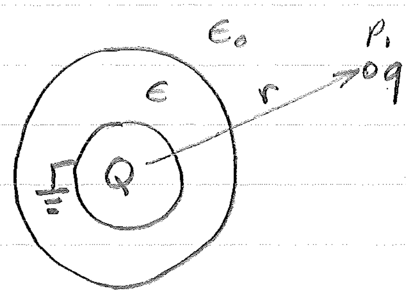


EMPLOYING GAUSS' FLUX THEM:

POINT $\begin{cases} V_{P_1} = \dfrac{q}{4\pi\epsilon_0 r} \\ q_{P_1} = 0 \end{cases}$

SPHERE $\begin{cases} V_S = \dfrac{q}{C} \\ q_S = q \end{cases}$

HERE, C IS THE CAPACITANCE IN PROB 2.13.



POINT $\begin{cases} V'_{P_1} = \text{DON'T MATTER} \\ q'_{P_1} = q \end{cases}$

SPHERE $\begin{cases} V'_S = 0 \\ q'_S = Q \end{cases}$

THEN, DUE TO GREEN'S RECIP. THEOREM:

$$V_{P_1} q'_{P_1} + V_S q'_S = \underbrace{q_{P_1}}_{0} V'_{P_1} + q_S \underbrace{V'_S}_{0}$$

$$\frac{q^2}{4\pi\epsilon_0 r} + \frac{qQ}{C} = 0 \Rightarrow Q = \frac{-qC}{4\pi\epsilon_0 r}$$

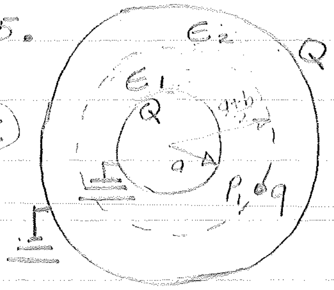
WHERE $C = 4\pi kab\epsilon_0 / (b - a + ka)$

$$\therefore Q = \frac{-q}{\cancel{4\pi\epsilon_0} r} \frac{\cancel{4\pi} kab \cancel{\epsilon_0}}{b + (k-1)a}$$

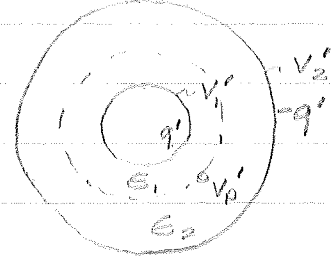
$$= -qabk / r(b + (k-1)a)$$

2.15. WE REQUIRE $Q_1 = Q_2 = Q$ = CHARGE INDUCED BY THE POINT CHARGE q AT P_1. TO APPLY GREENS RECIP. THEOREM, WE WISH TO FIND THE POTENTIAL V_p' TO WHICH POINT P_1 WOULD BE RAISED DUE TO V_1' AND V_2' PLACED ON INNER & OUTER CONDUCTORS RESPECTIVELY.

(I)



(II)



SINCE ALL THE FLUX FROM q IN (I) LANDS ON CONDUCTORS:

$$q = -2Q$$

APPLYING GREEN'S RECIP. THEOREM:

$$QV_1' + qV_p' + QV_2' = 0$$

OR $$QV_1' - 2QV_p' + QV_2' = (V_1' - 2V_p' + V_2')Q = 0$$

THE REQUIREMENT OF EQUAL INDUCED CHARGES IS THUS: $V_1' - 2V_p' + V_2' = 0$

OR, EQUIVALENTLY: $V_1' - V_p' = V_p' - V_2'$ (1)

WE FIND THIS FROM CONFIGURATION (II). PLACE A GAUSSIAN SPHERICAL SURFACE AT $r = \frac{a+b}{2}$. THEN

$$q' = \int_S \epsilon_1 \vec{E}\cdot\hat{n}\,ds = 4\pi r^2 E \epsilon_1$$

$\frac{\partial V}{\partial r}$

$$\Rightarrow -E = \frac{+\delta V}{\delta r} = \frac{-q'}{4\pi\epsilon_1 r^2}$$

$$\Rightarrow V_1' - V_p' = \frac{+q'}{4\pi\epsilon_1}\int_a^{(a+b)/2}\frac{dr}{r^2} = \frac{-q'}{4\pi\epsilon_1}\frac{1}{r}\Big|_a^{\frac{a+b}{2}}$$

$$= \frac{-q'}{4\pi\epsilon_1}\left[\frac{2}{a+b} - \frac{1}{a}\right] = \frac{q'}{4\pi\epsilon_1}\frac{b-a}{a(a+b)} \quad (2)$$

SIMALARLY, WE HAVE

$$V_p' - V_2' = \frac{+q'}{4\pi\epsilon_2}\int_{\frac{a+b}{2}}^{b}\frac{1}{r^2} = \frac{-q'}{4\pi\epsilon_2}\frac{1}{r}\Big|_{(a+b)/2}^{b}$$

$$= \frac{-q'}{4\pi\epsilon_2}\left[\frac{1}{b} - \frac{2}{a+b}\right] = \frac{-q'}{4\pi\epsilon_2}\left[\frac{(a+b)-2b}{b(a+b)}\right]$$

$$= \frac{q'}{4\pi\epsilon_2}\left[\frac{b-a}{b(a+b)}\right] \quad (3)$$

TO SATISFY Eq. (1) WE EQUATE Eqs. (2) & (3)

$$\frac{q'}{4\pi\epsilon_1}\,\frac{b-a}{a(a+b)} = \frac{q'}{4\pi\epsilon_2}\,\frac{b-a}{b(a+b)}$$

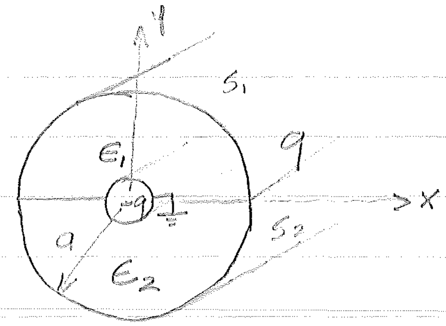
OR

$$\frac{1}{\epsilon_1 a} = \frac{1}{\epsilon_2 b}$$

THE NECESSARY RATIO OF DIELECTRICS IS THUS

$$\frac{\epsilon_1}{\epsilon_2} = \frac{b}{a}$$

2.49C.



WE HAVE FROM 2.20, THE GENERAL EXPRESSION FOR FORCE ON A CONDUCTOR:

$$F_p = \int_s \frac{\vec{D}\cdot\vec{E}}{2}\ \vec{p}\cdot\vec{n}\,ds$$

CONSIDER THE PART OF THE CAPACITOR CONTAINING ϵ_1. FOR A LENGTH ℓ, THE SURFACE CHARGE DENSITY ON THE CONDUCTOR IS $\sigma = \frac{Q}{2\pi a \ell} = \frac{q}{2\pi a}$.

THE DISPLACEMENT, D, IS NORMAL TO THE CYLINDER AND EQUAL IN MAGNITUDE TO σ. THUS

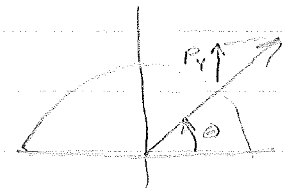
$$F_1 = \int_{S_1} \frac{1}{2}\epsilon_1 D^2 ds \quad \Leftarrow \text{ what happened to vector } \vec{p}\,?$$

$$= \frac{1}{2\epsilon_1}\int_{S_1} \frac{q^2}{(2\pi a)^2} ds$$

WE CAN SPEAK OF THE "PRESSURE" (NORMAL) TO THE TOP HALF OF THE CYLINDER:

$$P_n = \frac{dF_1}{ds} = \frac{1}{2\epsilon_1}\frac{q^2}{(2\pi a)^2}$$

DUE TO SYMMETRY, THE RESULTING FORCE ON THE UPPER HALF WILL BE IN THE Y DIRECTION:



THE COMPONENT OF PRESSURE IN THIS DIRECTION IS vector $\bar{p}$

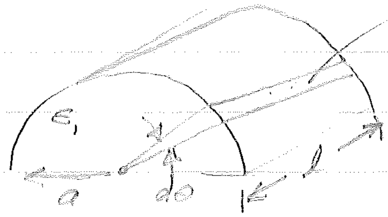
$$P_Y = P_n \sin\theta = \frac{1}{2\epsilon_1} \frac{q^2}{(2\pi a)^2} \sin\theta$$

THE TOTAL FORCE ACTING ON THE TOP CYLINDER IS THEN

$$F_{Y_1} = \int_{S_1} P_Y \, ds = \frac{1}{2\epsilon_1} \frac{q^2}{(2\pi a)^2} \int_{S_1} \sin\theta \, d\theta \cdot l \cdot a \qquad !$$

WE TAKE ds TO BE A LONG THIN SLICE OF LENGTH l

$ds = l \, a \, d\theta$



THUS

$$F_{Y_1} = \frac{1}{2\epsilon_1} \frac{q^2 \, l a}{(2\pi a)^2} \int_0^{\pi} \sin\theta \, d\theta$$

$$= l q^2 / (2\pi)^2 a \epsilon_1$$

DUE TO SYMMETRY, THE COMPONENT OF FORCE ACTING ON THE LOWER CYLINDER IS (IN $-Y$ DIRECTION)

$$F_{Y_2} = l q^2 / (2\pi)^2 a \epsilon_2$$

THE TOTAL FORCE ON THE CYLINDER, F, (PER UNIT LENGTH) IS THUS

$$F = \frac{1}{l}\left[F_{Y_1} - F_{Y_2}\right] \qquad \text{(IN + Y DIRECTION)}$$

$$= \frac{q^2}{(2\pi)^2 a}\left[\frac{1}{\epsilon_1} - \frac{1}{\epsilon_2}\right]$$

$$= \frac{q^2 (\epsilon_2 - \epsilon_1)}{(2\pi)^2 a \epsilon_1 \epsilon_2}$$

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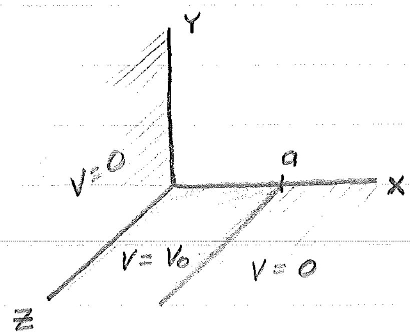
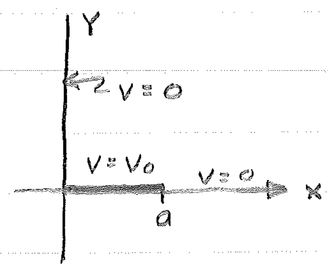
4.1.



⇐ GIVEN POTENTIAL BOUNDRY CONDITIONS
⇓



WILL NOW SHOW THAT THE GIVEN BOUNDRY CONDITIONS ARE SATISFIED BY THE LINE CHARGE DISTRIBUTION:

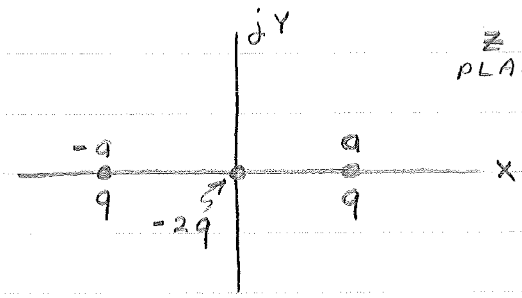


FIG 1

THE VALUES OF q WILL BE SHOWN TO BE DETERMINED BY V_0.

IF U IS THE POTENTIAL ON THE Z PLANE:

$$W = V + jU = -\frac{1}{2\pi\epsilon}\sum_{s=1}^{n} q_s \ln(z - z_s) \qquad \text{(Eq. 4.12 (4))}$$

THE POTENTIAL U IS THUS GIVEN BY

$$U = \frac{-1}{2\pi\epsilon}\sum_{s=1}^{n} q_s \arg(z - z_s) = \operatorname{Im}[W]$$

WHERE, FROM FIG 1:

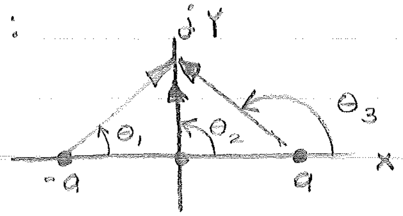
$n = 3$

$z_1 = -a \quad , q_1 = q$

$z_2 = 0 \quad , q_2 = -2q$

$z_3 = a \quad , q_3 = q$

CONSIDER FIRST, THE POTENTIAL ON THE Y AXIS:

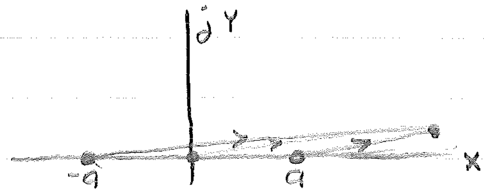


SINCE $\theta_1 + \theta_3 = \pi$, AND $\theta_2 = \frac{\pi}{2}$, WE HAVE

$$U = \frac{-q}{2\pi\epsilon}\left[\theta_1 - 2\theta_2 + \theta_3\right] = 0$$

THIS BOUNDRY CONDITION IS SATISFIED.

CONSIDER NEXT, THE X AXIS FOR $|X| > a$:



HERE, $\theta_1 = \theta_2 = \theta_3 = 0 \Rightarrow U = 0$

FOR $0 < x < a$, $\theta_1 = \theta_2 = 0$ AND $\theta_3 = \pi$. THUS

$$U = V_0 = \frac{-q}{2\pi\epsilon}(\pi) = \frac{-q}{2\epsilon}$$

THIS GIVES OUR RELATIONSHIP BETWEEN THE IMAGE CHARGE VALUE q, AND THE STATED BOUNDRY POTENTIAL V_0:

$$q = -2\epsilon V_0$$

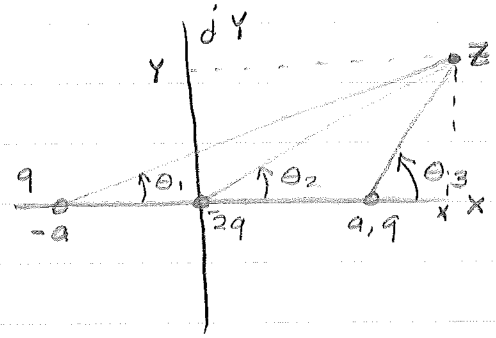
DUE TO SYMMETRY, $U = -V_0$ FOR $-a < x < a$.

DUE TO THE UNIQUENESS THEOREM, EVALUATION OF THE FIELD DUE TO THE "IMAGE" CHARGES IS EQUIVALENT TO ANALYSIS OF THE GIVEN BOUNDRY VALUE:

PROBLEM. THUS, THE ANALYSIS TO FOLLOW MAY BE SAID TO CONSTITUTE A "PROOF."

AGAIN, WE HAVE

$$U = \frac{-1}{2\pi\epsilon} \sum_{n=1}^{3} q_n \arg(z - z_n)$$



FOR $x > 0$ AND $Y > 0$, WE HAVE (FROM THE FIGURE)

$$\begin{aligned} U &= \frac{-q}{2\pi\epsilon}\left[\theta_1 - 2\theta_2 + \theta_3\right] \\ &= \frac{V_0}{\pi}\left[\theta_1 - 2\theta_2 + \theta_3\right] \\ &= \frac{V_0}{\pi}\left[\tan^{-1}\frac{Y}{X+a} - 2\tan^{-1}\frac{Y}{X} + \tan^{-1}\frac{Y}{X-a}\right] \end{aligned}$$

$$\Rightarrow \pi U = V_0\left[\tan^{-1}\frac{Y}{X+a} - 2\tan^{-1}\frac{Y}{X} + \tan^{-1}\frac{Y}{X-a}\right]$$

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403. WITHOUT LOSS OF GENERALITY, ASSUME THAT q IS ON THE REAL AXIS

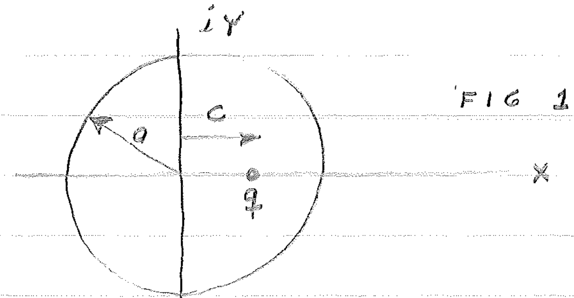
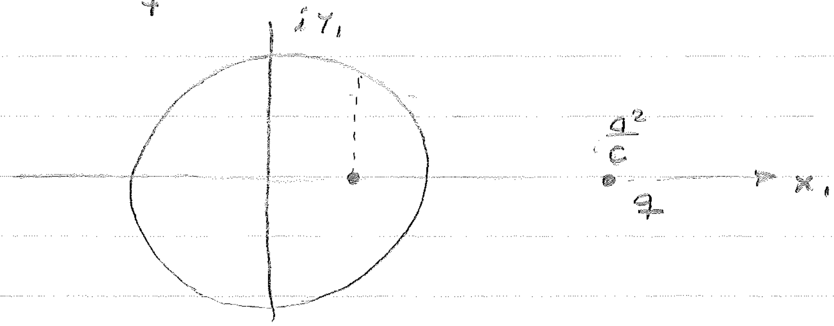


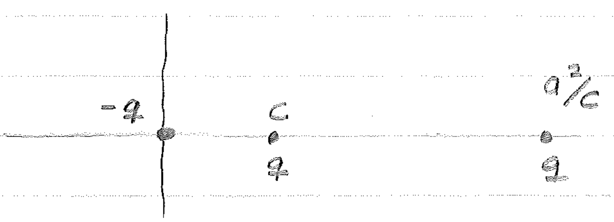
FIG 1.

NOW PERFORM AN INVERSION: $Z_1 = \frac{a^2}{Z}$ THE CHARGE q IS NOW AT $X_1 = \frac{a^2}{c}$:

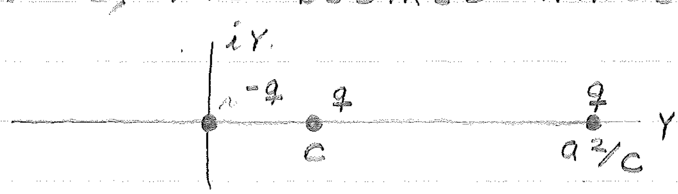


This is ~~[illegible]~~ [illegible] only for grounded cylinder
No it will look like for ungrounded cylinder

THE IMAGE FOR THIS CASE IS IN 4.05, AND IS



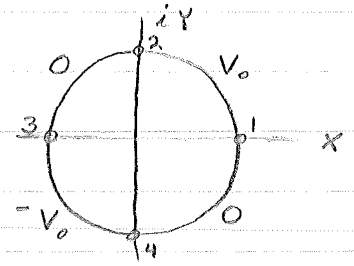
PERFORMING AN INVERSE MAPPING, $0 \to \infty$ & $c \to \frac{a^2}{c}$.
THUS, FROM THE FIRST SENTENCE ON TOP OF pg. 88, THE DESIRED IMAGE OF FIG. 1 IS



WILL GIVE W WITHIN CYLINDER

THIS IS REASONABLE. ITS JUST THE DUAL OF SEC. 4.05. ie, AN EXTERNAL LINE CHARGE @ a^2/c SHOULD GIVE EQUIVALENT IMAGES.

4.8.



PERFORM AN INVERSION

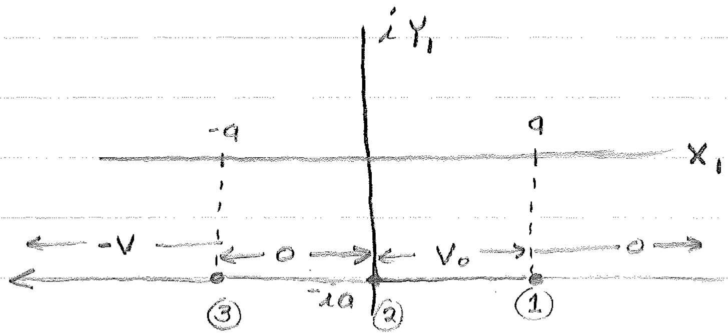
$$z_1 = \frac{2a^2}{z + ia}$$

1. $z = a \Rightarrow z_1 = \frac{2a^2}{a+ia}\left(\frac{a-ia}{a-ia}\right) = \frac{2a^2}{2a^2}(a-ia) = a(1-i)$

2. $z = ia \Rightarrow z_1 = \frac{2a^2}{i2a} = \frac{a}{i} = -ia$

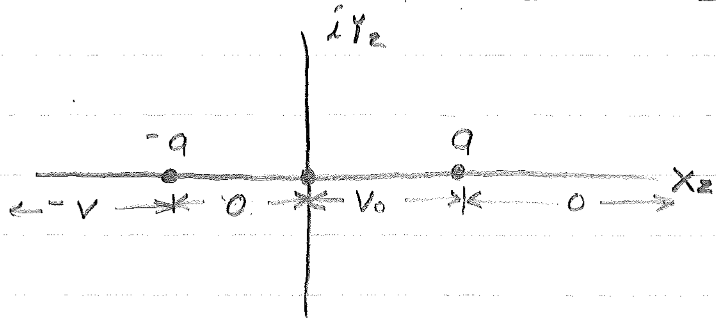
3. $z = -a \Rightarrow z_1 = \frac{-2a^2}{(a-ia)}\left(\frac{a+ia}{a+ia}\right) = \frac{-2a^2}{2a^2}(a+ia) = -a(1+i)$

4. $z = -ia \Rightarrow z_1 = \pm i\infty$ $\qquad (z = 0 \Rightarrow z_1 = \frac{2a^2}{ia} = -i2a)$

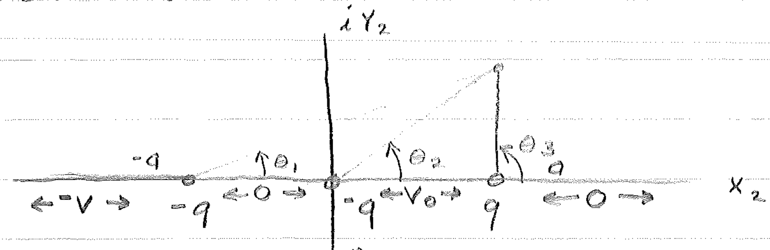


PERFORM A TRANSLATION:

$$z_2 = z_1 + ia = \frac{2a^2}{z+ia} + ia$$
$$= \frac{2a^2 + ia(z+ia)}{z+ia}$$
$$= \frac{2a^2 - a^2 + iaz}{z+ia}$$
$$= \frac{a^2 + iaz}{z+ia}$$
$$= a\left[\frac{a+iz}{z+ia}\right] = ia\left[\frac{z-ia}{z+ia}\right]$$



THE FOLLOWING POINT CHARGE DISTRIBUTION WILL SATISFY THE GIVEN POTENTIAL BOUNDRY CONDITIONS



BOTTOM HALF OF $\Leftarrow$ Z_2 PLANE CORRESPONDS TO INSIDE OF CIRCLE

$$W = V + jU = \frac{-1}{2\pi\epsilon} \sum_{s=1}^{n} q_s \ln(z - z_s)$$

$$U = \frac{-1}{2\pi\epsilon} \sum q_s \arg(z - z_s)$$

$$= \frac{-q}{2\pi\epsilon}\left[-\theta_1 - \theta_2 + \theta_3\right]$$

ON X_2 AXIS:

- $X_2 > a \Rightarrow \theta_1 = \theta_2 = \theta_3 = 0$
  $\Rightarrow U = 0$
- $0 < X_2 < a \Rightarrow \theta_1 = \theta_2 = 0 \quad \theta_3 = -\pi$
  $\Rightarrow U = \frac{+q}{2\epsilon} = V_0 \qquad \Longrightarrow V_0 = \frac{q}{2\epsilon}$
- $-a < X_2 < 0 \Rightarrow \theta_1 = 0 \;,\; \theta_2 = \theta_3 = -\pi$
  $\Rightarrow U = 0$
- $X_2 < -a \Rightarrow \theta_1 = \theta_2 = \theta_3 = -\pi$
  $U = \frac{-q}{2\pi\epsilon}\left[\pi + \pi - \pi\right] = \frac{-q}{2\epsilon} = -V_0$

THE CORRESPONDING COMPLEX DISTRIBUTION IS THUS

$$W = \frac{V_0}{\pi}\left[-\ln(z_2 + a) - \ln z_2 + \ln(z_2 - a)\right]$$

NOW, GO BACK TO THE Z PLANE:

$$W = \frac{V_0}{\pi}\left[\ln \frac{z_2 - a}{z_2 + a} - \ln z_2\right]$$

$$= \frac{V_0}{\pi}\left[\ln\left\{\frac{ia\left(\frac{z-ia}{z+ia}\right) - a}{ia\left(\frac{z-ia}{z+ia}\right) + a}\right\} - \ln ia\left(\frac{z-ia}{z+ia}\right)\right]$$

$$= \frac{V_0}{\pi}\left[\ln\left\{\frac{ia(z-ia) - a(z+ia)}{ia(z-ia) + a(z+ia)}\right\} + \ln \frac{z+ia}{ia(z-ia)}\right]$$

$$= \frac{V_0}{\pi}\left[\ln \frac{i(z-ia) - (z+ia)}{i(z-ia) + (z+ia)} + \ln \frac{z+ia}{ia(z-ia)}\right]$$

$$= \frac{V_0}{\pi}\left[\ln\left\{\frac{z(i-1) - a(i-1)}{z(i+1) + a(i+1)}\right\} + \ln \frac{z+ia}{ia(z-ia)}\right]$$

$$= \frac{V_0}{\pi}\left[\ln \frac{(i-1)}{(i+1)}\frac{(z-a)}{(z+a)} + \ln \frac{z+ia}{ia(z-ia)}\right]$$

$$= \frac{V_0}{\pi}\left[\ln \frac{i(1+i)}{(1+i)}\frac{(z-a)}{(z+a)} + \ln \frac{z+ia}{ia(z-ia)}\right]$$

$$= \frac{V_0}{\pi}\left[\ln i + \ln \frac{z-a}{z+a} - \ln i + \ln \frac{z+ia}{a(z-ia)}\right]$$

$$= \frac{V_0}{\pi}\left[\ln \frac{z-a}{z+a} + \ln \frac{z+ia}{a(z-ia)}\right]$$

NOW

- $$\ln \frac{z-a}{z+a} = \ln \frac{(x-a)+jY}{(x+a)+jY}$$

$$= \ln \frac{[(x-a)+jY][(x+a)-jY]}{(x+a)^2 + Y^2}$$

$$\Rightarrow \arg \frac{z-a}{z+a} = \arg\left([(x-a)+jY][(x+a)-jY]\right)$$

$$= \arg\left[(x+a)(x-a) + Y^2 + jY[(x+a)-(x-a)]\right]$$

$$= \arg\left[(x+a)(x-a) + Y^2 + j\,2aY\right]$$

$$= \arg\left[X^2 - a^2 + Y^2 + j\,2aY\right]$$

$$= \arg\left[r^2 - a^2 + j\,2a\right] \ni r^2 = X^2 + Y^2$$

$$= \tan^{-1} \frac{2aY}{r^2 - a^2}$$

Y ?

$$\bullet \ \ln \frac{z+ia}{a(z-ia)} = \ln \frac{x+i(y+a)}{a[x+i(y-a)]}$$

$$= \ln \frac{(x+i(y+a))(x-i(y-a))}{a[x^2+(y-a)^2]}$$

$$\Rightarrow \arg \frac{z+ia}{a(z-ia)} = \arg \frac{z+ia}{z-ia}$$

$$= \arg [x+i(y+a)][x-i(y-a)]$$

$$= \arg [x^2+(y+a)(y-a)+ix\{(y+a)-(y-a)\}$$

$$= \arg [x^2+y^2-a^2+i2ax]$$

$$= \arg [r^2-a^2+i2ax]$$

$$= \tan^{-1} \frac{2ax}{r^2-a^2}$$

THEN, SINCE

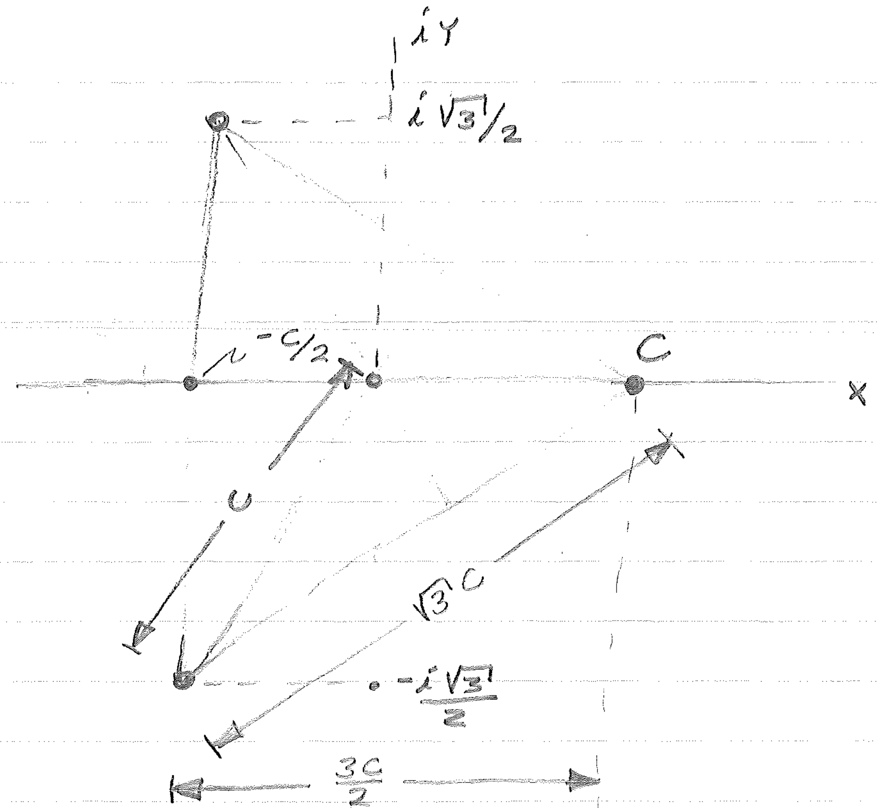
$$W = V + jU = \frac{V_0}{\pi} \left[\ln \frac{z-a}{z+a} + \ln \frac{z+ia}{a(z-ia)} \right]$$

WE HAVE

$$U = \operatorname{Im} W = \frac{V_0}{\pi} \left[\arg \left\{ \frac{z-a}{z+a} \right\} + \arg \left\{ \frac{z+ia}{a(z-ia)} \right\} \right]$$

$$= \frac{V_0}{\pi} \left[\tan^{-1} \frac{2ay}{r^2-a^2} + \tan^{-1} \frac{2ax}{r^2-a^2} \right]$$

4.10C



FROM THE GEOMETRY AND 4.12 (4):

$$W = \frac{-q}{2\pi\epsilon} \ln\left\{\left[z-\left(-\tfrac{1}{2}+\tfrac{i\sqrt{3}}{2}\right)c\right]\left[z-\left(\tfrac{1}{2}-\tfrac{i\sqrt{3}}{2}\right)c\right]\left[z-c\right]\right\}$$

BUT

$$-\left(\tfrac{1}{2}+\tfrac{i\sqrt{3}}{2}\right) = e^{i\,2\pi/3}$$

$$\left(\tfrac{1}{2}-\tfrac{i\sqrt{3}}{2}\right) = e^{i\,4\pi/3}$$

THUS

$$W = \frac{-q}{2\pi\epsilon} \ln\left[\left(z-c\,e^{i\,2\pi/3}\right)\left(z-c\,e^{i\,4\pi/3}\right)\left(z-c\right)\right]$$

$W = U + iV$

IN THE CONVENTION WE'VE WRITTEN W, U CORRESPONDS TO EQUAPOTENTIAL LINES

THUS

$U = \mathrm{Re}\, W = K = \text{CONSTANT}$

$\Rightarrow K' = \ln\left[|z - ce^{i2\pi/3}||z - ce^{i4\pi/3}||z - c|\right]$

$K'' = |(z - ce^{i2\pi/3})(z - ce^{i4\pi/3})(z - c)|$

WHERE K' AND K'' ARE ALSO CONSTANTS. $(K'' = e^{K'})$

IT REMAINS TO EXPAND THIS RELATIONSHIP USING THE RELATIONSHIP

$$z = re^{i\theta}:$$

$$K'' = |(re^{i\theta} - ce^{i2\pi/3})(re^{i\theta} - ce^{i4\pi/3})(re^{i\theta} - c)|$$

$$= |(r^2e^{i2\theta} + c^2e^{i6\pi/3} - cre^{i(\theta + \frac{2\pi}{3})} - cre^{i(\theta + \frac{4\pi}{3})})(re^{i\theta} - c)|$$

$$K''^2 = |r^3e^{i3\theta} + c^2re^{i(\theta + 2\pi)} - cr^2e^{i(2\theta + \frac{2\pi}{3})} - cr^2e^{i(2\theta + \frac{4\pi}{3})} - cr^2e^{i2\theta} - c^3e^{i2\pi} + c^2re^{i(\theta + 2\pi/3)} + c^2re^{i(\theta + \frac{4\pi}{3})}|^2$$

$$= [r^3e^{i3\theta} + c^2re^{i2\theta} - cr^2e^{i(2\theta + \frac{2\pi}{3})} - cr^2e^{i(2\theta + \frac{4\pi}{3})} - cr^2e^{i2\theta} - c^3 + c^2re^{i(\theta + 2/3\pi)} + c^2re^{i(\theta + \frac{4\pi}{3})}]$$

$$\times [r^3e^{-i3\theta} + c^2re^{-i2\theta} - cr^2e^{-i(2\theta + \frac{2\pi}{3})} - cr^2e^{-i(2\theta + \frac{4\pi}{3})} - cr^2e^{-i2\theta} - c^3 + c^2re^{-i(\theta + \frac{2\pi}{3})} + c^2re^{-i(\theta + \frac{4\pi}{3})}]$$

THIS UNGODLY EXPRESSION <u>WAS</u> <u>EVALUATED</u> ON THREE SHEETS OF COMPUTER PAPER, AND GIVES THE DESIRED ANSWER:

$$K''^2 = \text{CONST}$$

$$= r^6 + c^6 - r^3c^3[e^{i3\theta} + e^{-i3\theta}]$$

$$= r^6 + c^6 - 2r^3c^3\cos 3\theta$$

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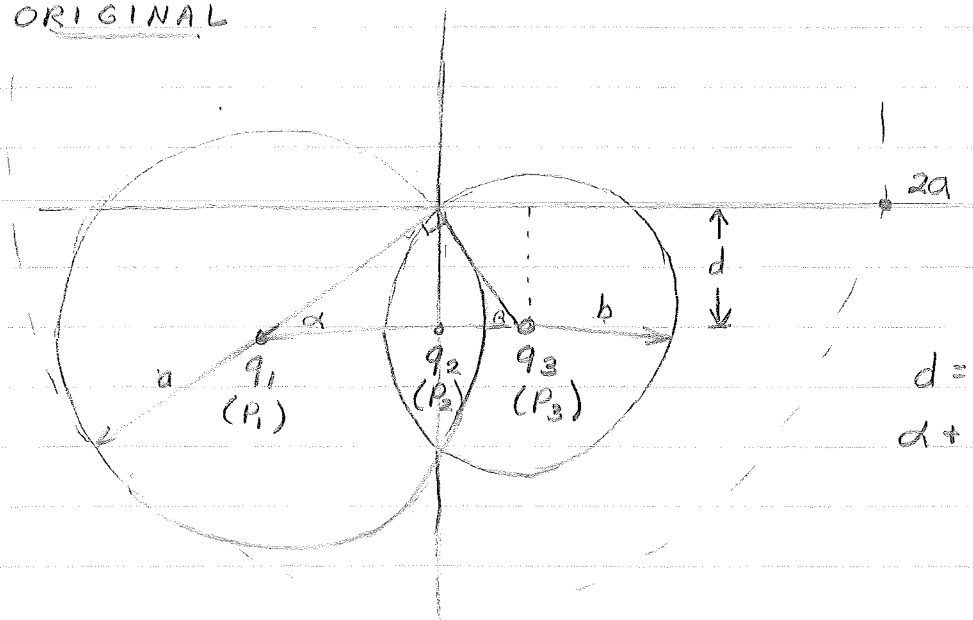
EE-5341

COMPUTATION OF POTENTIAL FROM 5.103

ORIGINAL

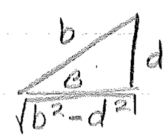


$$d = \frac{+ab}{\sqrt{a^2+b^2}}$$

$$\alpha + \beta = \frac{\pi}{2}$$

WE MUST DETERMINE COORDINATES OF P_1, P_2, P_3.

- $P_2 = (x_2, Y_2) = (0, -d) = \left(0, \frac{-ab}{\sqrt{a^2+b^2}}\right)$
- $P_3 = (x_3, Y_3) = (+b\cos\beta, -b\sin\beta)$

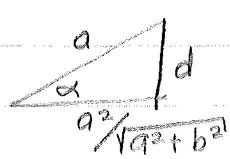


$$b^2 - d^2 = b^2 - \frac{a^2b^2}{a^2+b^2} = \frac{b^4}{a^2+b^2}$$

$$\Rightarrow \sqrt{b^2-d^2} = b^2/\sqrt{a^2+b^2}$$

$$\Rightarrow P_3 = \left(\frac{+b^2}{\sqrt{a^2+b^2}}, \frac{-ab}{\sqrt{a^2+b^2}}\right)$$

- $P_1 = (x_1, Y_1) = (-a\cos\alpha, -a\sin\alpha)$



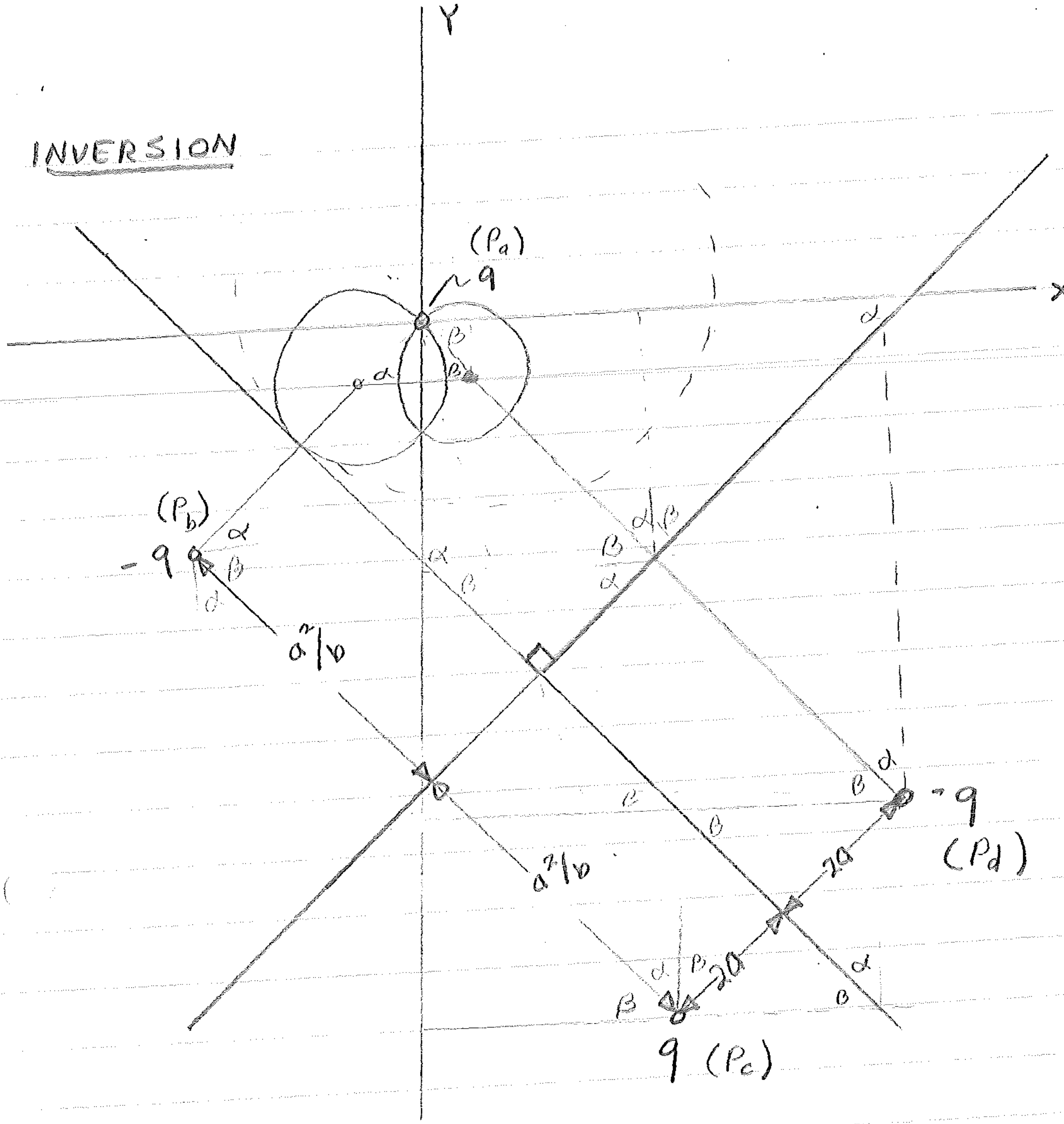
$$a^2 - d^2 = \frac{a^4}{a^2+b^2}$$

$$\Rightarrow P_1 = (x_1, Y_1) = \left(\frac{-a^2}{\sqrt{a^2+b^2}}, \frac{-ab}{\sqrt{a^2+b^2}}\right)$$

OBVIOUSLY, $Y_1 = Y_2 = Y_3 = \frac{-ab}{\sqrt{a^2+b^2}}$

INVERSION

$$r_1 = \frac{(2a)^2}{r}$$



$P_a = (x_a, y_a)$, $P_b = (x_b, y_b)$

$P_c = (x_c, y_c)$, $P_d = (x_d, y_d)$

WE MAY FIND COORDINATES OF P_a, P_b, P_c, P_d

$$P_a = (0,0)$$

$$P_d = \left(+\frac{2a^2}{b}\cos\beta,\ -\frac{2a^2}{b}\sin\beta\right) = \left(\frac{+2a^2}{\sqrt{a^2+b^2}},\ \frac{-2a^3}{b\sqrt{a^2+b^2}}\right)$$

$$P_b = (-4a\cos\alpha,\ -4a\sin\alpha) = \left(\frac{-4a^2}{\sqrt{a^2+b^2}},\ \frac{-4ab}{\sqrt{a^2+b^2}}\right)$$

$$P_c = (x_b + x_d,\ y_b + y_d) = \left(\frac{-2a^2}{\sqrt{a^2+b^2}},\ \frac{-2a}{\sqrt{a^2+b^2}}\left(\frac{a^2}{b} + b\right)\right)$$

IN MAPPING THE INVERSION BACK TO THE ORIGINAL, THE CHARGE q AT P_4 GOES TO INFINITY AND HAS NO EFFECT ON THE ORIGINAL POTENTIAL FIELD. THE CHARGE $-q$ AT P_d MAPS TO P_3:

$$\frac{-q}{q_3} = \frac{2a^2}{b(2a)} = \frac{a}{2b} \qquad \text{(FROM 5.10 (1))}$$

$$\Rightarrow q_3 = -\frac{2b}{a}q = -\frac{2b}{a}(4\pi\epsilon)$$

SIMILAR MAPPING GIVES (P_c TO P_d)

$$q_2 = 4\pi\epsilon \frac{ab}{2a\sqrt{a^2+b^2}} = 4\pi\epsilon \frac{b}{2\sqrt{a^2+b^2}}$$

$$q_1 = \frac{-a}{2a}(4\pi\epsilon) = (4\pi\epsilon)\left(-\frac{1}{2}\right)$$

WE MAY NOW COMPUTE THE POTENTIAL VIA 1.06 Eq. 3:

$$V(X,Y) = \frac{1}{4\pi\epsilon}\sum \frac{q_i}{r_i}$$

$$= \frac{\left(-\frac{1}{2}\right)}{\left[\left(X+\frac{a}{\sqrt{a^2+b^2}}\right)^2 + \left(Y+\frac{ab}{\sqrt{a^2+b^2}}\right)\right]^{1/2}}$$

$$+ \frac{\frac{b}{2\sqrt{a^2+b^2}}}{\left[X^2 + \left(Y+\frac{ab}{\sqrt{a^2+b^2}}\right)^2\right]^{\frac{1}{2}}}$$

$$+ \frac{\left(-2b/a\right)}{\left[\left(X \div \frac{b^2}{\sqrt{a^2+b^2}}\right)^2 + \left(Y+\frac{ab}{\sqrt{a^2+b^2}}\right)\right]^{\frac{1}{2}}}$$

I was asking you to find the pot. in the inverse system and then transform it to the original. Obviously result will be the same.

IN MAPPING THE INVERSION BACK TO THE ORIGINAL, THE CHARGE q AT P_4 GOES TO INFINITY AND HAS NO EFFECT ON THE ORIGINAL POTENTIAL FIELD. THE CHARGE $-q$ AT P_d MAPS TO P_3:

$$\frac{-q}{q_3} = \frac{2a^2}{b(2a)} = \frac{a}{2b} \qquad \text{(FROM 5.10 (1))}$$

$$\Rightarrow q_3 = -\frac{2b}{a} q = -\frac{2b}{a}(4\pi\epsilon)$$

SIMILAR MAPPING GIVES (P_c TO P_d)

$$q_2 = 4\pi\epsilon \frac{ab}{2a\sqrt{a^2+b^2}} = 4\pi\epsilon \frac{b}{2\sqrt{a^2+b^2}}$$

$$q_1 = \frac{-a}{2a}(4\pi\epsilon) = (4\pi\epsilon)(-\tfrac{1}{2})$$

WE MAY NOW COMPUTE THE POTENTIAL VIA 1.06 Eq. 3:

$$V(X,Y) = \frac{1}{4\pi\epsilon} \sum \frac{q_i}{r_i}$$

$$= \frac{(-\frac{1}{2})}{\left[(X+\frac{a}{\sqrt{a^2+b^2}})^2 + (Y+\frac{ab}{\sqrt{a^2+b^2}})\right]^{1/2}}$$

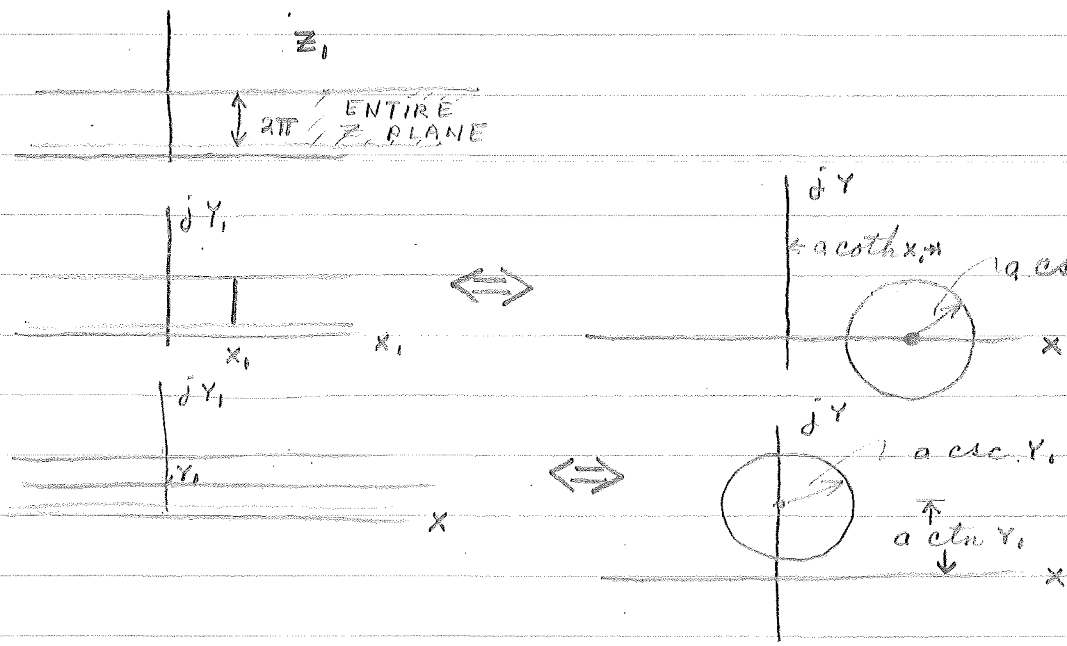
$$+ \frac{\frac{b}{2\sqrt{a^2+b^2}}}{\left[X^2 + (Y+\frac{ab}{\sqrt{a^2+b^2}})^2\right]^{\frac{1}{2}}}$$

$$+ \frac{(-2b/a)}{\left[(X-\frac{b^2}{\sqrt{a^2+b^2}})^2 + (Y+\frac{ab}{\sqrt{a^2+b^2}})\right]^{\frac{1}{2}}}$$

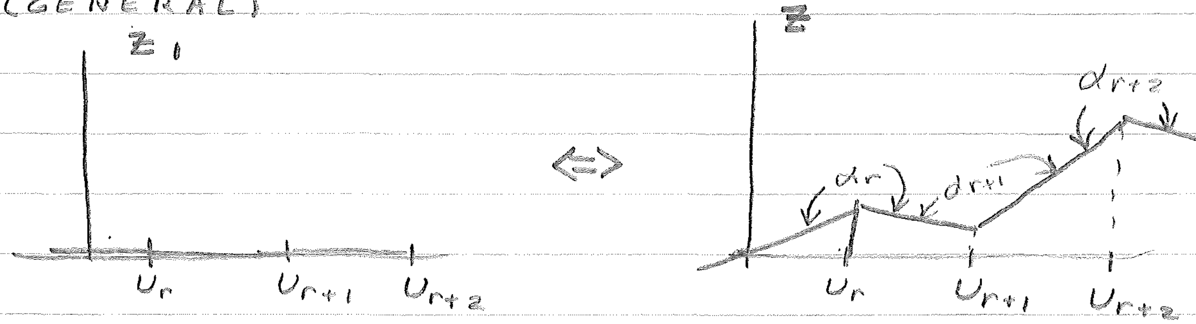
I was asking you to find the pot. in the inverse system and then transform it to the original. Obviously result will be the same.

CONFORMAL MAPPINGS (CHAPT. 4)

4.17. $Z_1 = \ln \dfrac{Z + ja}{Z - ja}$



4.18. THE SCHWARZ TRANSFORM (GENERAL)



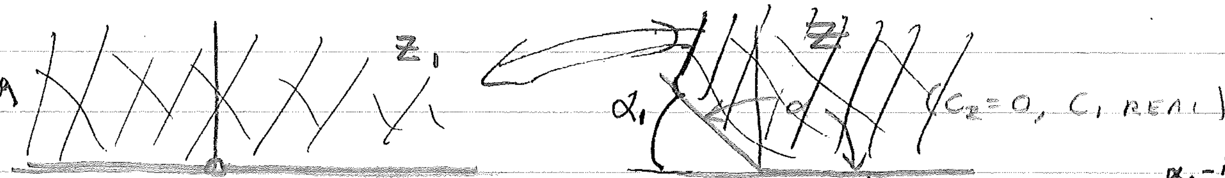
$$Z = C_1 \int (Z_1 - U_1)^{\frac{\alpha_1}{\pi} - 1} (Z_1 - U_2)^{\frac{\alpha_2}{\pi} - 1} \ldots dZ_1 + C_2$$

C_1 SCALES AND ROTATES

C_2 TRANSLATES

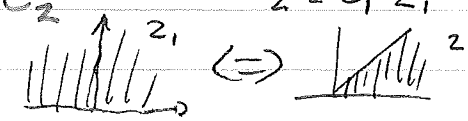
4.19. POLYGONS WITH ONE POSITIVE ANGLE

it was interesting to show the regions related in both planes

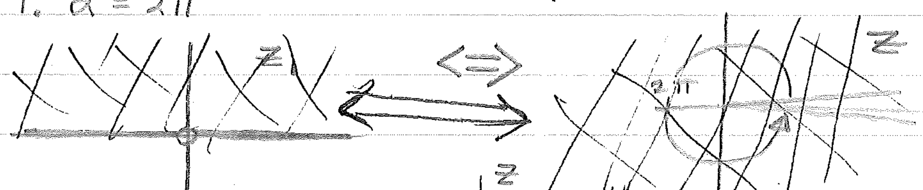


$$z = C_1 z_1^{\frac{\alpha}{\pi}} + C_2 \qquad z = C_1 z_1^{\frac{\alpha_1}{\pi}-1} + C_2 = C_1 z_1^{\frac{\alpha_1-\pi}{\pi}} + C_2 =$$

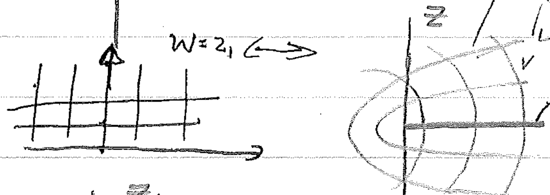
$$= C_1 z_1^{\frac{\alpha}{\pi}} + C_2$$



1. $\alpha = 2\pi$



field uniform

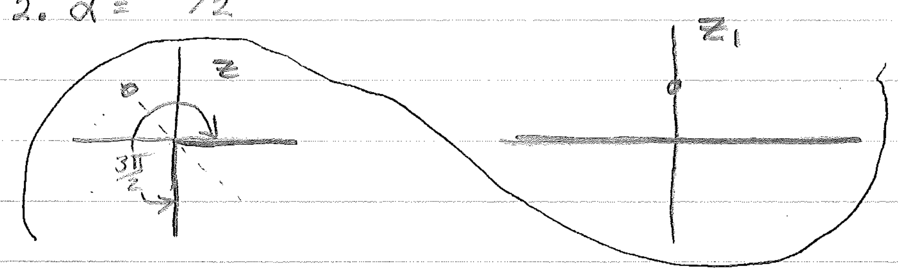


semi infinite charged conducting plane

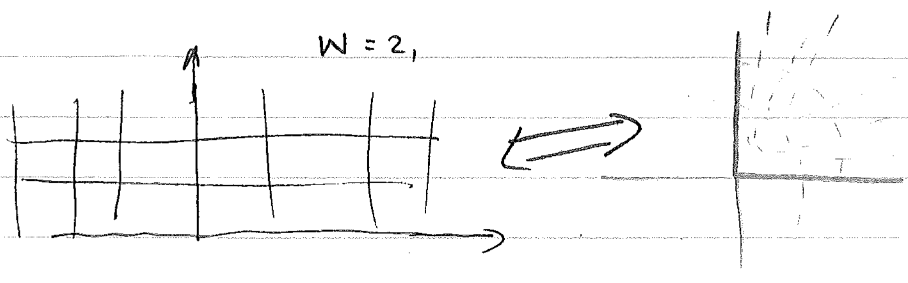
PARABOLIC

it is interesting to point out that the same problem is transformed to two different problems in z using two different α

2. $\alpha = 3\pi/2$



3. $\alpha = \frac{\pi}{2}$

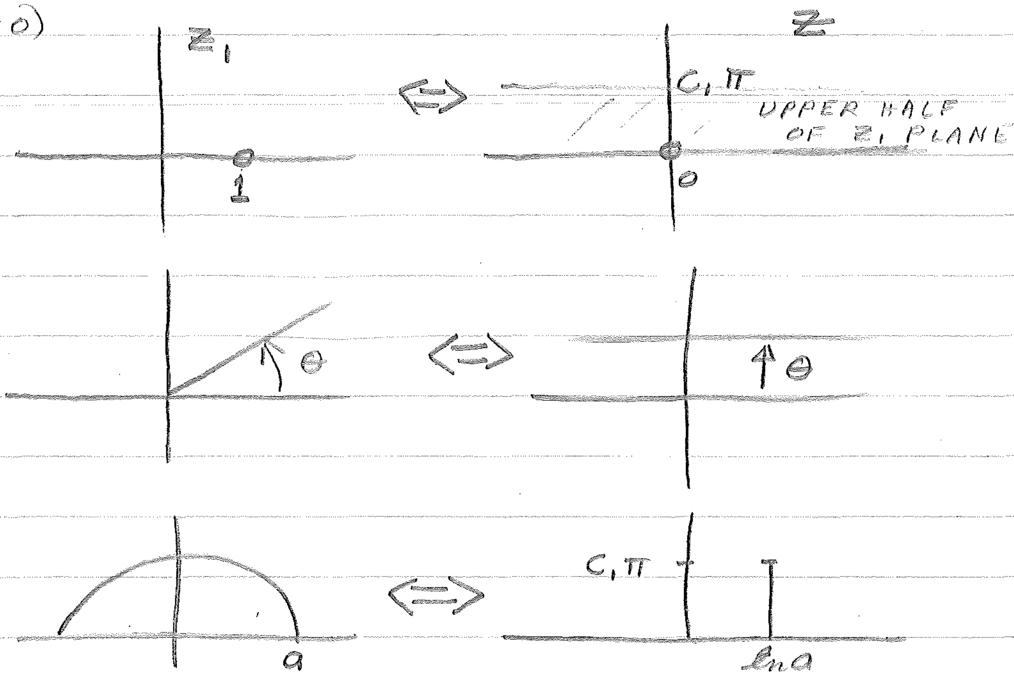


HYPERBOLOIDS

4.20 POLYGON WITH ZERO ANGLE ($\alpha = 0$)

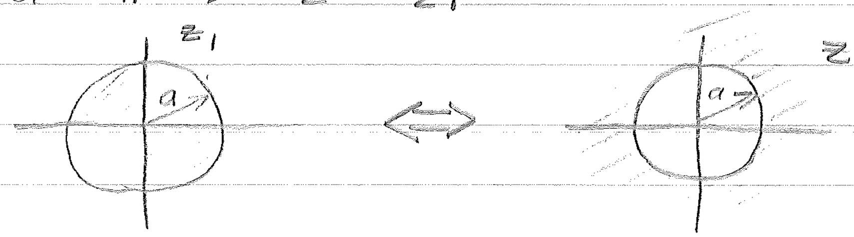
$$Z = C_1 \ln Z_1 + C_2 \quad \text{(A SCHWARZ XFORM)}$$

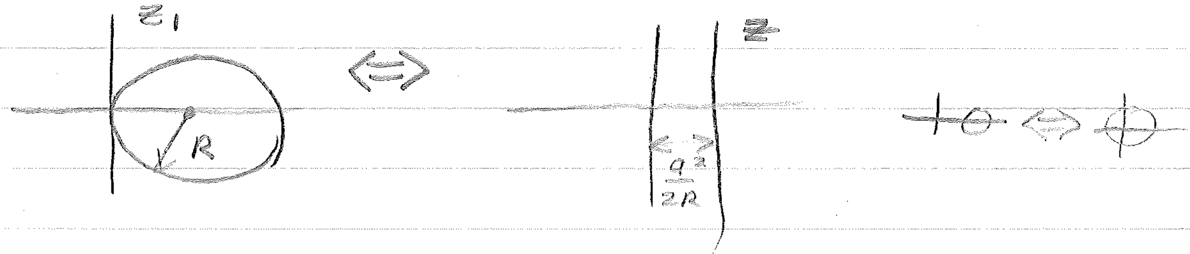
($C_2 = 0$)



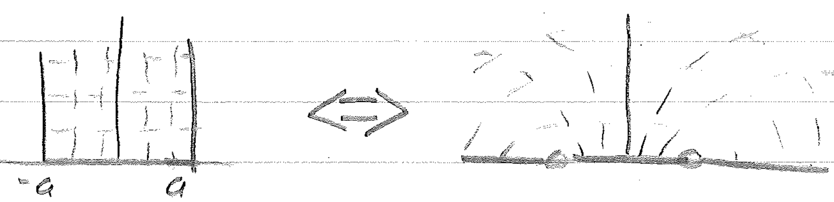
4.21 ● POLYGONS WITH ONE NEGATIVE ANGLE. INVERSION

$\alpha = -\pi \Rightarrow \quad z = \frac{a^2}{z_1}$





4.22 ● POLYGONS WITH TWO ANGLES



5.60. IT WAS SHOWN IN 5.17 THAT THE POTENTIAL INSIDE THE SPHERE, FOR $a < r \le b$ IS GIVEN BY

$$V = \frac{Q}{4\pi\epsilon a} \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!} \left[\left(\frac{a}{r}\right)^{2n+1} - \left(\frac{a}{b}\right)^{2n+1} \left(\frac{r}{b}\right)^{2n} \right] P_{2n}(\cos\theta)$$

ON THE SPHERE, THE $\vec{E}$ FIELD IS NORMAL (SINCE THE SPHERE IS A CONDUCTOR). THUS, WE MAY OBTAIN THE FIELD STRENGTH, E. THERE, MERELY BY DIFFERENTIATING THE ABOVE EXPRESSION WITH RESPECT TO r AND EVALUATE @ $r = b$.

$$E_b = \left. \frac{-\delta V}{\delta r} \right|_{r=b}$$

AND NOW, THE MATH:

$$E_b = \frac{-Q}{4\pi\epsilon a} \sum_n (-1)^n \frac{(2n-1)!!}{(2n)!!} \left. \left[-(2n+1) \frac{a^{2n+1}}{r^{2n+2}} - 2n \left(\frac{a}{b}\right)^{2n+1} \frac{r^{2n-1}}{b^{2n}} \right] \right|_{r=b} P_{2n}(\cos\theta)$$

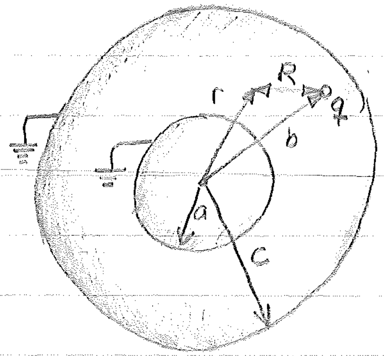
$$= \frac{Q}{4\pi\epsilon a} \sum_n (-1)^n \frac{(2n-1)!!}{(2n)!!} \left[(2n+1) \frac{a^{2n+1}}{b^{2n+2}} + 2n \frac{a^{2n+1}}{b^{2n+2}} \right] P_{2n}(\cos\theta)$$

$$= \frac{Q}{4\pi\epsilon a} \sum_n (-1)^n \frac{(2n-1)!!}{(2n)!!} \frac{a^{2n+1}}{b^{2n+2}} \left[2n+1+2n \right] P_{2n}(\cos\theta)$$

$$= \frac{Q}{4\pi\epsilon} \sum_n (-1)^n \frac{(2n-1)!!}{(2n)!!} \frac{a^{2n}}{b^{2n}} \frac{1}{b^2} [4n+1] P_{2n}(\cos\theta)$$

$$= \frac{Q}{4\pi\epsilon} \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!} \frac{4n+1}{b^2} \left(\frac{a}{b}\right)^{2n} P_{2n}(\cos\theta)$$

5.61.



[ATTENTION HEREON RESTRICTED TO $a \le r \le b$ UNLESS STATED OTHERWISE]

$\mu = \cos\theta$; $\sum_n = \sum_{n=0}^{\infty}$

- LET THE POTENTIAL DUE TO THE INNER SPHERE BE

$$V_{IN} = \sum_n \left(A_n r^n + \frac{B_n}{r^{n+1}}\right) P_n(\mu)$$

SINCE IT'S GROUNDED, $V_{IN}(a) = 0$

$$\Rightarrow A_n a^n = \frac{-B_n}{a^{n+1}} \Rightarrow B_n = -a^{2n+1} A_n$$

THUS

$$V_{IN} = \sum_n A_n \left[r^n - \frac{a^{2n+1}}{r^{n+1}}\right] P_n(\mu) \qquad (1)$$

- LET THE POTENTIAL DUE TO THE OUTER SPHERE BE

$$V_{OUT} = \sum_n \left(D_n r^n + \frac{E_n}{r^{n+1}}\right) P_n(\mu)$$

SINCE IT TOO IS GROUNDED, $V_{OUT}(c) = 0$

$$\Rightarrow D_n c^n = E_n / c^{n+1} \Rightarrow E_n = -D_n c^{2n+1}$$

THUS

$$V_{OUT} = \sum_n D_n \left[r^n - \frac{c^{2n+1}}{r^{n+1}}\right] P_n(\mu) \qquad (2)$$

- THE POTENTIAL DUE TO THE CHARGE q IS

$$V_c = \frac{q}{4\pi\epsilon R}$$

FOR $a \le r < b$

$$V_c = \frac{q}{4\pi\epsilon r} \sum_n \left(\frac{b}{r}\right)^n P_n(\mu) \qquad (3)$$

FOR $b < r \le c$

$$V_c = \frac{q}{4\pi\epsilon b} \sum_n \left(\frac{r}{b}\right)^n P_n(\mu) \qquad (4)$$

✓

MEETING BOUNDRY CONDITIONS

THE TOTAL POTENTENTIAL BETWEEN INNER AND OUTTER SPHERES IS $V = V_{IN} + V_{OUT} + V_0$

- $V(a) = 0$. EQUATING COEFFICIENTS OF $P_n(\mu)$

USING Eqs. (1), (2), AND (3):

$$\frac{-q}{4\pi\epsilon a}\left(\frac{b}{a}\right)^n = A_n\left[a^n - \frac{a^{2n+1}}{a^{n+1}}\right] + D_n\left[a^n - \frac{c^{2n+1}}{a^{n+1}}\right]$$

$$= D_n\left[a^n - \frac{c^{2n+1}}{a^{n+1}}\right]$$

$$= D_n\left[\frac{a^{2n+1} - c^{2n+1}}{a^{n+1}}\right]$$

$$\Rightarrow D_n = \frac{-q}{4\pi\epsilon a}\left(\frac{b}{a}\right)^n \frac{a^{n+1}}{a^{2n+1} - c^{2n+1}}$$

$$= \frac{-q\, b^n}{4\pi\epsilon\left[a^{2n+1} - c^{2n+1}\right]} \qquad (5)$$

- $V(c) = 0$. EQUATING COEFFICIENTS OF $P_n(\mu)$

USING Eqs. (1), (2), AND (4)

$$\frac{-q}{4\pi\epsilon b}\left(\frac{c}{b}\right)^n = A_n\left[c^n - \frac{a^{2n+1}}{c^{n+1}}\right] + D_n\left[c^n - \frac{c^{2n+1}}{c^{n+1}}\right]$$

$$= A_n\left[c^n - \frac{a^{2n+1}}{c^{n+1}}\right]$$

$$= A_n\left[\frac{c^{2n+1} - a^{2n+1}}{c^{n+1}}\right]$$

$$\Rightarrow A_n = \frac{-q}{4\pi\epsilon\, b^{n+1}} \frac{c^{2n+1}}{c^{2n+1} - a^{2n+1}} \qquad (6)$$

(CONT) →

WE WISH TO FIND V FOR $a<r<b$. FROM Eqs. ①, ② & ③

$$V = V_c + V_{IN} + V_{OUT}$$
$$= \sum_n \left[\frac{q}{4\pi\epsilon r}\left(\frac{b}{r}\right)^n + A_n\left(r^n - \frac{a^{2n+1}}{r^{n+1}}\right) + D_n\left(r^n - \frac{c^{2n+1}}{r^{n+1}}\right)\right] P_n(\mu)$$

SUBSTITUTING Eqs. ⑤ AND ⑥:

$$V = \frac{q}{4\pi\epsilon}\left[\sum_n \frac{b^n}{r^{n+1}} + \frac{c^{2n+1}}{b^{n+1}}\frac{1}{(a^{2n+1}-c^{2n+1})}\left(r^n - \frac{a^{2n+1}}{r^{n+1}}\right) - \frac{b^n}{(a^{2n+1}-c^{2n+1})}\left(r^n - \frac{c^{2n+1}}{r^{n+1}}\right)\right] P_n(\mu)$$

$$= \frac{q}{4\pi\epsilon}\sum_n \left\{\frac{b^{2n+1}(a^{2n+1}-c^{2n+1})}{r^{n+1}(a^{2n+1}-c^{2n+1})b^{n+1}} + \frac{c^{2n+1}(r^{2n+1}-a^{2n+1})}{r^{n+1}(a^{2n+1}-c^{2n+1})b^{n+1}} - \frac{b^{2n+1}(r^{2n+1}-c^{2n+1})}{r^{n+1}(a^{2n+1}-c^{2n+1})b^{n+1}}\right\} P_n(\mu)$$

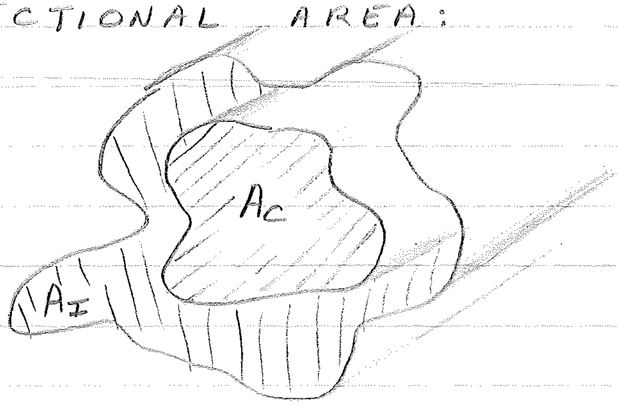
$$= \frac{q}{4\pi\epsilon}\sum_n \frac{1}{r^{n+1}(a^{2n+1}-c^{2n+1})b^{n+1}} \times \left[b^{2n+1}(a^{2n+1}-\cancel{c^{2n+1}}) + c^{2n+1}(r^{2n+1}-a^{2n+1}) + b^{2n+1}(\cancel{c^{2n+1}}-r^{2n+1})\right] P_n(\mu)$$

$$= \frac{q}{4\pi\epsilon}\sum_n \frac{b^{2n+1}(a^{2n+1}-r^{2n+1}) - c^{2n+1}(a^{2n+1}-r^{2n+1})}{r^{n+1}(a^{2n+1}-c^{2n+1})b^{n+1}} P_n(\mu)$$

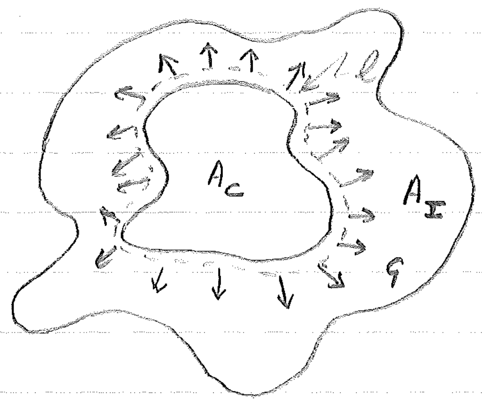
$$= \frac{q}{4\pi\epsilon}\sum_n \frac{(b^{2n+1}-c^{2n+1})}{b^{n+1}(a^{2n+1}-c^{2n+1})}\frac{(a^{2n+1}-r^{2n+1})}{r^{n+1}} P_n(\mu)$$

$$= \frac{-q}{4\pi\epsilon}\sum_{n=0}^{\infty} \frac{b^{2n+1}-c^{2n+1}}{b^{n+1}(a^{2n+1}-c^{2n+1})}\left[r^n - \frac{a^{2n+1}}{r^{n+1}}\right] P_n(\cos\theta)$$

602C. CONSIDER AN ARBITRALILY SHAPED SHEATHED CONDUCTOR WITH GIVEN CROSS SECTIONAL AREA:



TO MINIMIZE THE LEAKAGE CURRENT, CONSIDER THE FOLLOWING (TWO DIMENSIONAL) GAUSSIAN "SURFACE." THE (TWO DIMENSIONAL) LEAKAGE CURRENT IS GIVEN BY

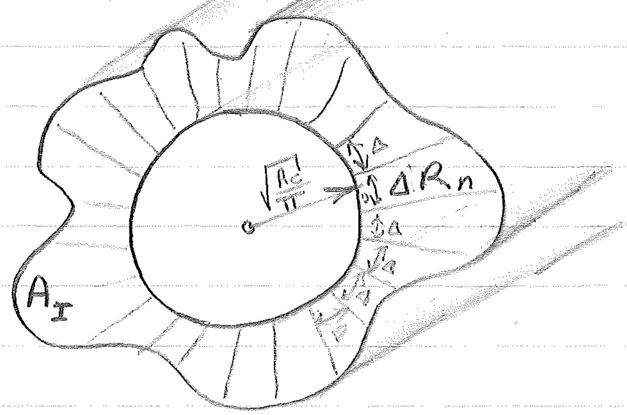


$$I_\ell = -\int_\ell \frac{1}{q} \frac{\delta V}{\delta n} \, d\ell$$

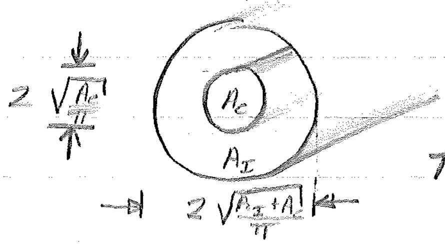
FOR THE INNER CONDUCTOR AT A GIVEN POTENTIAL, THE DOTTED SURFACE IS AT EQUAPOTENTIAL AND I_ℓ IS CLEARLY MINIMIZED BY MINIMIZING THE LENGTH OF THE PATH OF INTEGRATION. ie. BY MINIMIZING ℓ. UNDER THE CONSTRAINT A_c REMAIN CONSTANT, IT IS CLEAR THAT THE OPTIMAL SHAPE OF THE INNER CURVE IS CIRCULAR. $\Longrightarrow$

WE ARE HERE EMPLOYING THE FACT THAT MINIMAZATION OF LEAKAGE CURRENT CORRESPONDS TO MAXIMIZING LATERAL RESISTANCE.

IT REMAINS TO FIND THE OPTIMAL SHEATH SHAPE. CONSIDER THE FOLLOWING GEOMETRY

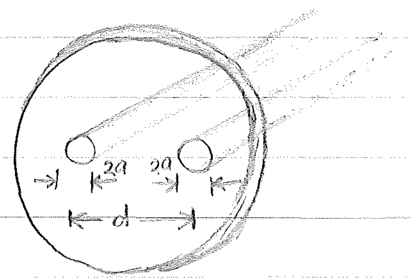


SUPPOSE WE NOW DIVIDE THE SHEATH INTO A NUMBER OF SMALL RESISTANCES WITH LATERAL RESISTANCES ΔR_n. THESE RESISTANCES ARE IN PARALLEL. TO MAXIMIZE THE TOTAL RESISTANCE, IT FOLLOWS FROM ELEMENTARY CIRCUIT THEORY THAT WE WISH TO MAKE ALL ΔR_n'S THE SAME. THUS, DUE TO SYMMETRY, WE REQUIRE THAT THE OUTER SURFACE OF THE SHEATH ALSO BE CIRCULAR:



THIS CONFIGURATION MAXIMIZES RESISTANCE TO CURRENT LEAKAGE.

6.3C



WE KNOW FROM 4.14(2) THAT THE CAPACITANCE BETWEEN TWO SIMILAR CYLINDERS EACH OF RADIUS a AND SEPARATED BY A DISTANCE d IS

$$C = \pi\epsilon\left[\cosh^{-1}\frac{d}{2a}\right]^{-1}$$

THE CORRESPONDING (TWO-DIMENSIONAL) RESISTANCE IS THEN GIVEN FROM 6.06(7) AS

$$R = \frac{\varsigma\epsilon}{C}$$
$$= \frac{\varsigma\epsilon}{\pi\epsilon}\cosh^{-1}\frac{d}{2a}$$
$$= \frac{\varsigma}{\pi}\cosh^{-1}\frac{d}{2a}$$

but in addition to this you have a dish

90%

BOB MARKS
4/15/76
ADV. FIELDS
(CHAPT. 7)

#1

WE HAVE FROM ART. 7.10 #27

$$A_\phi = \frac{\mu I}{4\pi} \oint \frac{ds_\phi}{r}$$

WHERE

$$ds_\phi = a\cos\phi\, d\phi$$

$$r = \sqrt{a^2+\rho^2+z^2-2a\rho\cos\phi}$$

BUT, FROM 5.297, WE CAN WRITE

$$\frac{1}{r} = \sum_{s=0}^{\infty}(2-\delta_s^0)\cos\phi-\phi_0 \int_0^\infty e^{-k|z-z_0|} J_s(ka)J_s(k\rho)dk$$

BUT, DUE TO SYMMETRY, THE RESULTING VECTOR POTENTIAL, A_ϕ, WILL BE INDEPENDENT OF THE VARIABLE ϕ. ALSO, WE HAVE THE CLEARLY OBVIOUS SITUATION OF $z_0=0$. THE EQUATION WRITTEN IMMEDIATELY ABOVE THUS BEOMES

$$\frac{1}{r} = \sum_{s=0}^{\infty}(2-\delta_s^0)\int_0^\infty e^{-k|z|}J_s(ka)J_s(k\rho)dk$$

THUS, THE VECTOR POTENTIAL BECOMES

$$A_\phi = \frac{\mu I a}{4\pi}\int_0^{2\pi}\sum_{s=0}^{\infty}(2-\delta_s^0)\int_0^\infty e^{-k|z|}J_s(ka)J_s(k\rho)dk\,d\phi$$

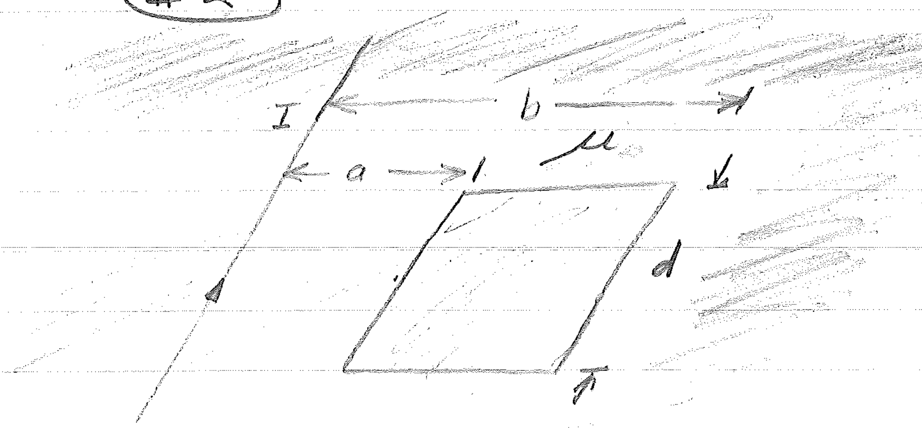
$$= \frac{\mu I a}{2}\sum_{s=0}^{\infty}(2-\delta_s^0)\int_0^\infty e^{-k|z|}J_s(ka)J_s(k\rho)dk$$

it is not clear

BUT, CLEARLY, DUE TO THE PROPOSED SOLUTION (ie, DUE TO B.C.), THE ONLY CONTRIBUTION TO THIS WILL BE $s=1$. CHOOSING $J_1(ka)=0$, THIS IS ?

$$A_\phi = \frac{\mu I a}{2}\int_0^\infty e^{-k|z|}J_1(ka)J_1(k\rho)dk$$

#2



what problem from Book it is?

WE ADDRESS THE PROBLEM OF FINDING B IN THE SQUARE LOOP SHOWN. THE CONTRIBUTION, $\vec{B}$, TO THE FLUX DENSITY AT A POINT P FROM THE SIDE OF THE CONDUCTING LOOP LOCATED AT $x = \frac{a}{2}$ IS DIRECTED

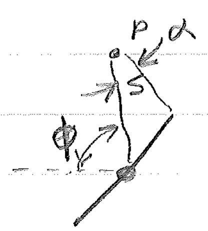
Is it current in the loop?

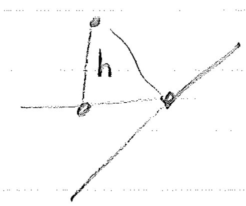
PERPENDICULAR TO THE PLANE FORMED BY THE LOOP AND THE POINT P. ITS MAGNITUDE IS

$$B = \frac{\mu I}{2\pi \underline{ds}} \sin\alpha$$

is it ds or d×s ?

WHERE S IS THE DISTANCE TO THE POINT, AND α THE ANGLE BETWEEN





THUS $B = \frac{\mu I}{2\pi s} \frac{d/2}{\sqrt{(d/2)^2 + s^2}}$

TAKING THE CONTRIBUTION FROM TWO SIDES:

$$B \cos\phi = B \frac{a/2}{s} = \frac{\mu I a d}{8\pi s^2 d} \frac{1}{\sqrt{(d/2)^2 + s^2}}$$

BUT $s^2 = h^2 + (a/2)^2$

$$\Rightarrow B \cos\phi = \frac{\mu a d I}{8\pi d \sqrt{h^2 + \frac{a^2+d^2}{4}}} \frac{1}{h^2 + a^2/4}$$

WE MAY IMMEDIATELY WRITE THE CONTRIBUTION FROM THE OTHER SIDES AS

$$\frac{\mu a d I}{8\pi d \sqrt{h^2 + \frac{a^2+d^2}{4}}} \frac{1}{h^2 + d^2/4}$$

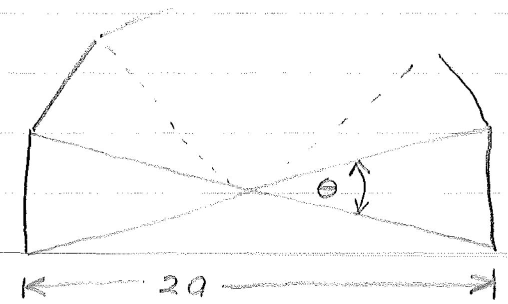
THE TOTAL FLUX IS THE SUM OF THE ABOVE RELATIONSHIPS

$$B = \frac{\mu a d I}{4\pi \sqrt{h^2 + \frac{a^2+d^2}{4}}} \left[\frac{1}{h^2 + \frac{a^2}{4}} + \frac{1}{h^2 + d^2/4} \right]$$

what was happend to ~~the~~ additional wire with current I ?

BOB MARKS

7.1.



WE KNOW THAT $2n\theta = 2\pi \Rightarrow \theta = \frac{n}{\pi}$? $\theta = \frac{\pi}{n}$

FROM BIOT AND SAVART'S LAW (ART. 7.14) WE KNOW THAT THE MAGNETIC INDUCTION DUE TO AN INFINITE WIRE AT A POINT P A DISTANCE a FROM IT IS

$$B = \mu I / 2\pi a$$

IT FOLLOWS FROM ART (7.14) THAT THE MAGNETIC INDUCTION FROM A WIRE OF FINITE LENGTH AT A POINT LYING ON ITS PERPENDICULAR BISECTOR IS

$$B = \frac{\mu I}{2\pi a} \sin \frac{\theta}{2}$$

WHERE θ IS THE ANGLE SUBTENDED BY THE LINE. IT FOLLOW THAT, FOR THE POLYGON CONSIDERED, THAT B DUE A SINGLE SEGMENT IS

$$B_s = \frac{\mu I}{2\pi a} \sin \left(\frac{n}{2\pi}\right)$$

SUPERIMPOSING ALL 2n SEGMENTS GIVES THE DESIRED ANSWER

$$B = 2n B_s$$
$$= n \frac{\mu I}{\pi a} \sin \left(\frac{n}{2\pi}\right)$$

because at the center all B_s are parallel.

BOB MARKS
DUE 4/22/78

50% + 50% - 100%

8032. WE HAVE FROM PROB. 31

$$M_{12} = 2\mu ab \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!!}{(2n+4)!!} \frac{(a^2-b^2)^{n+1}}{r^{2n+3}} P'_{n+1}\left(\frac{a^2+b^2}{a^2-b^2}\right) P_{2n+2}(\cos\alpha) \quad (1)$$

TO FIND THIS RELATIONSHIP FOR $a=b$, WE NEED ONLY TO EVALUATE $\lim_{b\to a} M_{12}$.
THE TERM OF INTEREST IS

$$T_n = \lim_{b\to a} ab(a^2-b^2)^{n+1} P'_{n+1}\left(\frac{a^2+b^2}{a^2-b^2}\right)$$

IN THIS LIMIT, THE ARGUMENT OF P'_{n+1} WILL TEND TO ∞. FROM 5.23(12):

$$P_n^m(\mu) \xrightarrow[\mu\to\infty]{} \frac{2n!}{2^n n!(n-m)!} \mu^n$$

IT FOLLOWS THAT

$$P'_{n+1}\left(\frac{a^2+b^2}{a^2-b^2}\right) \xrightarrow[b\to a]{} \frac{\left(\frac{a^2+b^2}{a^2-b^2}\right)^{n+1}}{2^{n+1}(n+1)!(n!)}$$

THUS

$$T_n = ab(a^2-b^2)^{n+1} \frac{\left(\frac{a^2+b^2}{a^2-b^2}\right)^{n+1}}{2^{n+1}(n+1)!\,n!}$$

$$= ab \frac{(a^2+b^2)^{n+1}}{2^{n+1}(n+1)!\,n!}$$

BUT, SINCE $a=b$:

$$T_n = a^2 \frac{(2a^2)^{n+1}}{2^{n+1}(n+1)!\,n!}$$

$$= a \frac{a^{2n+3}}{(n+1)!\,n!}$$

SUBSTITUTING THIS VALUE FOR

$ab(a^2-b^2)^{n+1} P'_{n+1}\left(\frac{a^2+b^2}{a^2-b^2}\right)$ IN EQ. 1

GIVES THE DESIRED ANSWER:

$$M_{12} = 2\pi\mu a \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!!}{n!(n+1)!(2n+4)!!} \left(\frac{a}{r}\right)^{2n+3} P_{2n+2}(\cos\alpha)$$

8.34. THIS IS ESSENTIALLY EQUIVALENT TO PROB. 32. OUR MUTUAL INDUCTANCE IS NOW

$$M_{12} = 2\pi\mu abmn \sum_{p=0}^{3}\sum_{q=0}^{\infty} \frac{(-1)^{p+q}(2q-1)!!}{(2q+2)(2q+4)!!} \frac{(a^2-b^2)^{q+1}}{r_p^{2q+1}} P_{q+1}^{1}\left(\frac{a^2+b^2}{a^2-b^2}\right) P_{2q}(\cos\alpha_p$$

FROM THE PREVIOUS PROBLEM, WE MAY WRITE

$$P_{q+1}^{1}\left(\frac{a^2+b^2}{a^2-b^2}\right) \xrightarrow[b\to a]{} \frac{\left(\frac{a^2+b^2}{a^2-b^2}\right)^{q+1}}{2^{q+1}(q+1)!\, q!}$$

SUBSTITUTING INTO M_{12}:

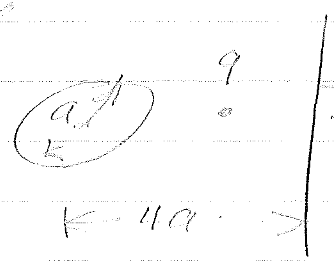
$$\lim_{b\to a} M_{12} = 2\pi\mu abmn \sum_{p=0}^{3}\sum_{q=0}^{\infty} \frac{(-1)^{p+q}(2q-1)!!}{(2q+2)(2q+4)!!\, q!\,(q+1)!\, 2^{q+1}} \frac{(a^2+b^2)^{q+1}}{r_p^{2q+1}} P_{2q}(\cos\alpha_p)$$

BUT $b=a$. THIS, GIVES THE DESIRED ANSWER:

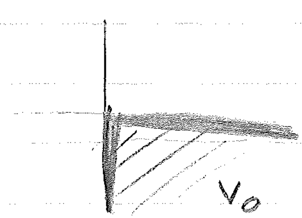
$$\lim_{b\to a} M_{12} = 2\pi\mu a^3 mn \sum_{p=0}^{3}\sum_{q=0}^{\infty} \frac{(-1)^{p+q}(2q-1)!!}{(2q+2)(2q+4)!!\, q!\,(q+1)!} \left(\frac{a}{r_p}\right)^{2q+1} P_{2q}(\cos\alpha_p)$$

1. Find potential outside grounded conducting cylinder of radius a due to line charge q placed @ r = b & θ = 0. Find the image law for this case. (15)

(20)
2. Find V_o outside & potential V_i inside using method of images for charge q placed half way between plane & center of sphere with dielectric capacitance K and radius a. The distance of sphere is distance 4a away from plate.

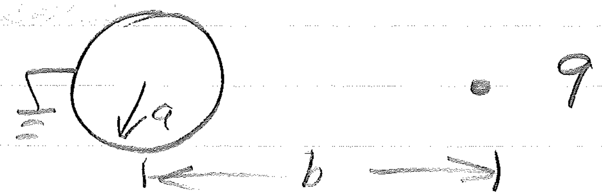


(15)
3. Find field near edge of conducting 90° wedge @ potential V_o whose sides coincide with +X & −Y axis.

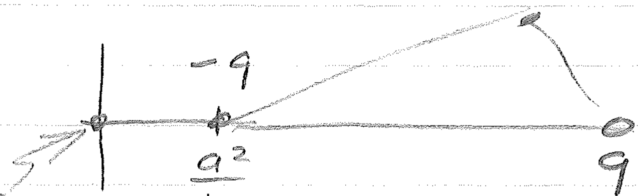


✓ 2 pg 22 (25)
2 pg 41 (25)
59 pg 229 (25)
71 pg 232 (25)
95 pg 235 (25)

1.



By pg. 70, the image is



No charge here. It's grounded!

∴ Potential is

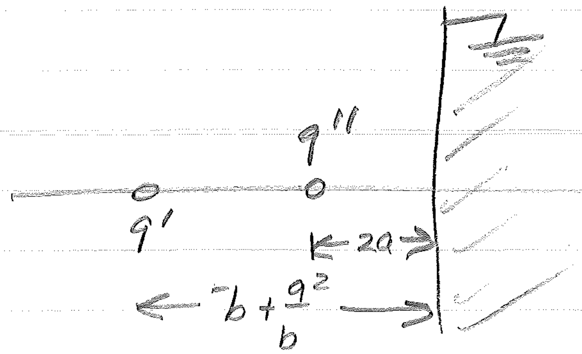
$$V = -\frac{1}{2\pi\epsilon}\left[\sum_s q_s \operatorname{Re} \ln (z - z_s)\right]$$

$$= -\frac{1}{2\pi\epsilon}\left[-q_1 \operatorname{Re} \ln\left(z - \frac{a^2}{b}\right) + q \operatorname{Re} \ln (z - b)\right]$$

$$= \frac{q}{2\pi\epsilon}\left[\ln\sqrt{\left(x - \frac{a^2}{b}\right)^2 + y^2} - \ln\sqrt{(x-b)^2 + y^2}\right]$$

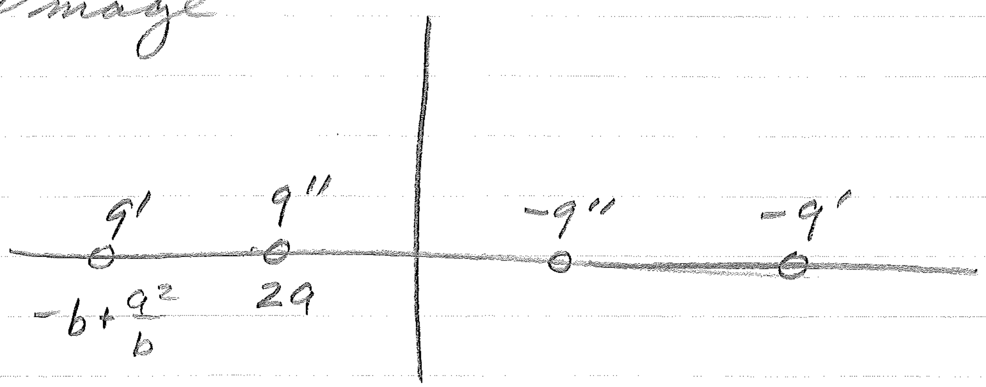
$V \neq 0$ at the cylinder

Inside, Image is, from pg 69



$$q'' = \frac{2}{1+K} q$$

Image



Use Eq. 1 with

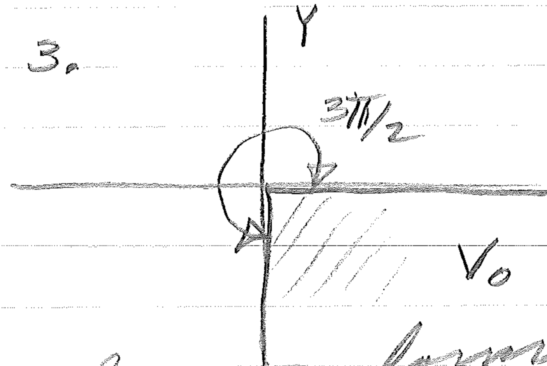
$$z_1 = -b + \frac{a^2}{b} \qquad q_1 = q' \qquad q' = \frac{1-K}{1+K} q$$

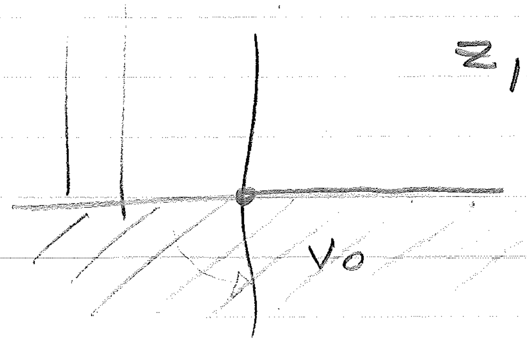
$$z_2 = -2a \qquad q_2 = q''$$

$$z_3 = 2a \qquad q_3 = q'' \qquad q'' = \frac{2}{(1+K)} q$$

$$z_4 = b - \frac{a^2}{b} \qquad q_4 = -q'$$

3.





Schwartz xform

$$\frac{dz}{dz_1} = \left(z_1^{\frac{3\pi}{2}\left(\frac{1}{\pi}\right)-1}\right) = z_1^{-\frac{1}{2}}$$

$$z = \int z_1^{-\frac{1}{2}}\, dz_1$$

$$z = \frac{1}{2} z_1^{\frac{1}{2}} \xRightarrow{\text{LET}} z_1 = z^2$$

$$z = 0 \Rightarrow z_1 = 0$$

$$z_0 = e^{i\frac{3\pi}{2}} \Rightarrow z = e^{i3\pi}$$

LET

$$z_1 = z^{\frac{2}{3}}$$

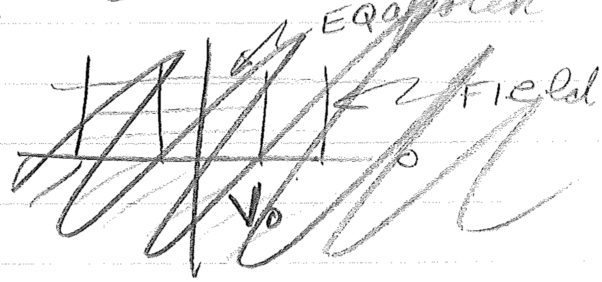
$$z = e^{i3\pi/2} \Rightarrow z_1 = e^{i\pi}$$

$$z = 0 \Rightarrow z_1 = 0$$

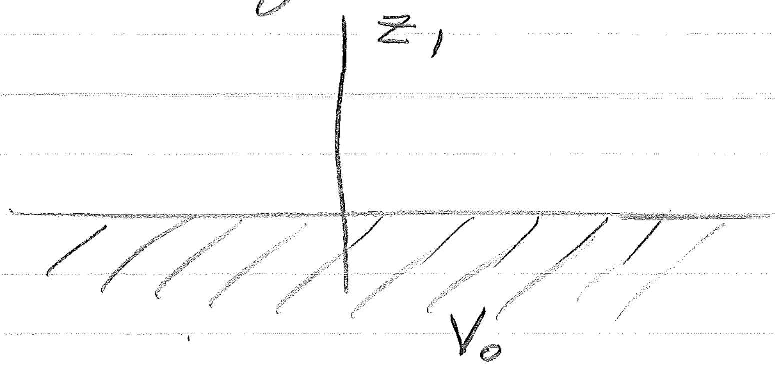
$$z = e^{i\pi/2} \Rightarrow z_1 = e^{i\pi/3}$$

$\cancel{W = z_1 = x_1 + iy_1}$

$\cancel{W_{x_1<0} = V_0}$

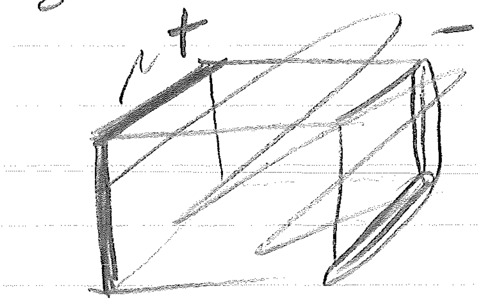


This xform opens the problem to a single sheet:

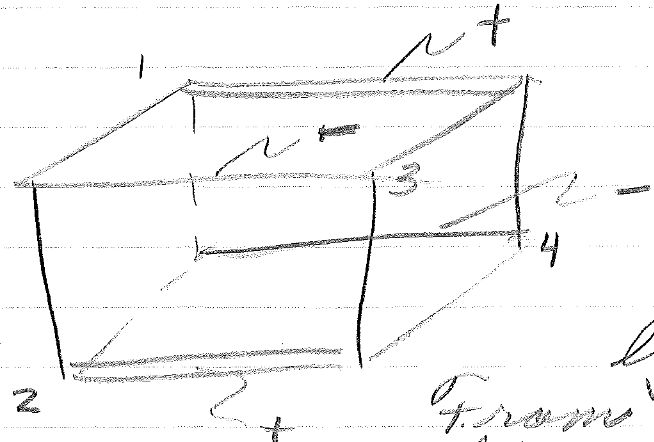


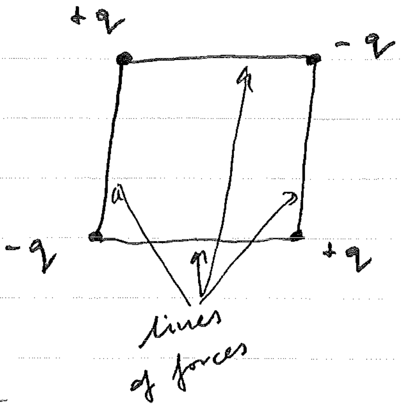
and? ....

Pg 22. # 2



Let total flux leaving a line charge $= \phi$



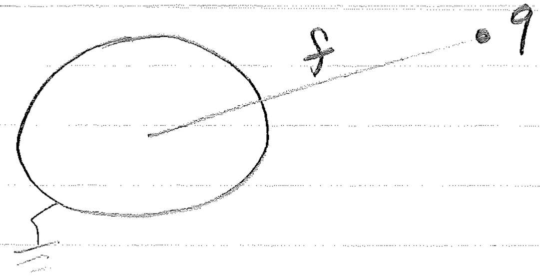


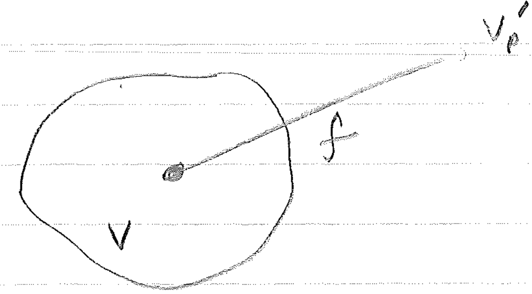
From lines 1 & 2, the total flux entering prism is (for both) $= \Phi/4$) why?
(This is due to symmetry).
Let $\rho = Q/l$ = linear charge density. If we take a gaussian surface enclosing box, we get zero. ~~i, on the~~

What is the conclution ?

71C Br pg. 231

$$r = a(1 + n S_n)$$





$$Q = \frac{-V_p'}{V'} q$$

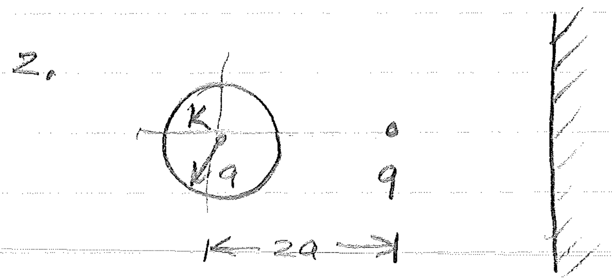
For a sphere: $\quad f \qquad Q = \frac{-a q}{f}$

$$Q \cong -\frac{q}{f} \; a(1 + n S_n)$$

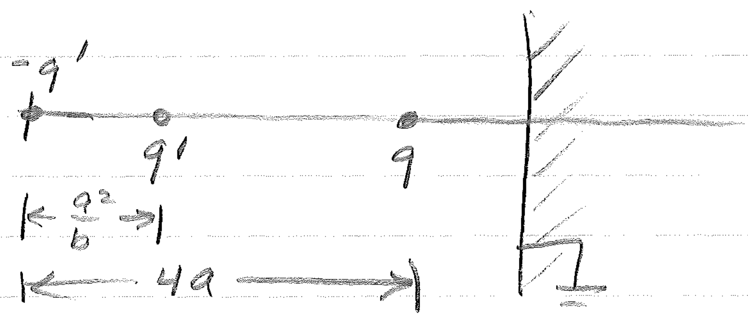
$q = 1$ (unit charge)

$$Q \simeq \frac{a}{f}(1 + n S_n)$$

2.

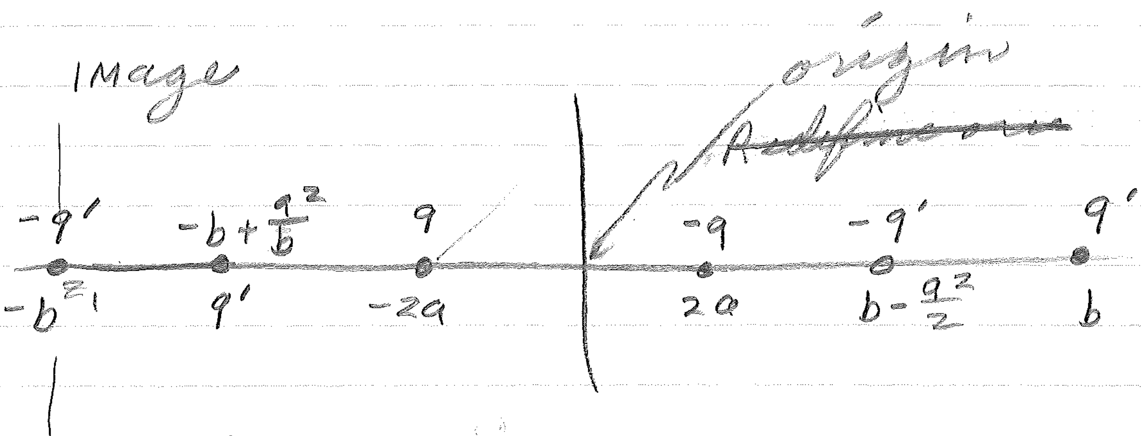


EQUAL TO (pg. 69)



$$q' = \frac{1-K}{1+K} q$$

Image



$$V_0 = \frac{-1}{2\pi\epsilon} \sum q_s \operatorname{Re} \ln (z - z_s) \qquad (1)$$

$$\begin{aligned}
z_1 &= -b & q_1 &= -q' \\
z_2 &= -b + \frac{a^2}{b} & q_2 &= q' \\
z_3 &= -2a & q_3 &= q \\
z_4 &= 2a & q_4 &= -q \\
z_5 &= b - \frac{a^2}{2} & q_5 &= b - \frac{a^2}{2} \\
z_6 &= b & q_6 &= q'
\end{aligned}$$

$$q' = \frac{1-K}{1+K} q$$

5

Pg 41, # 2

1 2

3 4.

$$S_{11} = S_{22} = S_{33} = S_{44} = S$$
$$S_{12} = S_{13} = S_{34} = S_{42} = S_s$$
$$S_{14} = S_{23} = S_\ell$$

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} S & S_s & S_s & S_\ell \\ S_s & S & S_\ell & S_s \\ S_s & S_\ell & S & S_s \\ S_\ell & S_s & S_s & S \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix}$$

$V_1 = V_4$ V_3

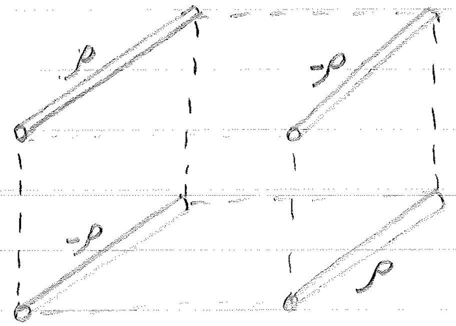
ARG!

1.2 5
2.2 10
5.71 10
5.95 10

$\} = 35\% + 58\% = 93\%$

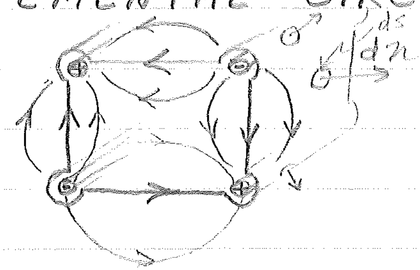
ROBERT J. MARKS II.
06 APR 1976
EE 5341

1.2.



ρ = CHARGE DENSITY

SUPPOSE WE PLACE A GAUSSIAN SURFACE IMMEDIATELY ON THE PRISM, SUCH THAT ALL THE LINE CHARGES ARE ENCLOSED BY INCREMENTAL CIRCLES:

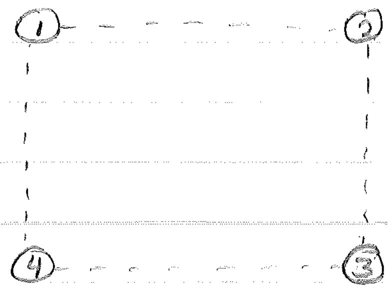


$$q = \epsilon \oint_s \vec{E} \cdot \vec{n}\, ds$$
= TOTAL CHARGE ENCLOSED
= 0

GAUSS' FLUX THEOREM STATES THAT THE TOTAL FLUX NORMAL TO THE SURFACE IS EQUIVALENT TO THE TOTAL CHARGE ENCLOSED. FOR THE SURFACE ABOVE, THIS IS CLEARLY ZERO.

IT FOLLOWS FROM SYMMETRY THAT THE TOTAL FLUX THAT IS LEAVING THE SURFACE IS THE SAME AS THAT ENTERING. THAT IS, $\frac{1}{2}$ (50%) OF THE TOTAL FLUX ENTERS THE BOX.

202.



IT FOLLOWS BY SYMMETRY, THAT

$$S_{12}=S_{21}=S_{14}=S_{41}=S_{23}=S_{32}=S_{34}=S_{43}$$

$$S_{11}=S_{22}=S_{33}$$

$$S_{13}=S_{31}=S_{24}=S_{42}$$

WE NOW GO THROUGH THE PERSCRIBED STEPS:

(a) PLACE CHARGE Q ON #1

(b) TOUCH #1 TO #2

$$\Rightarrow V_1 = V_2$$

$$V_1 = S_{11}Q_1 + S_{21}Q_2 \qquad V_2 = S_{12}Q_1 + S_{22}Q_2$$

THUS $\quad S_{11}Q_1 + S_{21}Q_2 = S_{12}Q_1 + S_{22}Q_2$

$$(S_{11}-S_{12})Q_1 = (S_{22}-S_{21})Q_1$$

BUT $\quad S_{11}-S_{12} = S_{22}-S_{21}$

$$\therefore Q_1 = Q_2$$

WE KNOW THAT, DUE TO CHARGE CONSERVATION: $\quad Q_1 + Q_2 = Q$

IT FOLLOWS THAT

$$Q_1 = Q_2 = Q/2$$

(c) TOUCH #1 TO #3

$$\Rightarrow V_1 = V_3$$

$$V_1 = S_{11} Q_1' + S_{21} Q_2' + S_{31} Q_3'$$

$$V_3 = S_{13} Q_1' + S_{23} Q_2' + S_{33} Q_3'$$

THUS $(S_{11} - S_{13}) Q_1' + (S_{21} - S_{23}) Q_2' + (S_{31} - S_{33}) Q_3' = 0$

BUT $S_{21} - S_{23} = 0$ AND $S_{11} - S_{13} = -(S_{31} - S_{33})$

$$\Rightarrow Q_1' = Q_3'$$

WE KNOW THAT $Q_1' + Q_3' = Q_1 = \frac{Q}{2}$

IT FOLLOWS THAT

$$Q_1' = Q_3' = Q/4 \Rightarrow Q_2' = \frac{3Q}{4}$$

(d) TOUCH #1 TO #4

$$\Rightarrow V_1 = V_4$$

$$V_1 = S_{11} Q_1'' + S_{21} Q_2'' + S_{31} Q_3'' + S_{41} Q_4''$$

$$V_4 = S_{14} Q_1'' + S_{24} Q_2'' + S_{34} Q_3'' + S_{44} Q_4''$$

OR

$$(S_{11} - S_{14}) Q_1'' + (S_{21} - S_{24}) Q_2'' = (S_{34} - S_{31}) Q_3'' + (S_{44} - S_{41}) Q_4'' \quad (2)$$

BUT $S_{11} - S_{14} = S_{44} - S_{41}$ & $(S_{21} - S_{24}) = (S_{34} - S_{31})$

ALSO WE KNOW THAT

$$Q_2'' - Q_3'' = \frac{Q}{2} \quad (3)$$

$$Q_1'' + Q_4'' = Q_1' = \frac{Q}{4} \quad (4)$$

REWRITING (2) USING (4) GIVES

$$Q_4'' = Q_1'' + \frac{S_{21} - S_{24}}{S_{11} - S_{14}} (Q_2'' - Q_3'')$$

SUBSTITUTING (3)

$$Q_4'' = Q_1'' + \frac{S_{21} - S_{24}}{S_{11} - S_{14}} \frac{Q}{2}$$

SUBSTITUTING (4) :

$$Q_4'' = \frac{Q}{4} - Q_4'' + \frac{S_{21} - S_{24}}{S_{11} - S_{14}} \frac{Q}{2}$$

$$2 Q_4'' = \frac{Q}{4} + \frac{S_{21} - S_{24}}{S_{11} - S_{14}} \frac{Q}{2}$$

$$= \frac{(S_{11} - S_{14}) + (S_{21} - S_{24})}{4 (S_{11} - S_{14})} Q$$

$$= \frac{S_{11} - (S_{14} - S_{21}) - S_{24}}{4(S_{11} - S_{14})} Q$$

BUT $S_{14} = S_{21}$. IT FOLLOWS THAT

$$Q_4'' = \frac{Q}{8} \frac{S_{11} - S_{24}}{S_{11} - S_{14}} \quad (5)$$

FROM (4)

$$Q_1'' = \frac{Q}{4} - Q_4''$$

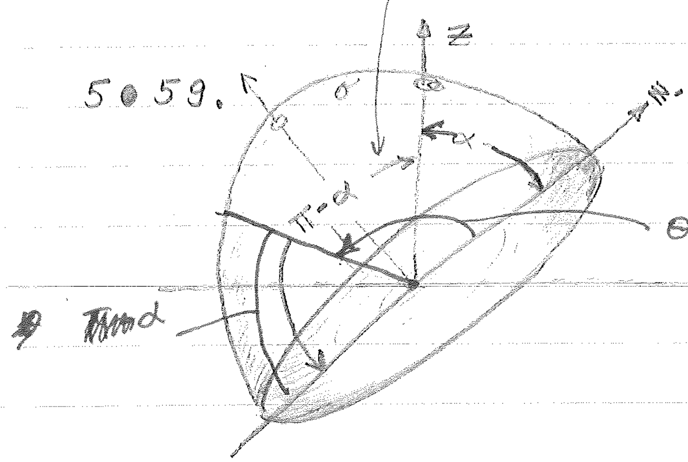
$$= \frac{Q}{4} - \frac{Q}{8} \left[\frac{S_{11} - S_{24}}{S_{11} - S_{14}} \right]$$

$$= \frac{Q}{4} \left[\frac{(S_{11} - S_{14})2}{(S_{11} - S_{14})2} - \frac{(S_{11} - S_{24})}{2(S_{11} - S_{14})} \right]$$

$$\therefore Q_1'' = \frac{Q}{8} \frac{S_{11} - 2 S_{14} + S_{24}}{(S_{11} - S_{14})}$$

this is not $\theta = \pi - \alpha$

5.59.



FOR $\alpha = 0$, AND A COMPLETE SPHERE, WE KNOW THAT THE EXTERNAL POTENTIAL IS OF THE FORM OF A POINT CHARGE AT THE ORIGIN:

$$V = \frac{q}{4\pi\epsilon r}$$
$$= \left(\frac{q}{4\pi a^2}\right)\frac{a^2}{\epsilon r}$$
$$= \frac{\sigma a^2}{\epsilon r} \qquad (1)$$

THIS IS EXACTLY WHAT THE GIVEN ANSWER BECOMES FOR $\alpha = 0$:

$$V = \frac{a\sigma}{\epsilon}\left[\frac{a}{r}\cos\alpha + \sum_{n=1}^{\infty}\left\{P_{2n+1}(\cos\alpha) - P_{2n-1}(\cos\alpha)\right\} \times \left(\frac{a}{r}\right)^{2n+1} P_{2n}(\cos\theta)\right] \qquad (2)$$

FOR $\alpha = 0$, $\cos\alpha = 1$

AND, FROM ART. 1.157, $P_n(1) = 1$ FOR ANY n.

THUS (2) BECOMES

$$V = \frac{a\sigma}{\epsilon}\left[\frac{a}{r}\right] = \frac{a^2\sigma}{\epsilon r} \qquad (3)$$

WHICH IS THE SAME AS IS GIVEN IN (1) ⇒

NOW, IT WOULD SEEM THAT IF THE POTENTIAL $V(r, \theta, \phi)$ IN (3) WERE VALID, THEN THE CORRESPONDING POTENTIAL FOR THE HEMISPHERE IN THE FIGURE COULD BE GIVEN SIMPLY BY A COORDINATE ROTATION ABOUT THE ORIGIN. SINCE $V(r, \theta, \phi)$ IS A FUNCTION ONLY OF r, AND SINCE r WOULD SUFFER NO IDENTITY CHANGE UPON SUCH A ROTATION, ∴ THE RESULTING POTENTIAL WOULD BE THE SAME. THAT IS, Eq. 3 IS SYMMETRIC ABOUT THE ORIGIN, AND A ROTATION ABOUT THE ORIGIN WOULD LEAVE IT UNALTERED.

OBVIOUSLY, Eq. 3 IS NOT THE GIVEN SOLUTION. WHAT IS THE FLAW IN LOGIC HERE?

5.71C. FIRST OFF, WE KNOW FROM 2.14(1) THAT THE INDUCED CHARGE ON A GROUNDED CONDUCTOR DUE TO A POINT CHARGE q IS

$$Q = -\frac{V_p'}{V'}q \qquad (1)$$

A FIRST ORDER APPROXIMATION FOR THE PROBLEM AT HAND, IS

$$Q = -\frac{aq}{f}$$

THIS FIRST ORDER APPROXIMATION IS VALID FOR A SPHERE OF RADIUS a AND A POINT CHARGE q A DISTANCE f FROM THE SPHERE'S CENTER, AND IS A VALID SOLUTION FOR THE PROBLEM AT HAND FOR $n=0$.

TO FIND A BETTER STATEMENT, WE CALL ON ART. 5.131, EQS. 6 AND 7. APPROXIMATIONS MADE AT THE BEGINNING OF THIS ARTICLE MAKE THESE RELATIONSHIPS APPROXIMATIONS. THUS, THE ANALYSIS TO FOLLOW SHOULD BE VIEWED AS A (SECOND ORDER) APPROXIMATION.

IN LINE WITH 5.131(6), THE SURFACE FOR THE PROBLEM AT HAND MAY BE CONSIDERED AS TWO SURFACES SUPERIMPOSED:

Σ_1 : a

Σ_2 : $a n S_n$ if so, then why * ?

THAT IS, TAKING $\sigma \cong \frac{q'}{4\pi a^2}$, WE MAY WRITE

$$S_0' = \frac{q'}{4\pi a^2}$$
$$S_n' = \frac{q'}{4\pi a^2} n S_n \qquad * \qquad (2)$$

WE WRITE 5.131(7) IN CLOSED FORM:

$$V_0 = \frac{a}{\epsilon} \sum_{n=0}^{\infty} \left(\frac{a}{r}\right)^{n+1} \frac{1}{2n+1} S_n'$$

THUS, THE POTENTIAL DISTRIBUTION EXTERNAL TO OUR GIVEN SURFACE IS

$$V_0 = \left(\frac{a}{\epsilon}\right)\left(\frac{q'}{4\pi a^2}\right)\left[\left(\frac{a}{r}\right) + \frac{n}{2n+1}\left(\frac{a}{r}\right)^{2n+1} S_n\right]$$
$$= \frac{q'}{4\pi\epsilon a}\left(\frac{a}{r}\right)\left[1 + \frac{n}{2n+1}\left(\frac{a}{r}\right)^{2n} S_n\right]$$

THE V_p' IN (1) IS SIMPLY OBTAINED BY EVALUATING THIS EXPRESSION AT $(r, \theta, \phi) = (f, \theta_0, \phi_0)$

$$V_p' = \frac{q'}{4\pi\epsilon a}\left(\frac{a}{f}\right)\left[1 + \frac{n}{2n+1}\left(\frac{a}{f}\right)^{2n} S_n(\theta_0, \phi)\right] \qquad (3)$$

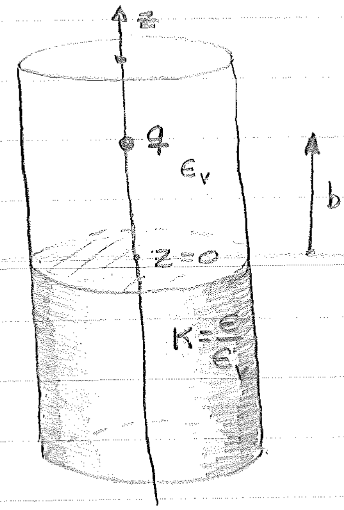
WE NOW NEED TO COMPUTE V' AS GIVEN IN (1). APPROXIMATING OUR SURFACE AS A SPHERE OF RADIUS a, WE HAVE

$$V' \cong \frac{q'}{4\pi\epsilon a} \qquad (4)$$

FOR A UNIT CHARGE, $q = 1$. THUS, SUBSTITUTING (3) AND (4) INTO (1) GIVES

$$Q = -\frac{a}{f}\left[1 + \frac{n}{2n+1}\left(\frac{a}{f}\right)^{2n} S_n(\theta_0, \phi_0)\right]$$

5.95.



WE FIRST LOOK AT THE CASE WHERE NO DIELECTRIC IS PRESENT IN THE LOWER HALF OF THE CYLINDER. THIS PROBLEM IS ADDRESSED IN ART. 5.298. WITH REFERENCE TO 5.298(4), WE NEED ONLY TO MAKE THE FOLLOWING OBSERVATIONS

① DUE TO SYMMETRY, THE POTENTIAL WILL NOT BE A FUNCTION OF ϕ.

② ALSO DUE TO SYMMETRY, ONLY ZEROTH ORDER BESSEL FUNCTIONS OF THE FIRST KIND (ie J_0) WILL APPEAR IN THE POTENTIAL EXPANSION. (SINCE J_0 IS THE ONLY INTEGRAL ORDER EVEN BESSEL FUNTION OF THE FIRST KIND. THAT IS $J_0(x) = J_0(-x)$)

③ THE POINT CHARGE q IS PLACED AT b ON THE Z AXIS. THUS, WE MUST ACCORDINGLY TRANSLATE THE POTENTIAL BY REPLACING z WITH $z - b$

FROM THESE OBSERVATIONS, WE REWRITE 5.298(4) AS

$$V_1 = \frac{q}{2\pi\epsilon a^2} \sum_{r=1}^{\infty} e^{-\mu_r |z-b|} \frac{J_0(\mu_r \rho)}{\mu_r J_1^2(\mu_r a)} \qquad (1)$$

WE HAVE HERE USED THE FACT THAT $J_0(0) = 1$ TO ELIMINATE THE $J_0(\mu_r b)$ TERM IN 5.298(4) (NOTE: THE b IN 5.298(4) IS NOT THE SAME AS HERE, BUT IS THE DISTANCE ρ OF THE POINT CHARGE FROM THE z AXIS, WHICH, HERE, IS ZERO).

SINCE WE ARE CONCERNED WITH THE POTENTIAL ABOVE THE DIELECTRIC AND WITHIN THE CYLINDER, WE MAY APPLY RESULTS GIVEN IN ART. 5.05. BY INTRODUCING THE DIELECTRIC, WE CHANGE THE POTENTIAL ACCORDING TO 5.05(3). SETTING THEIR K_2 TO OUR K, THEIR K_1 TO 1, AND THEIR $f(x,y,z)$ TO OUR $V_1(\rho,\phi,z)$ (GIVEN IN (1)) YIELDS

$$V(\rho,\phi,z) = V_1(\rho,\phi,z) - \frac{K-1}{K+1} V_1(\rho,\phi,-z)$$

$$= \frac{q}{2\pi\epsilon a^2} \sum_{r=1}^{\infty} e^{-\mu_r |z-b|} \frac{J_0(\mu_r \rho)}{\mu_r J_1^2(\mu_r a)} - \frac{K-1}{K+1} \frac{q}{2\pi\epsilon a^2} \sum_{r=1}^{\infty} e^{-\mu_r |-z-b|} \frac{J_0(\mu_r \rho)}{\mu_r J_1^2(\mu_r a)}$$

$$= \frac{q}{2\pi\epsilon a^2} \sum_{k=1}^{\infty} \left[e^{-\mu_r |z-b|} - \frac{K-1}{K+1} e^{-\mu_k (z+b)} \right] \frac{J_0(\mu_r \rho)}{\mu_r J_1^2(\mu_r a)}$$

WHERE, OF COURSE, $z \geq 0$, $\rho \leq a$, & $J_0(\mu_r a) = 0 \ \forall r$

TEST 2

15 1. List all properties that the magnetic induction and the electric current have in common

15 2. Explain the sense of polarization of the electromagnetic waves. List all types of polarization you know and the consequances it have on the wave propagation

30 3. 30 p. 278

30 4. 7 p. 320

30 5, 15 p. 410

$90 \leq g$ A

$80 \leq g$ B < 90 $70 \leq g$ C < 80

1. Smythe defined magnetic induction as a vector $\bar{B}$ in the magnetic field whose magnitude (in webers per square meter) is the torque in newton meters on a loop of moment one whose axis is normal to this direction. The magnetic field is determined by a small exploring loop of unit area and current yielding an unit moment.

The torque and therefore B, depends on the permeability μ of the medium. In a uniform medium, μ is constant. In a crystal, μ ~~is~~ is a tensor. In ferromagnetic materials, $\bar{B}$ + μ are related (hysteresis).

The properties of B can be summarized by the relationship

$$\bar{B} = \mu \bar{H}$$

and by the Maxwell's eq's:

$$\nabla \cdot \bar{B} = 0$$

$$\nabla \times E = -\frac{\delta B}{\delta t} \qquad \frac{\partial \bar{B}}{\partial t}$$

δ ← ~~this~~ symbol has other uses

$\nabla \cdot B = 0$ says that all the lines must [illegible] on each other. $\nabla \times E = -\delta B/\delta t$ says a moving magnetic field causes an electric field.

B is so many times expressed via the vector potential A such that

$$\bar{B} = \bar{\nabla} \times \bar{A}$$

A is unique to within a gradient of a scalar (this allows for gauges). For computational purposes, A is many times written

$$A = \nabla \times W$$

where ~~W~~ (A) has appropriately chosen (usually 2) components.

Rather $\bar{W}$; $\bar{A}$ is determined by current density or by $\bar{B}$.

It is not an answer for question (problem 1).

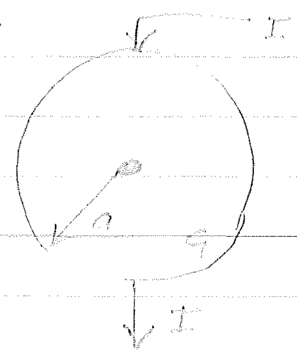
2. The sense of polarization of an ~~em~~ electromagnetic wave is a ~~statement~~, concerning the manner in which ~~the~~ the wave propagates specifically addressing the manner in which the fields change with respect to the direction of propagation. The plane of polarization, for example, is ~~[illegible] by~~ ~~[illegible]~~ in l polarized waves as ~~being~~ that which is formed by the E_x component of the wave and the direction of the wave normal. By linearly polarization (say in the x direction for a wave propagating in the z direction) means that the electric field has only a value in the x direction and varies sinusoidally in

in this manner for changing Z and to linearly an elliptically (or circularly which is a special case) polarized wave. The vector; ~~or~~, again oscillating in time may be thought of as rotating in an elliptical manner for increasing Z. I refer to Fig 11.09 on pg. 429 for a pictorial representation.

Polarization in anisotropic media such as a crystal will vary with propagation velocity. Thus for this case polarization is a function of direction (cosines). (Art 11-03)

PROB. 3

30.



p. 278.

Show

$$V = \frac{qI}{2\pi} \sum_n \frac{4n+3}{(2n+1)(2n+2)} \left(\frac{r}{a}\right)^{2n+1} P_{2n+1}(\mu)$$

$\mu = \cos\theta$

Find the potential inside of the shell.

From symmetry we know that the result will be independent of ϕ. (This is reflected in the answer).

~~Also we know that~~

We know the general potential (inside the sphere) is ~~write~~ $\cancel{?}$ due to arbitrary nature of A_n & B_n (may write $\frac{qI}{2\pi}$

$$V = \frac{qI}{2\pi} \sum_n \left[A_n r^n + B_n r^{-n-1}\right] P_n(\cos\theta)$$

~~Thus~~ The field inside must be finite. Thus we must set $B_n = 0$. Thus

$$V_i = \frac{qI}{2\pi} \sum_n A_n r^n P_n(\cos\theta) \quad (2)$$

We also see, by inspection, that V is an ~~even~~ odd function of θ such that $V(\theta) = -V(-\theta)$.
As such, due to the "oddness" of ~~the~~ only the odd order Legendre polynomials (See Fig 5.15?), ~~of~~ only odd indexed $P_n(\cos\theta)$ will contribute to the potential relationship.
As such, we may write (2) as

$$V_i = \frac{\sigma I}{2\pi} \sum_n A_{2n+1} (r)^{2n+1} P_{2n+1}(\cos\theta) \quad (3)$$

We ~~should~~ could proceed from here to find boundary conditions, such as finding the external potential and supply the current at the pole poles. Solving these boundary conditions would give

$$A_{2n+1} = \frac{4n+3}{(2n+1)(2n+2)} \frac{1}{a^n}$$

what from?

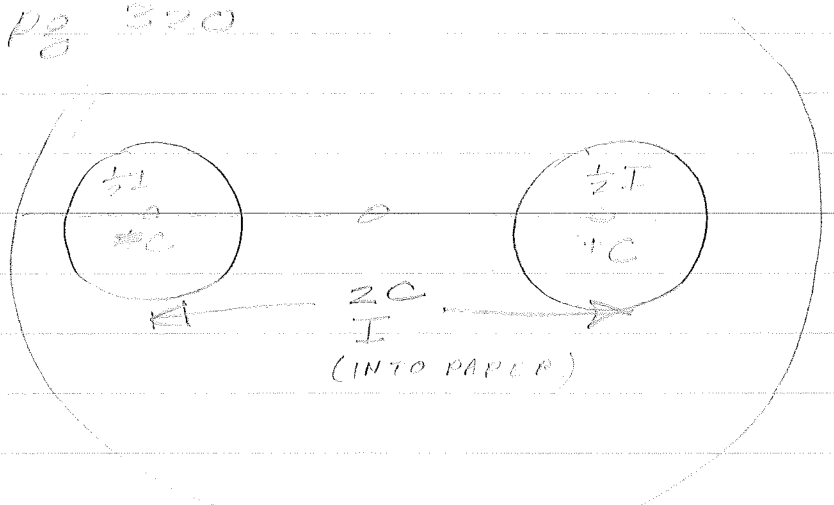
Substituting this into (3) would give the desired answer

$$V_i = \frac{1}{2\pi} \sum_n \frac{4n+3}{(2n+1)(2n+2)} \left(\frac{r}{a}\right)^{2n+1} \times P_{2n+1}(\cos\theta)$$

Are you sure that you will solve the problem if you don't have answer ~~at the~~ given?

4

7. pg 320



Show the force toward center is $\frac{\mu I^2}{16\pi c}$

We know from Art. 219 that the force between two wires separated by a distance a is $F = -\mu I I'/2\pi a$

Thus, the force F' acting between the two wires (a = SEPARATION = 2C)

is
$$F' = -\mu (I/2)(I/2)/2\pi(2c)$$
$$= -\mu I^2/16\pi c$$

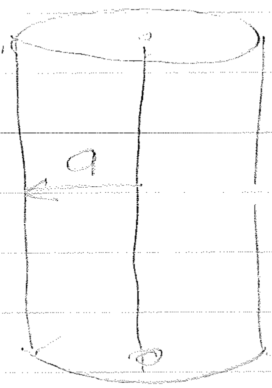
The wires are attracted to each other. Thus the force toward the center acting on (one of) the inner cylinders is simply

$$F = -F' = \frac{\mu I^2}{16\pi c} \qquad \text{QED}$$

What with outher cylinder?

5

#15, pg 410



$B\cos\omega t$

$\tan\epsilon = 2g/\mu_v\omega a$ (1)

The vector potential inside is given in Art 10.20:

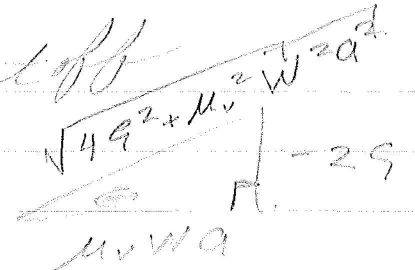
$$\vec{A} = -\vec{k}\sum_{n=1}^{\infty} C_n\rho^n a^{-n}\cos(n\theta+\delta_n) \times \sin\epsilon_n \sin(\omega t - \epsilon_n)$$

For our case, from (1), only one of these terms is applicable.

$$\vec{A} = -\vec{k}\,C\rho^n a^{-n}\cos(n\theta+\delta_n)\,\sin\epsilon\,\sin(\omega t - \epsilon) \quad (2)$$

Now the scalar potential must be finite inside (which it is). We get to play around ~~a~~ now to see what we can throw away, and find out what n is. →

First off



it follows that

$$\sin\epsilon = \frac{-2q}{\sqrt{4q^2+\mu_r^2 w^2 a^2}}$$

Thus (2) becomes

$$\vec{A} = \hat{k}\,\overbrace{C\rho^n a^{-n}\cos(n\theta+\delta_n)(-2q)}\,[4q^2+\mu_r^2 w^2 a^2]^{-\frac{1}{2}}\sin(wt-\epsilon) \qquad (3)$$

But, we are given the external B field

$$\bar{B} = \hat{k}\cos wt \qquad (4)$$

The ~~first~~ terms in the bracketed portion of (3) are recognized as ~~[illegible]~~ the negative shifted equivalent for this reliability whose [illegible] [illegible] we are for all intents and purposes [illegible] the integral [illegible] ([illegible])

As such we may replace the bracketed portion in (3) by the negative of (4) to give

$$A = (-B)(-2q)\left[4q^2 + \mu_v^2 w^2 a^2\right]^{-\frac{1}{2}} \sin wt - \epsilon$$

$$= 2qB\left[4q^2 + \mu_v^2 w^2 a^2\right]^{-\frac{1}{2}}$$

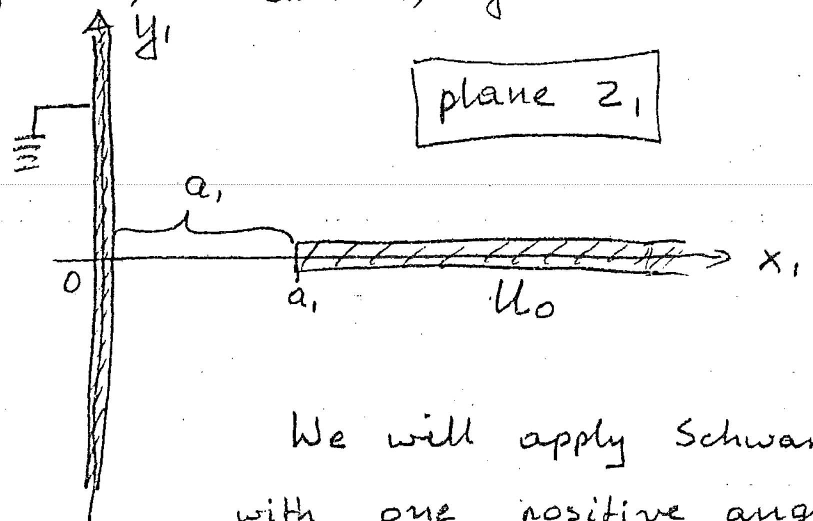
$$\times \sin wt - \epsilon$$

MARKS

①

Example

Find the field for the following configuration of two planes, one earthed, infinite and one at potential U_0, semiinfinite



We will apply Schwarz transformation with one positive angle $\alpha = \frac{\pi}{2}$ and $u_1 = a_1$ which will give us field between parallel plates

$$\frac{dz}{dz_1} = C_1 (z_1 - u_1)^{\frac{\alpha}{\pi} - 1}$$

$$u_1 = a_1 \qquad \alpha = \frac{\pi}{2}$$

$$z = 2C_1 (z_1 - a_1)^{1/2} + C_2 \qquad (*)$$

Now we should finde C_1 and C_2

1) for $z_1 = a_1$ we wish to have $z = a$

2) for $z_1 = 0$ we wish to have $z = 0$

From 1) if apply to (*) we have

$$\boxed{C_2 = a}$$

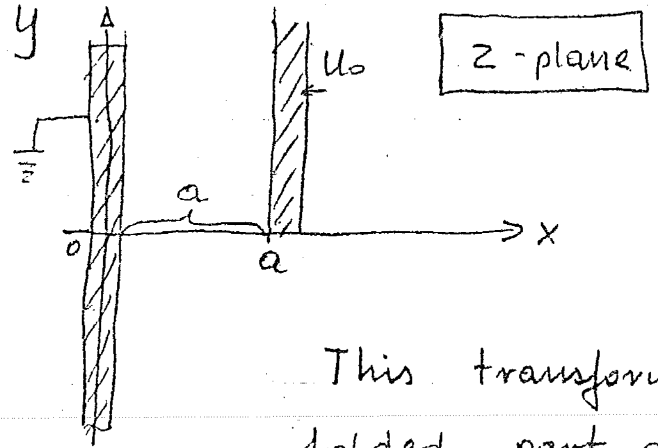
From 2) if apply to (*) we have

$$\cancel{0 = 2 C_1 a} \quad 0 = 2C_1 j\sqrt{a_1} + a \rightarrow C_1 = -\frac{a}{2j\sqrt{a_1}} = \frac{aj}{2\sqrt{a_1}}$$

Thus we have our transformation:

$$\boxed{z = j\frac{a}{\sqrt{a_1}}(z_1 - a_1)^{1/2} + a} \qquad (**)$$

Now our problem is reduced to following



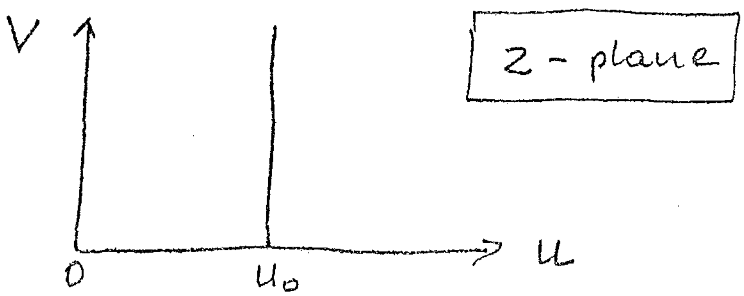
This transformation as you see did not folded part of real axis x_1 : $0 \leq x_1 \leq a_1$

That line before transformation was a line of force then after transformation ~~all~~ this will be also line of force and all other for $y > 0$ will be parallel. (plates for $y>0$ are infinite).

Originally problem was symmetric with regard to axis x and we will solve the problem only for $y > 0$, however after transformation back to z_1 plane we could extend solution for $y_1 < 0$.

Becouse all line of forces are parallel and eqipotential line perpendicular to them and parallel each to other and to plates, therefore enstead talking about z-plane with $z = x + jy$ we could understand it as a z-plane with $W = u + jV$ becouse u = const are parallel to x = const and V = const. to y = const.

We have



in our transformation we should now put $\boxed{a = u_0}$

In this system equipotential lines are stight lines, $U = \text{const}$ and $0 \le U \le U_0$, also stream functions (line of forces) are stright lines, $V = \text{const.}$ and $0 \le V \le \infty$

From (* *) we have

$$(***) \quad W = \underline{U + jV} = j \frac{U_0}{\sqrt{a_1}} (z_1 - a_1)^{1/2} + U_0 = \underline{\frac{U_0}{\sqrt{a_1}} (a_1 - z_1)^{1/2} + U_0}$$

Now knowing that equipotential lines are given $(0 \le U \le U_0)$ in z-plane by $U = \text{const}$ and stream functions are given in z-plane by $V = \text{const}$ $(0 \le V \le \infty)$, we can find $f_1(x_1, y_1) = \mathcal{L}(U = \text{const})$ and $f_2(x_1, y_1) = \mathcal{L}(V = \text{const})$ which are equipotential and stream lines in original ~~system~~ system, respectivly.

In other words we now are transforming W back to the original system using (* * *)

For this we should make equal real and imaginare parts in (* * *):

$$(U - U_0) + jV = \frac{U_0}{\sqrt{a_1}} [(a_1 - x_1) + j y_1]^{1/2}$$

$$(U - U_0)^2 + 2jV(U - U_0) - V^2 = \frac{U_0^2}{a_1} [(a_1 - x_1) + j y_1]$$

$$\begin{cases} (U - U_0)^2 - V^2 = \dfrac{U_0^2}{a_1} (a_1 - x_1) & \text{(1)} \\ 2(U - U_0)V = \dfrac{U_0^2}{a_1} y_1 & \text{(2)} \end{cases}$$

Now we should saparated U and V

from (2) $\quad V = \dfrac{U_0^2}{2a_1(U - U_0)} y_1$

$$(U - U_0)^2 - \frac{U_0^4}{4a_1^2 (U - U_0)^2} y_1^2 = \frac{U_0^2}{a_1} (a_1 - x_1)$$

and Lines of equipotentials are given by

$$\boxed{y_1^2 = 4a_1 \left(\frac{U_0 - U}{U_0}\right)^2 (x_1 - a_1) + 4a^2 \left(\frac{U_0 - U}{U_0}\right)^4}$$

taking $0 \le U \le U_0$ for each $U = \text{const}$ we have an equipotential line in original system

Similary for stream functions:

$$(u - u_0) = \frac{u_0^2}{2a_1 V} y_1$$

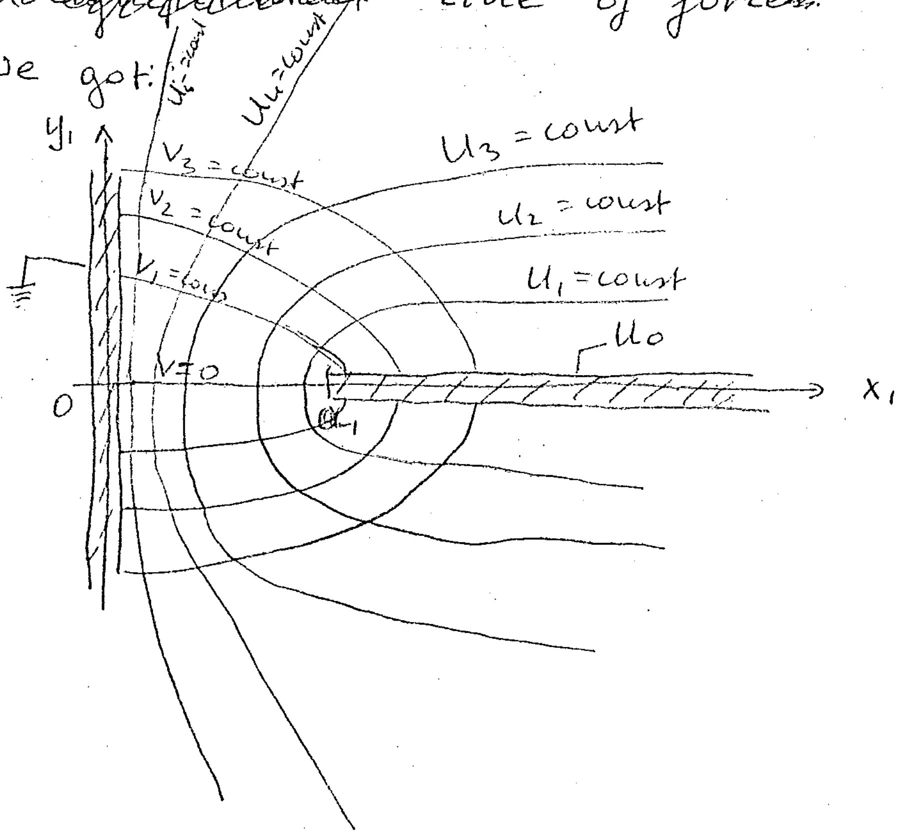
$$\frac{u_0^4}{4a_1^2 V^2} y_1^2 - V^2 = \frac{u_0^2}{a_1}(a_1 - x_1)$$

rearanging

$$\boxed{y_1^2 = -4a_1\left(\frac{V}{u_0}\right)^2(x_1 - a_1) + 4a_1^2\left(\frac{V}{u_0}\right)^4}$$ stream lines

Taking $0 \le V \le \infty$ for each $V = \text{const}$ we have ~~equipotential~~ line of forces.

We got:



In every point in real system we have now u and V.

For instance we could determined induced charge on granded plate and distribution on plate rised ~~changed~~ to u_0 (How?)

THIS OUTLINE TAKEN FROM
"FUNDEMENTALS OF ELECTRIC WAVES" 2nd ED
by H.H. SKILLING
(WILEY, NEW YORK, 1948)

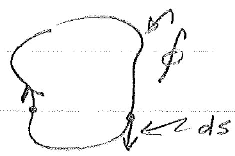
I. EXPERIMENTS ON THE ELECTROSTATIC FIELDS

A. FIELDS

1. EXP. 1:

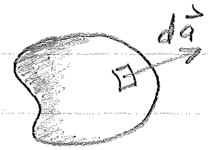
Q_0 $\vec{F}$ $\qquad \vec{F} = Q\vec{E}$; $\vec{E}$ = ELEC. FIELD STRENGTH IN V/m

2. EXP. 2:



$$\oint \vec{F} \cdot ds = \oint \vec{E} \cdot d\vec{s} = 0$$

3. EXP. 3: $\quad \epsilon_0 \oint \vec{E} \cdot d\vec{a} = Q \quad$ (IN VACUUM)



4. EXP. 4: $\quad K\epsilon_0 = \epsilon$

K = RELATIVE DIELECTRIC CONSTION

ϵ = PERMITTIVITY

ϵ_0 = " OF FREE SPACE

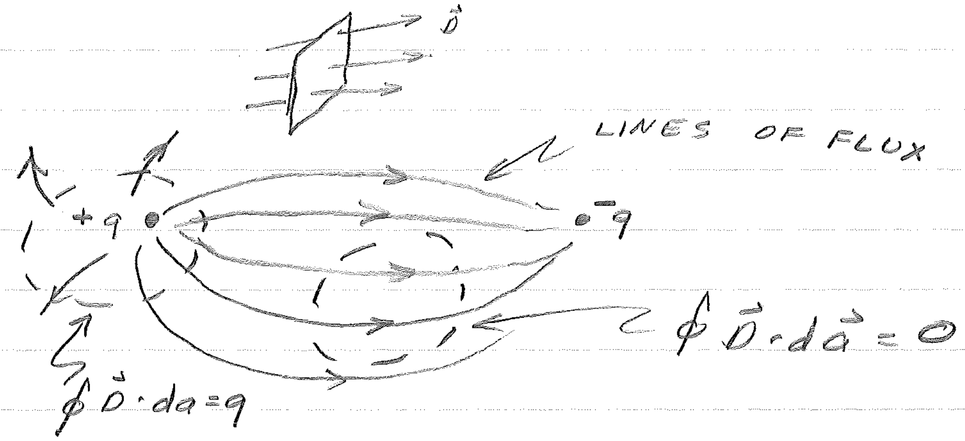
$$\oint \epsilon \vec{E} \cdot d\vec{a} = Q \quad \text{(IN } \epsilon \text{ MATERIAL)}$$

B. ELECTROSTATIC FLUX

$\vec{D} = \epsilon \vec{E} \Rightarrow \oint \vec{D} \cdot d\vec{a} = Q$

$\vec{D}$ = ELECTROSTATIC FLUX DENSITY

ELECTROSTATIC FLUX = $\int \vec{D} \cdot d\vec{a}$



C. UNITS

$\vec{E}$ = ELECTRIC FIELD STRENGTH ~ V/m

$\vec{D}$ = FLUX DENSITY ~ $\frac{\text{COULOMBS}}{m^2}$

ϵ = PERMITTIVITY

ϵ_0 = " OF FREE SPACE

$K = \frac{\epsilon}{\epsilon_0}$ = RELATIVE DIELECTRIC CONSTANT.

PROB

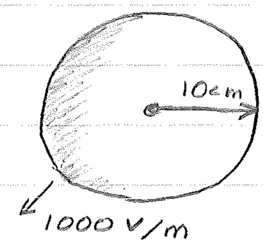
I (1) A BODY CARRYING A POSITIVE ELECTRIC CHARGE OF 1000 $\mu\mu$ COULOMBS IS IN AN E FIELD OF 5000 V/CM. WHAT IS THE ELECTRIC FORCE OF THE BODY IN NEWTONS?

ANS.

$$F = QE$$
$$= (1000 \times 10^{-12} \text{ COUL})\left(5000 \tfrac{V}{cm} \times 100 \, cm/m\right)$$
$$= 5 \times 10^{-4} \text{ NEWTONS}$$

I (2) E FIELD STRENGTH IS MEASURED AT ALL POINTS OF A SPHERE OF 10cm RADIUS IN AIR. IT IS FOUND TO BE EVERYWHERE NORMAL TO THE SURFACE, 10,000 $\frac{V}{cm}$ IN MAGNITUDE & DIRECTED OUTWARD. HOW MUCH CHARGE IS CONTAINED WITHIN THE SPHERE?

ANS:



$$Q = \epsilon_0 \oint E \cdot ds$$
$$= \epsilon_0 \left(\tfrac{4}{3}\pi r^3\right)(1000\, V/m)$$
$$= (8.855 \times 10^{-12})\ \tfrac{4}{3}\pi\ (0.1)^3 m\ \ (1000\ V/m)$$
$$= 3.709 \times 10^{-10} \text{ COULOMBS}$$
$$= 0.0371\ \mu\mu \text{ COULOMBS}$$
$$= 37100\ \mu \text{ COULOMBS}$$

I (3). HOW MUCH FLUX COMES OUT OF THE SPHERICAL SURFACE IN THE PREVIOUS PROBLEM? WHAT QUANTITY WOULD COME OUT OF THE SAME CHARGE IN PETROLEUM OIL? ($\epsilon \cong 2\epsilon_0$) WHAT WOULD BE THE VALUE OF E AT THE SURFACE IN PETROLEUM OIL?

ANS.

$\vec{D} = \epsilon_0 \vec{E}$

FLUX $= \oint \vec{D} \cdot ds = 37100$ μ COLOMBS

(SAME AS IN 2)

FOR PETROLEUM OIL

$\vec{D} = 2\epsilon_0 \vec{E}$

WE MUST STILL HAVE

$Q = \oint \vec{D} \cdot d\vec{s} \Rightarrow$ ELECTROSTATIC FLUX IS STILL THE SAME, BUT THE E FIELD IS NOW HALF OF WHAT IT WAS IN AIR, SINSE

$$Q = 2\epsilon_0 \oint \vec{E} \cdot ds$$

D. VECTOR FIELDS

E. GRADIENT

PROPERTIES: 1. GRADIENT LINE IS ALWAYS $\perp$ TO CONTOUR LINES. (IT IS STEEPEST SLOPE) 2. THE CLOSER THE CONTOUR LINES, THE STEEPER THE GRADIENT.

$$\vec{\nabla} P = \vec{i}\frac{\delta P}{\delta x} + \vec{j}\frac{\delta P}{\delta y} + \vec{k}\frac{\delta P}{\delta z}$$

$$(\vec{\nabla} P)\cdot ds = \frac{\delta P}{\delta x}dx + \frac{\delta P}{\delta y}dy + \frac{\delta P}{\delta z}dz$$

F. DIVERGENCE

$$\vec{\nabla}\cdot\vec{A} = \frac{\delta A_x}{\delta x} + \frac{\delta A_y}{\delta y} + \frac{\delta A_z}{\delta z}$$

(AN INCOMPRESSIBLE FLOW HAS ZERO DIVERGENCE)

G. CURL

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ A_x & A_y & A_z \end{vmatrix}$$

H. DEL (NABLA): $\nabla = \vec{i}\frac{\delta}{\delta x} + \vec{j}\frac{\delta}{\delta y} + \vec{k}\frac{\delta}{\delta z}$

GRADIENT: $\vec{\nabla} P \rightarrow$ (VECTOR)

DIVERGENCE: $\vec{\nabla}\cdot\vec{A} \rightarrow$ (SCALAR)

CURL : $\vec{\nabla}\times\vec{A} \rightarrow$ (VECTOR)

- DIVERGENCE IS A RATE OF CHANGE OF A FIELD STRENGTH IN THE DIRECTION OF THE VEC. FIELD
- CURL IS A RATE OF CHANGE OF THE FIELD STRENGTH IN A DIRECTION AT RIGHT ANGLES TO THE FIELD.

I. $\left.\begin{aligned} \nabla\times\nabla F &= 0 \\ \nabla\cdot\nabla\times\vec{A} &= 0 \end{aligned}\right\}$ ALWAYS!

$\nabla^2 F \triangleq \nabla\cdot\nabla F$; ∇^2 = LAPLACIAN

SCALAR: $\nabla^2 F = \left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}\right)F$

VECTOR: $\nabla^2\vec{A} = \nabla^2(\vec{i}A_x + \vec{j}A_y + \vec{k}A_z)$

II. VECTOR ANALYSIS

A. VECTOR MULTIPICATION

- SCALAR OR DOT PRODUCT:

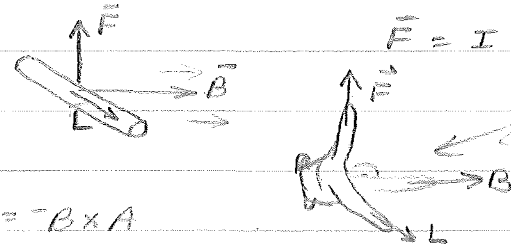
$C = \vec{A} \cdot \vec{B} = AB \cos\theta$

- VECTOR OR CROSS PRODUCT

$\vec{C} = \vec{A} \times \vec{B}$

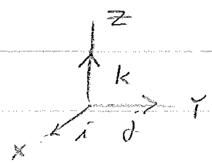
EXAMBLE: FORCE ON A WIRE IN A B FIELD

$\vec{F} = I \vec{L} \times B$



$A \times B = -B \times A$

B. UNIT VECTORS



$\vec{i} \cdot \vec{i} = 1$ $\qquad$ $\vec{i} \times \vec{i} = 0$

$\vec{i} \cdot j = 0$ $\qquad$ $\vec{i} \times \vec{j} = k$

$\vec{i} \times \vec{k} = -j$

$\vec{A} = \vec{i} A_x + \vec{j} A_Y + k \vec{A}_z$

$\vec{A} \cdot \vec{B} = A_x B_x + A_Y B_Y + A_z B_z$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_Y & A_z \\ B_x & B_Y & B_z \end{vmatrix}$$

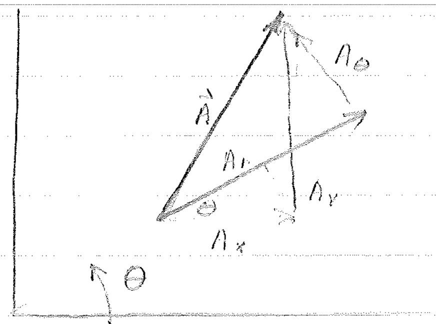
C. TRIPLE PRODUCTS

$[ABC] = A \times B \cdot C = {}^{+}(B \times C \cdot A) = -[CBA]$

J. POLAR COORDINATES

$$X = r\cos\theta \qquad Y = r\sin\theta$$

$$r = \sqrt{X^2 + Y^2} \qquad \theta = \tan^{-1}\left(\frac{Y}{X}\right)$$



$$A_r = A_X \cos\theta + A_Y \sin\theta$$

$$A_\theta = A_Y \cos\theta - A_X \sin\theta$$

$$A_X = A_r \cos\theta - A_\theta \sin\theta$$

$$A_Y = A_r \sin\theta + A_\theta \cos\theta$$

$$\nabla P = i\frac{\delta P}{\delta X} + j\frac{\delta P}{\delta Y}$$

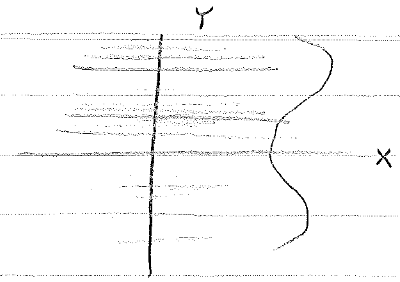
NOW

$$\frac{\delta P}{\delta X} = \frac{\delta P}{\delta r}\frac{\delta r}{\delta X} + \frac{\delta P}{\delta \theta}\frac{\delta \theta}{\delta X} = \frac{\delta P}{\delta r}\frac{X}{\sqrt{X^2+Y^2}} + \frac{\delta P}{\delta \theta}\frac{-\frac{Y}{X^2}}{\left(\frac{Y}{X}\right)^2 + 1}$$

$$= \frac{\delta P}{\delta r}\cos\theta - \frac{\delta P}{\delta \theta}\frac{\sin\theta}{r}$$

II. PROB.

$V_x = \sin Y, \quad V_Y = 0$ SKETCH THE FIELD OF V & FIND ITS DIVERGENCE & CURL



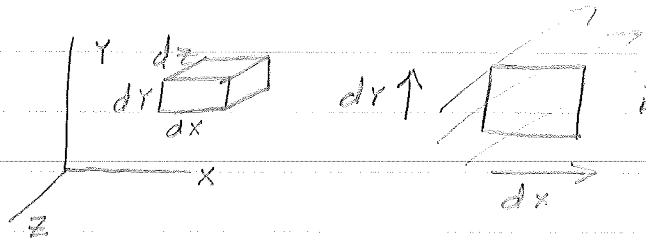
$$\bar{\nabla} \cdot \bar{V} = \frac{\delta}{\delta x} \sin Y = 0$$

$$\vec{\nabla} \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta Y} & \frac{\delta}{\delta z} \\ \sin Y & 0 & 0 \end{vmatrix}$$

$$= -\vec{K} \frac{\delta}{\delta Y} \sin Y = -K \cos Y$$

III. CERTAIN THEOREMS RELATING TO FIELDS

A. DIVERGENCE



$\vec{D} = i D_x + j D_y + k D_z$

AMOUNT OF CHANGE OF D_x IN $dx = \frac{\partial D_x}{\partial x} dx$

$\Rightarrow$ # OF FLUX LINES LEAVING RIGHT HAND SIDE IS

$$\left(D_x + \frac{\partial D_x}{\partial x} dx\right) dy\, dz$$

DIFFERENCE BETWEEN # OF LINES ON LEFT & RIGHT IS

$$\frac{\partial D_x}{\partial x} dx\, dy\, dz$$

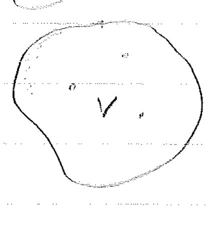
GENERALIZING, THE # OF FLUX LINES LEAVING WHICH DO NOT ENTER IS $\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) dx\, dy\, dz$

∴ DIVERGENCE IS # OF FLUX LINES ORIGINATING PER UNIT AREA

$$\vec{\nabla} \cdot \vec{D} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \quad \cancel{\frac{\delta x\, \delta y\, \delta z}{\delta y}}\ 1$$

B. GAUSS'S THEOREM

$$\oint \underset{(2)}{\vec{D} \cdot d\vec{a}} = \int_V \underset{(1)}{\vec{\nabla} \cdot \vec{D}}\, dv$$



IN A GIVEN VOLUME, THE # OF FLUX LINES ORIGINATING WITHIN CAN BE COMPUTED BY

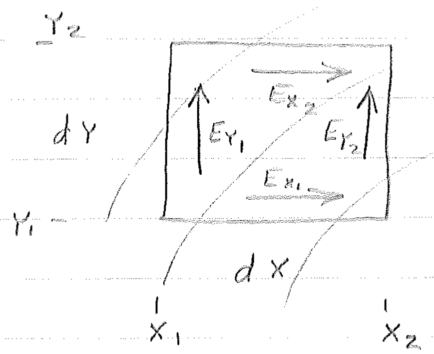
(1.) INTEGRATING THE DIVERGENCE OVER THE VOLUME

(2) SEEING HOW MUCH FLUX COMES OUT THAT DOESN'T GO IN.

B. CURL

$C = \text{CIRCULATION} = \oint_\ell \vec{E} \cdot d\vec{s}$

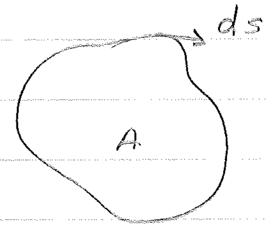
CURL IS A MICROSCOPIC CIRCULATION/AREA AND HAS A DIRECTION NORMAL TO THE PLANE IN WHICH IT IS MAXIMUM.



USING COUNTERCLOCKWISE SENSE:

$$C = -E_{Y_1}\,dY + E_{Y_2}\,dY - E_{X_2}\,dX + E_{X_1}\,dX$$
$$= (E_{Y_2} - E_{Y_1})\,dY - (E_{X_2} - E_{X_1})\,dX$$
$$= \frac{\delta E_Y}{\delta X}\,dX\,dY - \frac{\delta E_X}{\delta Y}\,dY\,dX$$
$$\Rightarrow \text{CURL} = \left(\frac{\delta E_Y}{\delta X} - \frac{\delta E_X}{\delta Y}\right)\vec{k}$$

C. STOKE'S THEOREM



$$\text{CIRC} = \underset{(1)}{\oint_\ell \vec{E} \cdot d\vec{s}} = \underset{(2)}{\int_A \vec{\nabla} \times \vec{E} \cdot d\vec{a}}$$

THE CIRCULATION CAN BE COMPUTED BY

(1.) THE LINE INTEGRAL DEFINITION

(2) SUMMING THE ITTY BITTY CURLS WITHIN THE AREA

NOTE: SURFACE NEED NOT NECESSARILY BE PLANAR.

D. COMPARISON OF THEOREMS

BOTH GAUSS' & STOKES THEOREM RELATE THE MICROSCOPIC TO THE MACROSCOPIC ASPECT.

E. SCALAR POTENTIAL

P = AN EQUAPOTENTIAL SURFACE

$\vec{F}$ = CORRESPONDING FIELD

$\vec{F} = -\nabla P$ (ASSURED IF $\vec{\nabla} \times \vec{F} = 0$

$$\begin{cases} \forall P \ni \vec{F} \\ \text{IF } \exists \vec{F} \ni \vec{\nabla} \times \vec{F} = 0 \Rightarrow \exists P \ni -\vec{\nabla} P = \vec{F} \end{cases}$$

- AN ELECTROSTATIC FIELD HAS NO CURL
- A STATIC MAGNETIC FIELD HAS NO CURL IN REGIONS THAT ARE NOT CARRYING CURRENT.
- CURLESS FIELDS HAVE EQUAPOTENTIAL LINES DIVIDING IT INTO "LAMELLERS"

$\Rightarrow \vec{\nabla} \times \vec{F} = 0 \Rightarrow \vec{F}$ IS "LAMELLAR" OR "IRROTATIONAL"

F. SOLENOIDAL FIELDS AND VECTOR POTENTIALS

IF $\vec{B} = \vec{\nabla} \times \vec{A}$, THEN $\overline{\nabla} \cdot \overline{B} = 0$

AND IF $\vec{\nabla} \cdot \vec{B} = 0$, THEN $\exists \vec{A} \ni \vec{B} = \vec{\nabla} \times \vec{A}$

$\vec{A}$ = VECTOR POTENTIAL

$\vec{B}$ = "SOLENOIDAL" OR "SOURCELESS"

- ALL MAGNETIC FIELDS ARE SOLENOIDAL

IV. THE ELECTROSTATIC FIELD

REVIEW: $\vec{F} = Q\vec{E}$

$\oint \vec{E} \cdot d\vec{s} = 0$

$\oint \vec{D} \cdot d\vec{a} = \oint \epsilon \vec{E} \cdot d\vec{a} = Q$

$\Big]\rightarrow$ FUNDEMENTAL LAWS OF ELECTROSTATICS

– USING STOKE'S THEOREM

$$\oint \vec{E} \cdot ds = \int (\bar{\nabla} \times \bar{E}) \cdot d\vec{a} = 0 \Longrightarrow \bar{\nabla} \times \bar{E} = 0$$

V = ELECTROSTATIC POTENTIAL

$$\vec{E} = -\nabla V$$

– USING GAUSS' THEOREM

$$\oint \bar{D} \cdot d\vec{a} = \int \bar{\nabla} \cdot \bar{D} \, dv = Q$$

$\therefore$ WHERE $Q = 0$, $\bar{\nabla} \cdot \bar{D} = 0$

$\vec{\nabla} \cdot \vec{D} = \rho$ = CHARGE DENSITY

$\bar{\nabla} \cdot \bar{E} = \rho/\epsilon = -\nabla^2 V \Rightarrow$ <u>POISSON'S</u> <u>EQ</u>

$\therefore$ WHEN THERE'S NO CHARGE, WE HAVE <u>LAPLACE EQ</u>

$$\nabla^2 V = 0$$

$$\left[\begin{array}{l} \vec{E} = -\nabla V \\ \nabla^2 V = -\rho/\epsilon \end{array}\right] \longleftarrow \text{SUMMARY}$$

A. CONDUCTORS

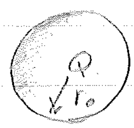
FOR THE ELECTROSTATIC CASE

$\vec{E} = 0$, $V = \text{CONST.}$ (IN CONDUCTOR)

Q WILL ALWAYS GO TO SURFACE

σ = SURFACE CHARGE DENSITY $= D_n = \epsilon E_n$

B. A CHARGED SPHERE



K

(1) $\nabla^2 V = 0$

(2) SURFACE EQUAPOTENTIAL

(3) TOTAL CHARGE = Q

IN SPHERICAL COORDINATES

$$\nabla^2 V = \frac{\delta^2 V}{\delta r^2} + \frac{1}{r^2}\frac{\delta^2 V}{\delta \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\delta^2 V}{\delta \phi} + \frac{2}{r}\frac{\delta V}{\delta r} + \frac{\operatorname{ctn}\theta}{r^2}\frac{\delta V}{\delta \theta}$$

DUE TO SYMMETRY: $\nabla^2 V = \frac{\delta^2 V}{\delta r^2} + \frac{2}{r}\frac{\delta V}{\delta r} = 0$

SOLUTION IS $V = \frac{a}{r} + b$

$V(\infty) = 0 \Rightarrow b = 0 \Rightarrow V = \frac{a}{r}$

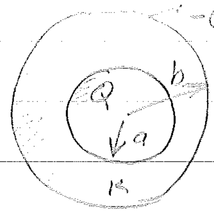
$\sigma = \frac{Q}{4\pi r_0^2} = \epsilon E_n \Rightarrow E_n = \frac{Q}{4\pi\epsilon r_0^2}$

$E = -\frac{\delta V}{\delta r} = \frac{a}{r^2}, \quad E(r_0) = \frac{a}{r_0^2} \Rightarrow a = \frac{Q}{4\pi\epsilon}$

OR $V = \frac{Q}{4\pi\epsilon_r r}$

$\vec{E} = \frac{Q}{4\pi K\epsilon\, r^2}$

C. SPHERICAL CONDENSER



$E = \frac{Q}{4\pi\epsilon r^2}$

D. VOLTAGE

$$V_{12} = \int_1^2 \vec{E} \cdot d\vec{s} = \frac{W_{12}}{Q}$$

FOR SPHERE: $V_{12} = \int_a^b \frac{Q}{4\pi\epsilon r^2} dr = \frac{Q(b-a)}{4\pi\epsilon ab}$

E. CAPACITANCE

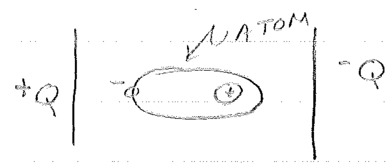
$$C = Q/V$$

FOR DOUBLE SPHERE: $C = \frac{4\pi\epsilon ab}{b-a}$

$C \sim \epsilon$ ALWAYS

F. POLARIZATION

(HOW COME E IS LESS IN ϵ THAN IN $\epsilon_0 < \epsilon$)



IN $\vec{E}$ FIELD, THE ATOMS DEVELOPE AN EQUIVALENT SURFACE CHARGE THRU POLARIZATION

G. INVERSE SQUARE LAW

$$F = \frac{Q_1 Q_2}{4\pi\epsilon r^2} \longleftarrow \text{COULOMB'S LAW}$$

H. FIELD IN A HOLLOW SPHERE

MUST BE ZERO

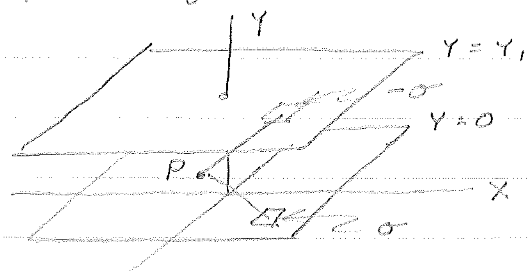
H. THE POTENTIAL INTEGRAL

POTENTIAL FROM A NUMBER OF CHARGES: $V = \frac{1}{4\pi\epsilon}\sum \frac{Q}{r}$

IN LIMIT, WE HAVE

$$V = \frac{1}{4\pi\epsilon}\int \frac{\rho}{r}\,dv$$

EX:



$P = (x_1, y_1, z_1)$

$$V = \frac{1}{4\pi\epsilon}\int \frac{\sigma\,da}{r}$$

$$= \frac{\sigma}{4\pi\epsilon}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{(x-x_1)^2 + y_1^2 + (z-z_1)^2}} - \frac{1}{\sqrt{(x-x_1)^2 + (y_2-y_1)^2 + (z-z_1)^2}}\,dx\,dz\right]$$

$$= -\frac{\sigma}{\epsilon}\left(y_1 - \frac{y_2}{2}\right)$$

I. ELECTROSTATIC ENERGY.

$$dW = V\,dQ = \frac{Q}{C}\,dQ \Rightarrow W = \int_0^Q \frac{Q}{C}\,dQ = \frac{Q^2}{2C} = \frac{1}{2}QV$$

FOR ENERGY DISTRIBUTION

$$W = \frac{1}{2}\int \vec{D}\cdot\vec{E}\,dv$$

V. ELECTRIC CURRENT

$$I = dQ/dt$$

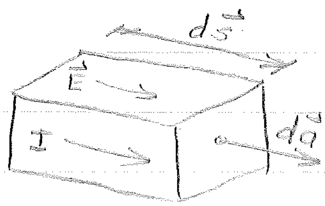
OHM'S LAW : $V = RI$

FOR CYLYNDER : $\frac{1}{R} = \gamma \frac{\text{AREA}}{\text{LENGTH}}$

γ = CONDUCTIVITY

$\Rightarrow$ VOLTAGE : $V = \int \vec{E} \cdot d\vec{s}$

CURRENT: $I = \int \vec{\imath} \cdot d\vec{a}$; $\vec{\imath}$ = CURRENT DENSITY



$$\frac{1}{R} = \gamma \frac{da}{ds}$$

$$\Rightarrow \underbrace{\vec{\imath} \cdot d\vec{a}}_{dI} = \gamma \frac{da}{dt} \underbrace{\vec{E} \cdot ds}_{dV}$$

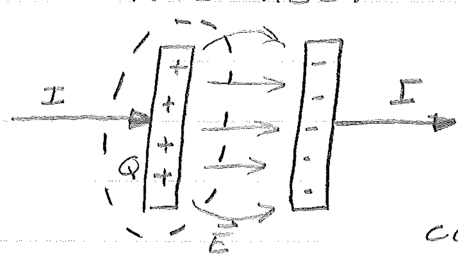
THUS:

$$\vec{\imath} = \gamma \vec{E} \Leftarrow \text{MICROSCOPIC OHM'S LAW}$$

WHEN $\vec{E}$ FIELD IS D.C., $\Longrightarrow \nabla \cdot \vec{\imath} = 0 \Leftarrow$ KIRCHOFF'S CURRENT LAW

(ie WHEN NO CURRENT IS FLOWING)

MORE GENERAL CASE:



FLUX FROM SURFACE

$$\frac{d}{dt}(\text{FLUX}) = \frac{d}{dt} Q = I$$

$$\text{FLUX} = \oint \vec{D} \cdot d\vec{a}$$

$$\Rightarrow I = \frac{d}{dt} \oint \vec{D} \cdot d\vec{a} = \oint \frac{d\vec{D}}{dt} \cdot d\vec{a}$$

CURRENT INTO VOLUME IS

$$I = \oint \vec{\imath} \cdot d\vec{a}$$

$$\Rightarrow \oint \left(\vec{\imath} + \frac{d\vec{D}}{dt} \right) \cdot d\vec{a} = 0$$

BY GAUSS' THEOREM: $\int \nabla \cdot \left(\vec{\imath} + \frac{d\vec{D}}{dt} \right) dV = 0$

OR $\nabla \cdot \left(\vec{\imath} + d\vec{D}/dt \right) = 0$

OR $\nabla \cdot \left(\gamma \vec{E} + \epsilon \frac{d\vec{E}}{dt} \right) = 0$

$\vec{\imath}_d = \frac{d\vec{D}}{dt} = \epsilon \frac{d\vec{E}}{dt}$ = DISPLACEMENT CURRENT

$\vec{\imath}_c = \gamma \vec{E}$ = CONDUCTION CURRENT

$\vec{\imath}_t = \vec{\imath}_d + \vec{\imath}_c$; $\nabla \cdot \vec{\imath}_T = 0$

$\vec{\imath}_T$ IS SOLENOIDAL

A. ELECTROMOTIVE FORCE

$\vec{E}_s = \vec{E}$ DUE TO Q

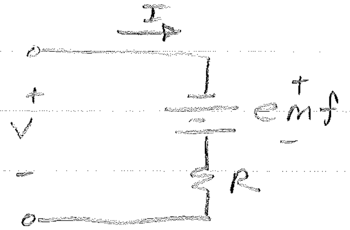
$\vec{E}_m = \vec{E}$ DUE TO CHANGING MAGNETIC FIELD

$\vec{E} = \vec{E}_s + \vec{E}_m$

$$V = \int \vec{E}_s \cdot d\vec{s}$$

$$emf = \int \vec{E}_m \cdot d\vec{s}$$

$\vec{E}_m$ FROM CHEMICAL OR MECH. ENERGY XFERED TO $\vec{E}$ ENERGY



$$V + emf = RI$$
$$= \int \vec{E} \cdot ds$$
$$= \int (\vec{E}_m + \vec{E}_s) \cdot ds$$

VI. THE MAGNETIC FIELD

A. MAGNETIC FORCE

$$\bar{F} = I\bar{L} \times \bar{B}$$

$\bar{L}$ = LENGTH OF WIRE

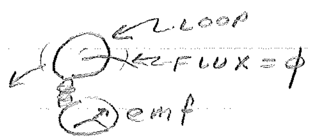
I = CURRENT THRU WIRE

$\bar{B}$ = MAGNETIC INDUCTION (FLUX DENTSITY) $\sim$ (WEBER/M^2)

B. MAGNETIC FLUX $\Rightarrow \phi = \int \bar{B} \cdot d\bar{a}$

A $\bar{B}$ FIELD CAN BE PRODUCED BY AN $\bar{E}$ FIELD

$$EMF = \oint \vec{E} \cdot d\vec{s} = -d\phi/dt \Leftarrow \text{FARADAY'S LAW}$$



APPLYING STOKE'S THEOREM

$$\oint \vec{E} \cdot d\vec{s} = \int \bar{\nabla} \times \bar{E} \cdot d\vec{a} = \int \frac{d}{dt} \vec{B} \cdot da$$

$$\Rightarrow \bar{\nabla} \times \bar{E} = -\frac{d}{dt} \bar{B}$$

(NOTE: IF $\bar{E}$ IS ELECTROSTATIC, $\bar{\nabla} \times \bar{E} = 0$)

C. VOLTAGE INDUCED BY MOTION

LET LOOP BE TRAVELING WITH SPEED $\vec{v}$

$$\Rightarrow \bar{\nabla} \times \bar{E} = -\frac{\delta}{\delta t} \bar{B} + \bar{\nabla} \times (\bar{v} \times \bar{B})$$

IN ALL CASES

$$\oint \bar{B} \cdot d\vec{a} = \int \vec{\nabla} \cdot \vec{B} \, dV = 0 \Rightarrow \bar{\nabla} \cdot \bar{B} = 0 \quad \text{(A MAXWELL)}$$

ALSO $\frac{1}{\mu} \oint \bar{B} \cdot d\bar{s} = I$

μ = PERMEABILITY $\quad \mu_0 = 4\pi \times 10^{-7}$

FERROMAGNETIC $\Rightarrow \mu >> \mu_0$

FOR NON-HOMOGENEOUS MATERIAL

$$\oint \frac{\bar{B} \cdot d\bar{s}}{\mu} = I$$

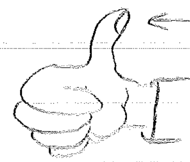
$\bar{B} = \mu \vec{H} \ni \vec{H}$ = MAGNETIC INTENSITY

$$\Rightarrow \oint \vec{H} \cdot d\bar{s} = I \quad \text{(A MAXWELL)}$$

NOW $I = \int \bar{J} \cdot d\bar{a} = \oint \vec{H} \cdot d\bar{s} = \int (\nabla \times \vec{H}) \cdot da$

$\Rightarrow \quad \vec{\nabla} \times \vec{H} = \vec{J}$

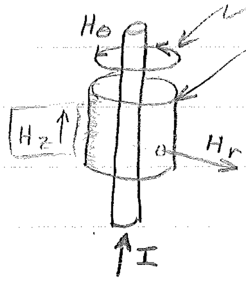
D. CONVENTION REGARDING SIGN



THUMB IS "BOREAL" TO FINGERS

BOREAL, BY CONVENTION, IS POSITIVE

EXAMPLE:



PATH OF INTEGRATION

SURFACE OF INTEGRATION. ON THIS PATH, $\vec{H}$ IS CONSTANT

NOW

$$\oint H_\theta \, ds = I = H_\theta \oint ds = H_\theta \, 2\pi r$$

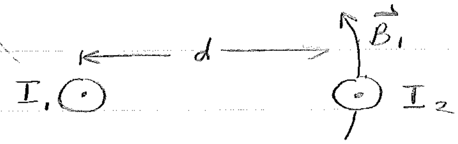
THIS GIVES "BIOT-SAVART LAW": $H_\theta = \dfrac{I}{2\pi r}$

$H_r = H_z = 0$ (WHY?)

$H_r = 0$ SINCE, FIRST, IT MUST BE THE SAME AT A DISTANCE r FROM THE WIRE. TOP AND BOTTOM OF PILL BOX WOULD CANCEL. MUST BE ZERO ALSO AROUND SURFACE, SINCE $\int \bar{B} \cdot d\bar{s} = 0$. THERE.

$H_z = 0$ SINCE INTEGRATION ABOUT ANY LINE EXTERNAL TO WIRE MUST BE ZERO

E. FORCE BETWEEN CURRENTS



$\Leftarrow$ TWO PARALLEL WIRES

FROM BIOT SAVART LAW, THE MAGNETIC FIELD FROM I_1 IS

$$B_1 = \mu H_1 = \mu I_1 / 2\pi d$$

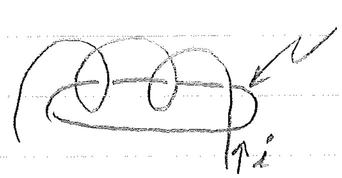
NOW $\bar{F} = I\bar{L} \times \bar{B} \Rightarrow F = I_2 L_2 \left(\frac{\mu I_1}{2\pi d} \right) = \frac{\mu}{2\pi} \frac{I_1 I_2}{d} L_2$

SOLVING FOR FORCE PER UNIT LENGTH:

$F/L_2 = \frac{\mu}{2\pi d} I_1 I_2 \Leftarrow$ A FORM OF AMPERE'S LAW

NOTE "LIKE CURRENTS ATTRACT"

F. MAGNETIC FLUX LINKAGES



EACH SURFACE PENETRATION IS A "FLUX LINKAGE"

G. MAGNETIC POTENTIAL

IN ELECTROSTATICS, $\bar{\nabla} \times \bar{E} = 0 \Rightarrow \bar{E} = -\vec{\nabla} V$

$\bar{\nabla} \times \bar{H} = 0$ ONLY WHEN $\exists$ NO CURRENT

BUT, $\bar{\nabla} \cdot \bar{H} = 0 \Rightarrow H = \bar{\nabla} \times \bar{A}$

$\vec{A}$ = MAGNETIC VECTOR POTENTIAL

$$\bar{J} = \bar{\nabla} \times \bar{H} \Rightarrow \bar{J} = \bar{\nabla} \times \bar{\nabla} \times \vec{A}$$
$$= \nabla(\nabla \cdot \bar{A}) - \nabla^2 A$$

FOR MAGNETOSTATIC FIELD, LET $\bar{\nabla} \cdot \bar{A} = 0$

$$\Rightarrow \nabla^2 \bar{A} = -\bar{J}$$

(ANALAGOUS TO POISON'S EQ: $\nabla^2 V = -\rho/\epsilon$)

IN COMPONENT FORM

$$\nabla^2 A_u = -i_u \qquad (u = x, y, z)$$

$$\Rightarrow A_u = \frac{1}{4\pi}\int i_u\, dv/r$$

(ANALOGOUS TO $V = \frac{1}{4\pi\epsilon}\int \rho\, dv/r$)

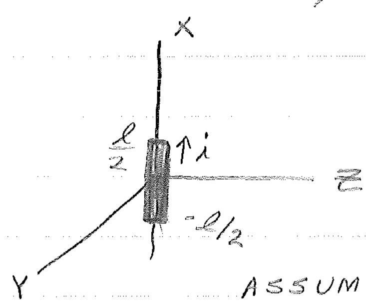
IN VECTOR FORM:

$$\vec{A} = \frac{1}{4\pi}\int \frac{\vec{J}}{r}\, dv$$ ⇐ NOTE $\vec{A}$ AND $\vec{J}$ HAVE SAME DIREC.

("$\vec{A}$ IS LIKE $\vec{J}$, BUT "FUZZY" AROUND THE EDGES")

EXAMPLE: IN A SHORT WIRE; $i = I \sin \omega t$

ASSUME ω IS SO SMALL, MAGNETOSTATIC CONDITION PREVAILS (ie, "QUASI-STATIONARY" STATE)



$$\vec{A} = \frac{1}{4\pi}\int \vec{J}/r\, dv$$

$$A_x = \frac{1}{4\pi}\int \frac{J_x\, dv}{r}$$

$$= \frac{1}{4\pi}\int \frac{J_x\, da\, dx}{r}$$

$$= \frac{1}{4\pi}\int \frac{i\, dx}{r} = \frac{1}{4\pi}\int \frac{I \sin \omega t\, dx}{r}$$

ASSUME $r >> l$

$$\Rightarrow A_x = \frac{I \sin \omega t}{4\pi r}\int_{-l/2}^{l/2} dx = \frac{i l}{4\pi r}$$

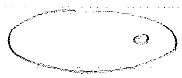
H. MAGNETIC ENERGY

MAGNETIC ENERGY DENSITY $= \frac{1}{2}\bar{B}\cdot\bar{H}$

$$\Rightarrow \text{MAG. EN.} = \frac{1}{2}\int \bar{B}\cdot\bar{H}\, dv$$

(WE ASSUME HERE THAT ENERGY IS STORED IN THE MAGNETIC FIELD)

I. THEORIES

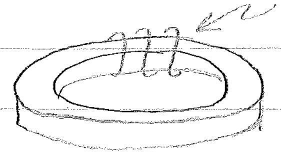
 ← FERROMAGNETIC POLARIZATION

J. DIAMAGNETIC MATERIALS

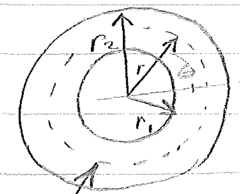
EXPLAINS RELUCTANCE OF SOME MATERIALS TO MAGNETIC FIELDS.

VII. EXAMPLES AND INTERPRETATION

A. EXAMPLE



N TURNS
CURRENT = I



← THE MAGNETIC FIELD HERE WILL ONLY HAVE A COMPONENT IN THE Θ DIRECTION (SINCE $\vec{J} = \vec{\nabla} \times H$)

$$\oint \vec{H} \cdot d\vec{s} = \int \vec{J} \cdot d\vec{a} = I_{CUT}$$

- FOR $r < r_1$ OR $r > r_2$, NO CURRENT PASSES WITHIN THE CIRCLE AND $\vec{H} = 0$.
- IF THE SURFACE LIES WITHIN THE SURFACE:

$$\oint \vec{H} \cdot d\vec{s} = \int \vec{J} \cdot d\vec{a} = NI$$

$$= H_\Theta \oint ds = 2\pi r H_\Theta \Rightarrow H_\Theta = \frac{NI}{2\pi r}; r_1 \le r \le r_2$$

$$\begin{cases} \int \bar{H} \cdot d\bar{s} = \text{MAGNETIC POTENTIAL DIFFERENCE} \\ \oint \bar{H} \cdot d\bar{s} = \text{MAGNETOMOTIVE FORCE} \end{cases}$$

COMPUTATION OF FLUX:

$$\Phi = \int \bar{B} \cdot d\bar{a} = \int \mu H_\theta \, da = \int_{r_1}^{r_2} \frac{\mu N I}{2\pi r} z_1 \, dr$$

$$= \frac{\mu N I z_1}{2\pi} \ln r_2/r_1$$

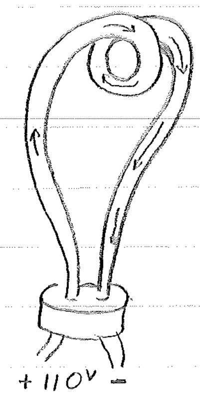
WHERE z_1 IS THE THICKNESS OF THE CORE

COMPUTATION OF INDUCTANCE:

$$L \triangleq N\Phi/I$$

$$= \frac{\mu N^2 z_1}{2\pi} \ln r_2/r_1$$

B. EXAMPLE



WE WISH TO FIND THE $\vec{E}$ FIELD ASSOCIATED WITH THIS OLD FASHIONED CARBON FILAMENT. $\vec{J}$ IS // TO WIRE. $\vec{J} = \gamma \vec{E}$. FOR 110V AND 10 INCHES OF WIRE, $E = 11$ V/INCH.

IN A CONDUCTOR NOT CARRYING CURRENT, $E = 0$.

ALSO, $\exists$ A (COMPLICATED) $\vec{E}$ FIELD OUTSIDE THE CONDUCTOR.

C. EXAMPLE; CONSIDER A CHANGING CURRENT IN THE TOROID OF EXAMPLE A. THE MAGNETIC FIELD CHANGES & THE CHANGING MAGNETIC FIELD INTRODUCES AN ELECTRIC FIELD.

$$\nabla \times \vec{E} = -\delta B/\delta t$$

$$\Rightarrow \underbrace{\int \nabla \times \vec{E} \cdot da}_{\text{SURFACE}} = \underbrace{\oint \vec{E} \cdot d\vec{s}}_{\text{LINE}} = -\frac{\delta}{\delta t} \int \vec{B} \cdot da = -\frac{\delta}{\delta t} \Phi$$

(FROM STOKE'S THEOREM)

CHOOSE LINE INTEGRAL AROUND ONE WRAPPING. SUMMING UP ALL N WINDINGS GIVES

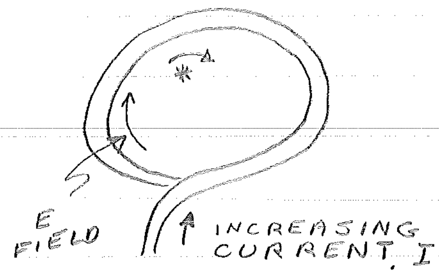
$$EMF = -N \, d\Phi/dt$$

OR, SINCE $L = N\Phi/I$

$$EMF = -L \, dI/dt$$

NOTE: EMF OPPOSES CURRENT FLOW

D. EXAMPE: "INDUCTION ACCELERATOR"



INCREASING I WILL INDUCE INCREASING MAGNETIC FIELD (OUT OF PAGE) WHICH WILL INDUCE AN E FIELD.

ELECTRONS WILL BE SPIRALLY ACCELERATED.

THE $\vec{E}$ FIELD CURL WILL BE OUT OF PAGE.

ASSUME $E_\theta = -Ar$

$$\nabla \times \bar{E} = -\vec{k}\left(A + \frac{Ar}{r}\right) = -\hat{k}\,2A$$

$$\frac{\delta \bar{B}}{\delta t} = \hat{k}\,2A \quad \text{OR} \quad \frac{\delta B}{\delta t} = 2A$$

$$\Rightarrow E_\theta = -\frac{r}{2}\,\delta B/\delta t \quad \text{INSIDE}$$

TURNS OUT THAT $E_\theta = -\dfrac{r_0^2}{2r}\dfrac{\delta B}{\delta t}$ OUTSIDE

XIII. MAXWELL'S HYPOTHESIS

ASSUMPTIONS AND EXPERIMENTS

E1: $\vec{E}$ FIELD DEFINED

E2: ELECTROSTATIC FIELD IS WITHOUT CURL (LAMELLAR) $(\bar{\nabla}\times\bar{E}=0)$

E3: $\vec{\nabla}\cdot\vec{E}=\rho/\epsilon$ WHEN $\vec{E}$ IS ELECTROSTATIC

E4: ϵ ESTABLISHED

E5: OHM'S LAW: $V=IR$

E6: $\vec{B}$, THE MAGNETIC FIELD, DEFINED

E7: A CHANGING $\bar{B}$ FIELD INDUCES AN $\bar{E}$ FIELD.

E8: A MAGNETOSTATIC FIELD IS WITHOUT DIV (SOLENOIDAL) $(\bar{\nabla}\cdot\bar{B}=0)$

E9: $\bar{\nabla}\times\bar{H}=\bar{J}$ WHEN $\bar{B}$ IS MAGNETOSTATIC

A1: ASSUME $\vec{\nabla}\cdot\vec{E}$ IS PROPORTIONAL TO $\vec{J}$ WHEN $\vec{E}$ IS DYNAMIC

A2: DYNAMIC $\bar{B}$ FIELD HAS NO DIVERGENCE $(\bar{\nabla}\cdot\bar{B}=0)$

MAXWELLS ASSUMTION: WE KNOW

TOTAL CURRENT = CONDUCTION CURRENT + DISPLACEMENT CURRENT

CONDUCTION CURRENT $\Rightarrow$ MAGNETIC FIELD

$\therefore$ DISPLACEMENT " $\Rightarrow$ MAG FIELD (MAXWELL'S HYPOTH)

$\nabla\times\vec{H}=\vec{J}$ = CONDUCTION CURRENT

$\therefore$ WITH DISPLACEMENT CURRENT: $\bar{\nabla}\times\bar{H}=J+\frac{\delta\vec{D}}{\delta t}$

MAXWELL'S EQUATIONS:

$$\bar{\nabla}\times\bar{H}=\bar{J}+\frac{\delta\bar{D}}{\delta t}$$

$$\bar{\nabla}\times\bar{E}=-\delta\bar{B}/\delta t \qquad \bar{D}=\epsilon\bar{E}$$

$$\bar{\nabla}\cdot\bar{B}=0 \qquad \bar{B}=\mu\bar{H}$$

$$\bar{\nabla}\cdot\bar{D}=\rho \qquad \bar{J}=\gamma\bar{D}$$

IN HOMOGENEOS MEDIA WITH NO CHARGE OR CONDUCTIVITY (SUCH AS FREE SPACE)

$$\bar{\nabla} \times \bar{H} = \epsilon\, \delta \bar{E}/\delta t$$

$$\bar{\nabla} \times \bar{E} = -\mu\, \delta \bar{H}/\delta t$$

$$\bar{\nabla} \cdot \bar{H} = 0$$

$$\bar{\nabla} \cdot \bar{E} = 0$$

- DERIVATION OF WAVE EQUATIONS

$$\bar{\nabla} \times (\bar{\nabla} \times \bar{E}) = -\mu \nabla \times \frac{\delta \bar{H}}{\delta t}$$

$$= -\mu \frac{\delta}{\delta t} \bar{\nabla} \times \bar{H} = -\mu \frac{\delta}{\delta t} \epsilon \frac{\delta \bar{E}}{\delta t} = -\mu\epsilon \frac{\delta^2 \bar{E}}{\delta t^2}$$

A VECTOR IDENTITY: $\bar{\nabla} \times (\bar{\nabla} \times \bar{A}) = \bar{\nabla}(\bar{\nabla} \cdot \bar{A}) - \nabla^2 \bar{A}$

BUT $\bar{\nabla} \cdot \bar{E} = 0$ (SINCE THERE IS NO CHARGE)

$\Rightarrow \nabla^2 \bar{E} = +\mu\epsilon\, \delta^2 \bar{E}/\delta t^2$ $\Leftarrow$ 3 DIFF. EQUATIONS

FOR $E_y = E_z = 0$, $E_x = f(x - ct) \ni c = 1/\sqrt{\mu\epsilon}$

IN FREE SPACE $\Rightarrow$ c = SPEED OF LIGHT $= 1/\sqrt{\mu_0 \epsilon_0}$ $= 3 \times 10^8\ m/s$

WE CAN ALSO SHOW: $\nabla^2 \bar{H} = \mu\epsilon\, \delta^2 \bar{H}/\delta t^2$

(NOTE: $\bar{E}$ & $\bar{H}$ ARE PHYSICALLY INSEPARABLE QUANTITIES)

IX. PLANE WAVES

A. ELECTRIC FIELD

$$\nabla^2 \bar{E} = \mu\epsilon \frac{\delta^2 \bar{E}}{\delta t^2}$$

ASSUME $E_x = E_z = 0 \Rightarrow \nabla^2 E_Y = \frac{\delta^2 E_X}{\delta X^2} + \frac{\delta^2 E_Y}{\delta Y^2} + \frac{\delta^2 E_Z}{\delta Z^2}$

$= \mu\epsilon \, \delta^2 E_Y / \delta t^2$

ALSO ASSUME WAVE DOESN'T CHANGE IN Y OR Z:

$\Longrightarrow \delta \bar{E}/\delta Y = \delta \bar{E}/\delta Z = 0$

$\therefore \delta^2 E_Y/\delta X^2 = \mu\epsilon \, \delta^2 E_Y/\delta t^2$ ASSUME = 0

THUS $E_Y = f_1(x - vt) + \cancel{f_2(x+vt)}$

B. MAGNETIC FIELD

FROM A, ASSUME $E_Y = E_m \cos \omega t - \beta x$

$\omega = 2\pi f$, $v = \frac{\omega}{\beta}$, $\lambda = 2\pi/\beta = v/f$

NOW $\delta \bar{B}/\delta t = -\nabla \times \vec{E}$

OR $\delta B_Z/\delta t = -\delta E_Y/\delta x = -\beta E_m \sin(\omega t - \beta x)$

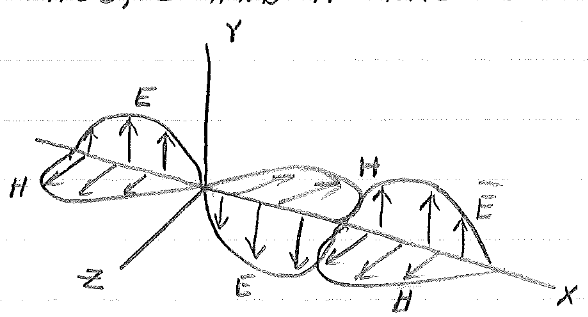
$\Rightarrow B_Z = \frac{\beta}{\omega} E_m \cos(\omega t - \beta x) = \frac{\beta}{\omega} E_Y$

THUS $E_Y = v B_Z$

OR $E_Y = \sqrt{\frac{\mu}{\epsilon}} H_Z$

$\eta = \sqrt{\frac{\mu}{\epsilon}} \equiv$ INTRINSIC IMPEDANCE (377Ω FREE SPACE)

THUS, E AND H ARE IN PHASE AND $\perp$



C. POLARIZATION

IF $\vec{E}$ VECTOR OSCILLATES, BUT MAINTAINS THE SAME DIRECTION, THE WAVE IS POLARIZED (FOR EXAMPLE, $E_x = E_z = 0$)

PLANE OF POLARIZATIN

- IN OPTICS, THE PLANE // $\vec{H}$ FIELD
- IN RADIO, " " " $\vec{E}$ FIELD

D. COMPLEX NOTATION

$$E_y = E_m e^{j(\omega t - \beta x)}$$

PHYSICALLY, WE WANT $Re\, e^{j\theta} = \cos\theta$

E. PROPAGATION IN A CONDUCTING MEDIA

UNTIL NOW, WE HAVE ASSUMED $\gamma = 0$ (ie A PERFECT DIELECTRIC).

RESTRICT ATTENTION TO SINUSOIDS:

$$\vec{E} = \vec{E}_0 e^{j\omega t} \qquad \vec{H} = \vec{H}_0 e^{j\omega t}$$

STILL ASSUME $\rho = 0$

MAXWELL:

$$\bar{\nabla} \times \bar{E} = -\mu \, \delta \bar{H}/\delta t$$

$$\bar{\nabla} \times \bar{H} = \gamma \bar{E} + \epsilon \, \delta \bar{E}/\delta t$$

OR

$$\bar{\nabla} \times \bar{E}_0 e^{j\omega t} = -j\omega\mu \bar{H}_0 e^{j\omega t}$$

$$\bar{\nabla} \times H_0 e^{j\omega t} = \gamma E_0 e^{j\omega t} + j\omega\epsilon \bar{E}_0 e^{j\omega t}$$

OR

$$\bar{\nabla} \times \bar{E}_0 = -j\omega\mu \bar{H}_0$$

$$\bar{\nabla} \times \bar{H}_0 = (\gamma + j\omega\epsilon)\bar{E}_0$$

WE NOW WISH TO DERIVE A WAVE EQUATION $\Longrightarrow$

$\bar{\nabla} \times \bar{\nabla} \times \bar{E}_0 = -j\omega\mu \, (\nabla \times H_0)$

$-\nabla^2 \bar{E}_0 = -j\omega\mu \, (\gamma + j\omega\epsilon) E_0$

DEFINE $\quad \Gamma^2 = j\omega\mu(\gamma + j\omega\epsilon)$

$\Rightarrow \nabla^2 \bar{E}_0 = \Gamma^2 E_0 \Leftarrow$ WAVE EQ. WITH CONDUCTIVITY

ASSUME, AS BEFORE: $E_X = E_Z = 0 = \delta E_Y/\delta Y = \frac{\delta E_Y}{\delta Z}$

$\Rightarrow \quad \delta^2 E_{0Y}/\delta x^2 = \Gamma^2 E_{0Y}$

SOLUTION IS $\quad E_{0Y} = E_m e^{\pm \Gamma x}$

$\Rightarrow E_Y = E_m e^{j\omega t \pm \Gamma x}$

NOTE: $\gamma = 0 \Rightarrow \Gamma = j\beta$ AND WE HAVE PREVIOUS RESULT

LET $\Gamma = j\omega\sqrt{\mu\epsilon(1 + \frac{\gamma}{j\omega\epsilon})} = \alpha + j\beta$

$\Rightarrow E_Y = E_m e^{-\alpha x} \; e^{j(\omega t - \beta x)}$

- MAGNETIC FIELD (NOT IN PHASE WITH $\bar{E}$ FOR $\gamma \neq 0$)

$\bar{\nabla} \times \bar{E}_0 = -j\omega\mu \bar{H}_0 \Rightarrow H_Z = \frac{\Gamma}{j\omega\mu} E_Y$

η = INTRINSIC IMPEDANCE $= E_Y / H_Z$

$$= \left[\frac{\mu}{\epsilon(1 + \frac{\gamma}{j\omega\epsilon})} \right]^{\frac{1}{2}}$$

NOTE: $\eta = \sqrt{\mu/\epsilon}$ FOR $\gamma = 0$

η IS COMPLEX IN FIRST QUADRANT

$\Rightarrow \vec{E}$ FIELD LEADS $\vec{H}$ FIELD IN PHASE

F. DIELECTRIC LOSS

DUE TO "DIELECTRIC HYSTERISIS"

$\sim$ PROPORTIONAL TO ω

G. POWER AND THE POYNTING VECTOR

$\bar{P}$ = POWER/m^2 ; $\bar{P} \cdot \bar{a}$ = POWER THRU AREA $\bar{a}$

FOR A GIVEN SURFACE:

OUTWARD FLOW OF POWER = $\oint \bar{P} \cdot d\bar{a}$

RECALL: (ENERGY FROM $\bar{E}$) = $\frac{1}{2}\int \bar{D} \cdot \bar{E}\, dv$

(" " $\bar{H}$) = $\frac{1}{2}\int \bar{B} \cdot \bar{H}\, dv$

POWER DECEASE = $-\frac{1}{2}\frac{\delta}{\delta t}\int (\bar{B} \cdot \bar{H} + D \cdot \bar{E})\, dv$

POWER DECREASE = OUTWARD FLOW OF POWER

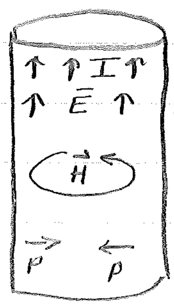
$$\Rightarrow \oint \bar{P} \cdot d\bar{a} = -\frac{1}{2}\int \frac{\delta}{\delta t}(\mu \bar{H} \cdot \bar{H} + \epsilon \bar{E} \cdot \bar{E})\, dv$$
$$= -\int (\mu \bar{H} \cdot \delta \bar{H}/\delta t + \epsilon \bar{E} \cdot \delta \bar{E}/\delta t)\, dv$$
$$= -\int (\vec{H} \cdot \delta B/\delta t + \bar{E} \cdot \delta \bar{D}/\delta t)\, dv$$
$$= \int [\bar{H} \cdot (\bar{\nabla} \times \bar{E}) - E \cdot (\bar{\nabla} \times \vec{H})]\, dv$$
$$= \int \nabla \cdot (E \times H)\, dv \Leftarrow \text{FROM VECTOR IDENTITY}$$

USING GAUSS' THEOREM

$$\oint \bar{P} \cdot d\bar{a} = \oint \bar{E} \times \bar{H} \cdot d\bar{a}$$

$\Longrightarrow \bar{P} = \bar{E} \times \bar{H}$ (NOTE: WE ASSUMED $\gamma = 0$ BUT IT IS ALSO TRUE FOR $\gamma > 0$)

EXAMPE:



X. REFLECTION

A. BOUNDRY SURFACES

NORMAL $\bar{D}$ FIELD MUST BE CONTINUOUS

TANGENTAL $\bar{E}$ FIELD THE SAME

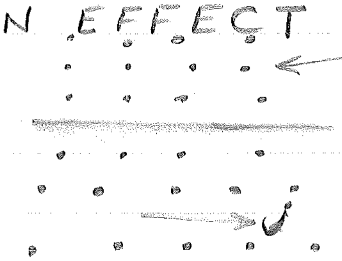
NORMAL $\bar{B}$ FIELD THE SAME

TANGENTAL $\bar{H}$ FIELD THE SAME

B. CONDUCTOR AS BOUNDRY

CAN BE NO E FIELD IN A PERFECT CONDUCTOR

SKIN EFFECT (INTRO) (SEE NEXT PAGE ⇒



INCREASING B FIELD (OUT OF PAGE)

⇒ CURRENT $\vec{J}$ IS PRODUCED

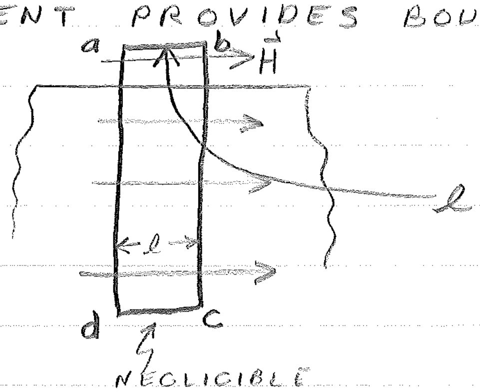
⇒ H FIELD WILL HAVE CURL

($\vec{J} \propto \bar{\nabla} \times \vec{H}$ FOR CONDUCTOR)

⇒ $\bar{B}$ WILL DIMINISH AT THE CONDUCTOR'S INARDS

⇒ MOST CURRENT WILL BE AT SURFACE

CURRENT PROVIDES BOUNDRY FOR $\bar{B}$ FIELD



$\oint \vec{H} \cdot d\vec{s} = I$ (SINCE $\delta D/\delta t \approx 0$ IN COND)

$= \ell H$

$\Rightarrow H = I/\ell$

C. SKIN EFFECT

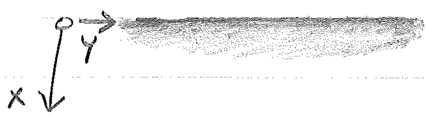
RECALL: $E_{oy} = E_m e^{\pm \Gamma x}$

THUS, IN A CONDUCTOR: $i_{oy} = J_m e^{\pm \Gamma x}$

$\Gamma^2 = j\omega\mu(\gamma + j\omega\epsilon)$

FOR $\omega \ll$ FREQ OF VIS. LIGHT: $\Gamma^2 \simeq j\omega\mu\gamma$

$\Rightarrow \Gamma = \frac{1+j}{\delta} \quad \Rightarrow \delta = 1/\sqrt{\pi f \mu \gamma}$



$\Rightarrow i_{oy} = i_m e^{-\Gamma x}$

$= i_m e^{-x/\delta} e^{-jx/\delta}$

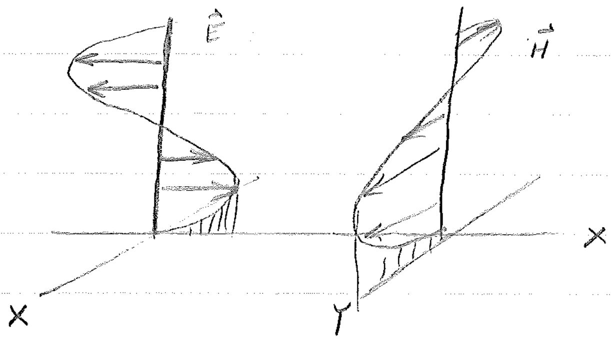
THUS, $I/\ell = \int_0^\infty i_{oy}\, dx = \frac{i_m}{\Gamma} = H$

PWR LOSS PER VOLUME $= |i|^2/\gamma = i_m^2 \delta / 4\gamma$

δ = "SKIN DEPTH"

$1/\gamma\delta$ = "SURFACE RESISTIVITY"

D. REFLECTION FROM A CONDUCTOR



- $E_x = E_m \cos\beta(vt - z) + f_2(vt + z)$

  $E_x(0) = 0 \Rightarrow f_2 = -E_m \cos\beta(vt + z) \quad ; \quad \omega = \beta v$

  $E_x = E_m[e^{-j\beta z} - e^{j\beta z}] e^{j\omega t}$
- $B_z = \frac{\beta}{\omega} E_y \Rightarrow H_y = \frac{E_m}{\eta}(e^{-j\beta z} + e^{j\beta z}) e^{j\omega t}$

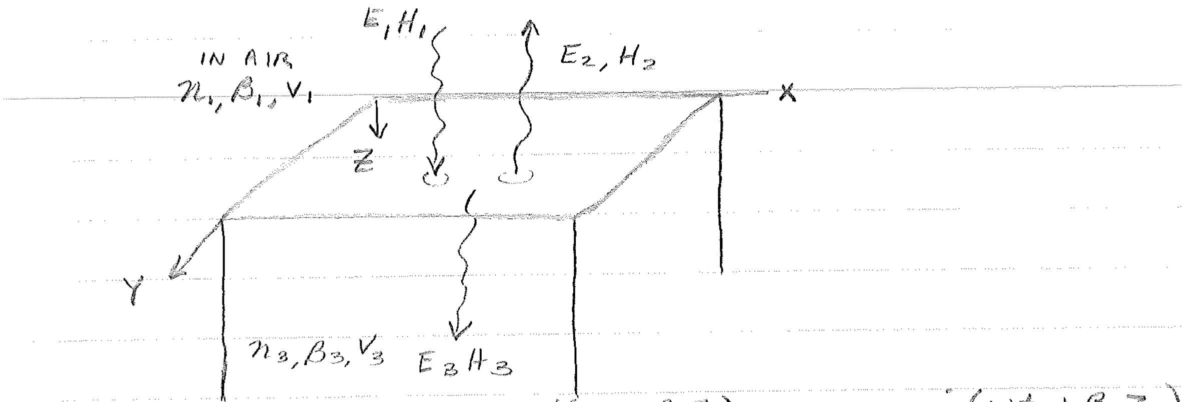
E_x & B_z ARE "STANDING WAVES" NOTE

$E_x = 2E_m \sin\beta z \sin\omega t$

$H_y = \frac{2E_m}{\eta} \cos\beta z \cos\omega t$

NODES 90° OUT OF PHASE.

E. DIELECTRIC DIFFRACTION



- $E_{X(AIR)} = E_{m_1} e^{j(\omega t - \beta_1 z)} + E_{m_2} e^{j(\omega t + \beta_1 z)}$

  $E_{X(D)} = E_{m_3} e^{j(\omega t - \beta_3 z)}$

  @ $Z = 0$ (SURFACE) $\Rightarrow E_{m_3} = E_{m_1} + E_{m_2}$

- $H_{Y(AIR)} = H_{m_1} e^{j(\omega t - \beta_1 z)} + H_{m_2} e^{j(\omega t + \beta_1 z)}$

  $H_{Y(d)} = H_{m_3} e^{j(\omega t - \beta_3 z)} \Rightarrow H_{m_3} = H_{m_1} + H_{m_2}$

H & E ARE RELATED BY INTRINSIC IMPEDENCE:

$$E_{m_1} = \eta_1 H_{m_1}, \quad E_{m_2} = -\eta_1 H_{m_2}, \quad E_{m_3} = \eta_3 H_{m_3}$$

(-) SIGN FROM $\Rightarrow E > 0 \rightarrow H < 0$

THIS GIVES

$$E_{m_2} = \frac{\eta_3 - \eta_1}{\eta_3 + \eta_1} E_{m_1}$$

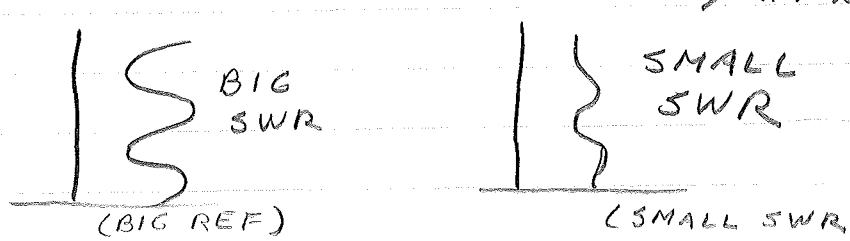
$$\rho = \frac{\eta_3 - \eta_1}{\eta_3 + \eta_1} \equiv \text{REFLECTION COEFFICIENT}$$

POWER FLOW: $P_1 = E_1 \times H_1 = E_{m_1} H_{m_1} \cos^2 \omega t$

WE HAVE A STANDING WAVE

SWR = STANDING WAVE RATIO

= FIELD STRENGTH AT LOOP / AT NODE



F. REFLECTION FROM A POOR CONDUCTOR

Γ = (COMPLEX) PROPAGATION FACTOR

η = (COMPLEX) INTRINSIC IMPEDANCE

$$E_{x(d)} = E_{m_3}\, e^{j\omega t - \Gamma_3 z}$$

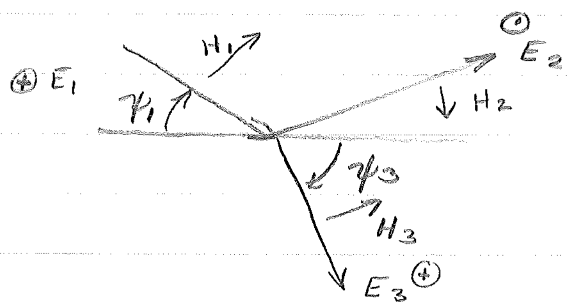
$$H_{y(d)} = H_{m_3}\, e^{j\omega t - \Gamma_3 z}$$

RECALL

$$\eta_3 = \left[\frac{\mu_3}{\epsilon_3\left(1 + \gamma_3/j\omega\epsilon_3\right)}\right]$$

XMITTED & REFL WAVES HAVE PHASE SHIFT

G. OBLIQUE REFLECTION

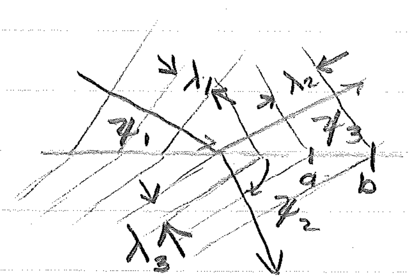


(ASSUME $\bar{E}$ IS HORIZONALLY POLARIZED)

TANGENTAL COMPONENTS

BOUNDRY CONDITIONS: $E_1 + E_2 = E_3$, $\underbrace{H_{t_1} + H_{t_2} = H_{t_3}}$

$$b - a = \frac{\lambda_3}{\cos\psi_3} = \frac{\lambda_1}{\cos\psi_3} = \frac{\lambda_2}{\cos\psi_2}$$



GIVES "SNELL'S LAW"

$$\frac{\cos\psi_1}{\cos\psi_3} = \frac{\lambda_1}{\lambda_3} = v_1/v_3$$

NOW

$$E_1 = \eta_1 H_1 \qquad E_2 = \eta_1 H_2 \qquad E_3 = \eta H_3$$

$$H_{t_1} = \frac{E_1}{\eta_1}\sin\psi_1 \qquad H_{t_2} = -\frac{E_2}{\eta_1}\sin\psi_1$$

$$H_{t_3} = E_3/\eta_3 \sin\psi_3$$

OR

$$\frac{E_2}{E_1} = \frac{\eta_3\sin\psi_1 - \eta_1\sin\psi_3}{\eta_3\sin\psi_1 + \eta_1\sin\psi_3}$$

XI. RADIATION

A. ELECTRODYNAMIC POTENTIALS

FOR ELECTROSTATIC CASE (WITH NO $\vec{i}$ OR q)

$$\nabla^2 V = 0 \quad ; \quad \nabla^2 \vec{A} = 0$$

FOR ELECTRODYNAMIC CASE (WITH NO $\vec{i}$ OR q)

$$\nabla^2 V - \mu\epsilon \frac{\delta^2 V}{\delta t^2} = 0$$

$$\nabla^2 \vec{A} - \mu\epsilon \, \delta^2 \vec{A} / \delta t^2 = 0 \quad ; \quad v = \frac{1}{\sqrt{\mu\epsilon}}$$

QUASISTATIONARY RELATIONS

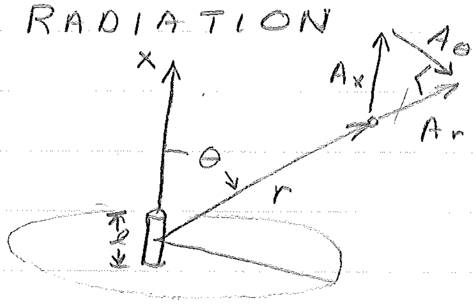
$$\vec{A} = \frac{1}{4\pi} \int \frac{\vec{i}}{r} dv \qquad V = \frac{1}{4\pi\epsilon} \int \frac{\rho}{r} dv$$

RETARDED POTENTIALS

$$V = \frac{1}{4\pi\epsilon} \int \frac{1}{r} \rho\left(t - \frac{r}{v}\right) dV$$

$$\vec{A} = \frac{1}{4\pi} \int \frac{1}{r} \vec{i}\left(t - \frac{r}{v}\right) dV$$

B. RADIATION



SHORT WIRE OF LENGTH l AND CURRENT

$$i = I \sin \omega t$$

SINE $\vec{i} = \vec{i}_x$, WE ONLY HAVE X COMPONENT OF A FIELD:

$$A_x = \frac{1}{4\pi} \int_{-l/2}^{l/2} \frac{1}{r} I \sin \omega\left(t - \frac{r}{v}\right) dx$$

FOR $r \gg l$, WE HAVE

$$A_x = \frac{1}{4\pi r} I l \sin \omega\left(t - \frac{r}{v}\right)$$

OR, IN SPHERICAL COORDINATES:

$$A_r = \frac{Il}{4\pi r} \cos\theta \sin \omega\left(t - \frac{r}{v}\right)$$

$$A_\theta = \frac{Il}{4\pi r} \sin\theta \sin \omega\left(t - \frac{r}{v}\right)$$

$$A_\phi = 0$$

TAKING CURL GIVES $\vec{H}$:

$$H_r = H_\theta = 0$$

$$H_\phi = \frac{Il}{4\pi r} \sin\theta \left[\frac{\omega}{v} \cos \omega\left(t - \frac{r}{v}\right) + \frac{1}{r} \sin \omega\left(t - \frac{r}{v}\right)\right]$$

NOW $\quad \nabla \cdot A = -\epsilon \frac{\delta V}{\delta t}$

$$\Rightarrow V = -\frac{1}{\epsilon} \vec{\nabla} \cdot \int \vec{A}\, dt$$

$$E = -\nabla \cdot V$$

GIVES

$$E_r = \frac{I\ell}{2\pi\epsilon r} \cos\theta \left[\frac{1}{vr} \sin\omega\left(t - \frac{r}{v}\right) - \frac{1}{\omega r^2} \cos\omega\left(t - \frac{r}{v}\right) \right]$$

$$E_\theta = \frac{I\ell}{4\pi\epsilon r} \sin\theta \left[\frac{\omega}{v^2} \cos\omega\left(t - \frac{r}{v}\right) + \frac{1}{rv} \sin\omega\left(t - \frac{r}{v}\right) - \frac{1}{\omega r^2} \cos\omega(t - r/v) \right.$$

$$E_\phi = 0$$

OR, WITH $\lambda = \frac{v}{f} = \frac{2\pi v}{\omega}$ AND $\eta = \sqrt{\frac{\mu}{\epsilon}}$:

$$\begin{cases} E_r = -\eta \dfrac{I\ell\cos\theta}{\lambda r} \left[\dfrac{\lambda^2}{4\pi^2 r^2} \cos\left(\dfrac{2\pi r}{\lambda} - \omega t\right) + \dfrac{\lambda}{2\pi r} \sin\left(\dfrac{2\pi r}{\lambda} - \omega t\right) \right] \\ E_\theta = \eta \dfrac{I\ell\sin\theta}{2r\lambda} \left[\dfrac{-\lambda^2}{4\pi^2 r^2} \cos\left(\dfrac{2\pi r}{\lambda} - \omega t\right) - \dfrac{\lambda}{2\pi r} \sin\left(\dfrac{2\pi r}{\lambda} - \omega t\right) + \cos\left(\dfrac{2\pi r}{\lambda} - \omega t\right) \right] \\ E_\phi = 0 \end{cases}$$

$$\begin{cases} H_r = 0 \\ H_\theta = 0 \\ H_\phi = \dfrac{I\ell}{2r\lambda} \left[\dfrac{-\lambda}{2\pi r} \sin\left(\dfrac{2\pi r}{\lambda} - \omega t\right) + \cos\left(\dfrac{2\pi r}{\lambda} - \omega t\right) \right] \end{cases}$$

FOR r SMALL: INDUCTION FIELD:

$$\begin{cases} E_r = \dfrac{-\eta I\ell\cos\theta}{r\lambda} \dfrac{1}{4\pi^2} \dfrac{\lambda^2}{r^2} \cos\left(\dfrac{2\pi r}{\lambda} - \omega t\right) \\ E_\theta = \dfrac{-\eta I\ell\sin\theta}{2r\lambda} \dfrac{1}{4\pi^2} \dfrac{\lambda^2}{r^2} \cos\left(\dfrac{2\pi r}{\lambda} - \omega t\right) \\ E_\phi = 0 \end{cases}$$

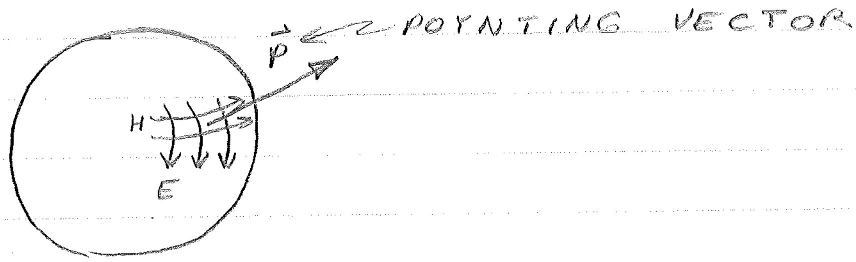
$$\begin{cases} H_r = H_\theta = 0 \\ H_\phi = \dfrac{-I\ell\sin\theta}{2\pi r\lambda} \dfrac{1}{2\pi} \dfrac{\lambda}{r} \sin\left(\dfrac{2\pi r}{\lambda} - \omega t\right) \end{cases}$$

FOR r LARGE: RADIATION FIELD:

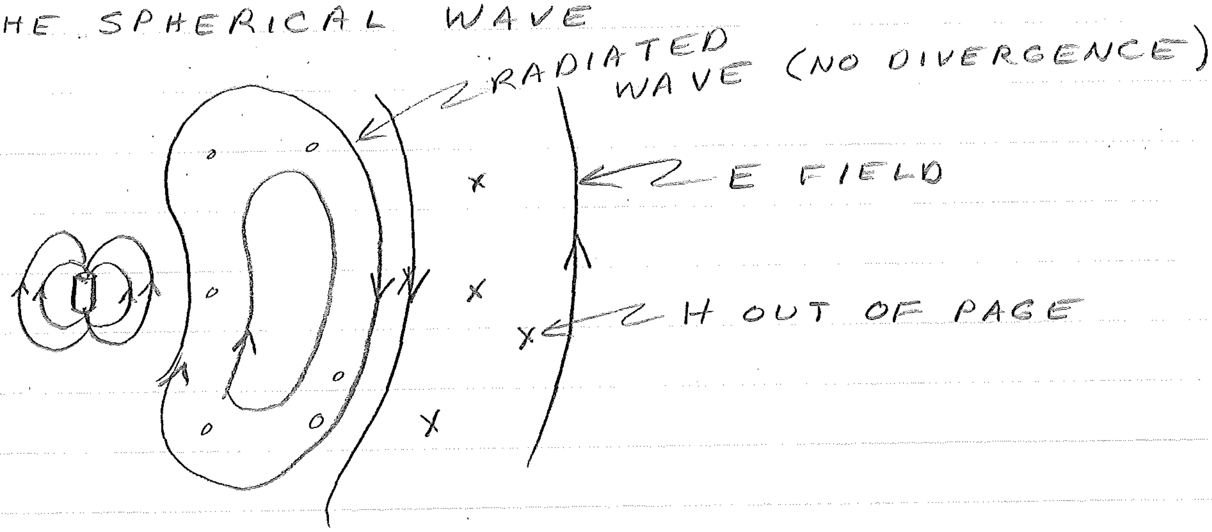
$$\begin{cases} E_r = E_\phi = 0 \\ E_\theta = \eta \dfrac{I\ell}{2r\lambda} \cos \dfrac{2\pi r}{\lambda} - \omega t \end{cases}$$

$$\begin{cases} H_r = H_\theta = 0 \\ H_\phi = \dfrac{I\ell \sin\theta}{2r\lambda} \cos\left(\dfrac{2\pi r}{\lambda} - \omega t\right) \end{cases}$$

NOTE: $E_\theta = \eta H_\phi$



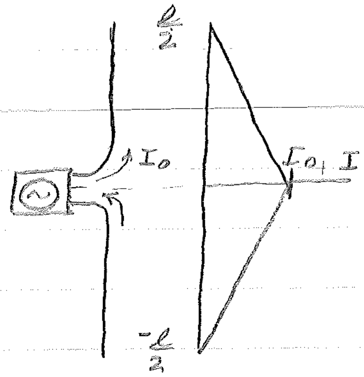
C. THE SPHERICAL WAVE



XII. ANTENNAS

A. SHORT ANTENNAS

LENGTH OF ANTENNA LESS THAN A λ



CURRENT FLOWS DUE TO ~~DISTRIBUTED CAPACITANCE~~ IN THE WIRE. AS A FIRST APPROXIMATION, LET IT TAPER OFF LINEARLY.

$$E_\theta = \frac{\eta I l \sin\theta}{2r\lambda} \cos\omega\left(t - \frac{r}{c}\right)$$

EFFECTIVE LENGTH $= l_e = l/2$, W CURRENT I_0

EFFECTIVE (RMS) FIELD STR $\left(\frac{V}{m}\right) = \frac{\eta I_0 l_e}{2r\lambda} \sin\theta$

$\eta = 377\,\Omega$ (FOR FREE SPACE)

I_0 = EFFECTIVE MIDPOINT CURRENT (AMPS)

l = ACTUAL ANTENNA LENGTH

l_e = EFF. ANT. LENGTH (IF $l \ll \lambda$, $l_e = \frac{1}{2} l$)

θ = ANGLE TWIXT XMITTING & REC. ANTENNA

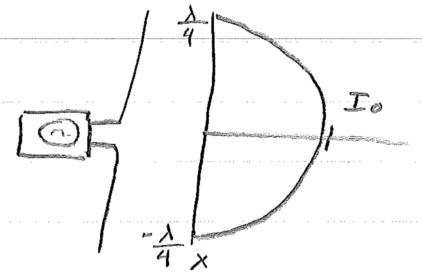
r = DISTANCE TWIXT ANTENNAS

λ = WAVELENGTH.

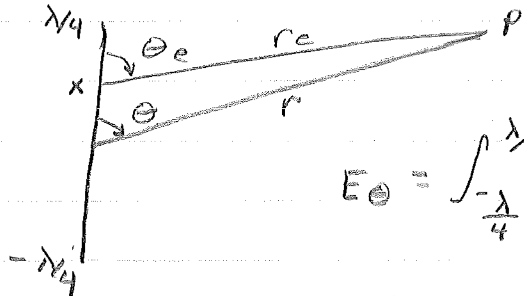
B. HALF WAVE LENGTH ANTENNAS

CURRENT IS NO LONGER LINEAR, BUT

$$I = I_0 \cos 2\pi \frac{x}{\lambda}$$



OBLIQUITY IS ANOTHER COMPLICATION



INTEGRATING EACH LITTLE dx GIVES

$$E_\theta = \int_{-\frac{\lambda}{4}}^{\lambda/4} \frac{\eta I_0 \sin\theta_e}{2 r_e \lambda} \cos\left(\frac{2\pi x}{\lambda}\right) \cos\omega\left(t - \frac{r_e}{c}\right) dx$$

$$r_e \approx r_e - x\cos\theta$$

$$\Rightarrow E_\theta = \frac{\eta I_0 \sin\theta}{2 r \lambda} \int_{-\lambda/4}^{\lambda/4} \cos\frac{2\pi x}{\lambda} \cos\omega\left(t - \frac{r}{c} + \frac{x\cos\theta}{c}\right) dx$$

$$= \frac{\eta I_0}{2\pi r} \cos\omega\left(t - \frac{r}{c}\right) \cos\left(\frac{\pi}{2}\cos\theta\right) / \sin\theta$$

$$H_\phi = \frac{1}{\eta} E_\theta$$

COMPARISON TO SHORT ANTENNA:

FOR $\frac{1}{2}$ WAVE (DIPOLE) : RMS FIELD $= \frac{\eta I_0}{2\pi r}$

FOR SHORT : RMS FIELD $= \frac{\eta I_0 l_e}{2 r \lambda}$

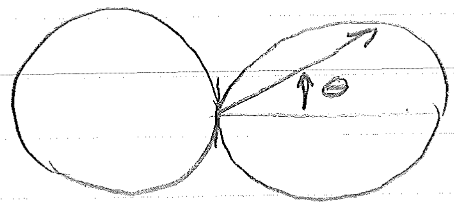
THUS $l_e = \lambda/\pi \; l$ FOR $\frac{1}{2}$ WAVE

ACT LENGTH $= \lambda/2$

$$\Rightarrow l_e = 2/\pi$$

C. RADIATION PATTERN

PLOT IS FIELD STRENGTH ($\frac{\mu V}{m}$) AT ONE MILE



← SHORT ANTENNA

$\eta \frac{I_0}{2r}\frac{l_e}{\lambda}\sin\theta$

TURNS OUT $\frac{1}{2}$ WAVE IS MORE EFFICIENT THAT SHORTY

D. RADIATED POWER

$\vec{P} = \vec{E}\times\vec{H}$

$$\int p\cdot da = \int E\times H\cdot da$$
$$= \int \eta\left[\frac{I l \sin\theta}{2r\lambda}\cos\omega\left(t-\frac{r}{c}\right)\right]^2 (2\pi r^2 \sin\theta)\,d\theta$$
$$= \frac{\eta\pi I^2 l^2 \cos^2\omega(t-\frac{r}{c})}{2\lambda^2}\int_0^\pi \sin^3\theta\, d\theta$$
$$= \frac{2\eta I^2 l^2 \pi}{3\lambda^2}\cos^2\omega\left(t-\frac{r}{c}\right)$$

THUS, FOR A SHORT UNIFORM i ANTENNA POWER (AVERAGE)

$$= \frac{\eta\pi I^2 l^2}{3\lambda^2}$$

E. RADIATION RESISTANCE

$= \text{AVE RAD PWR} / I_{EFF}^2$

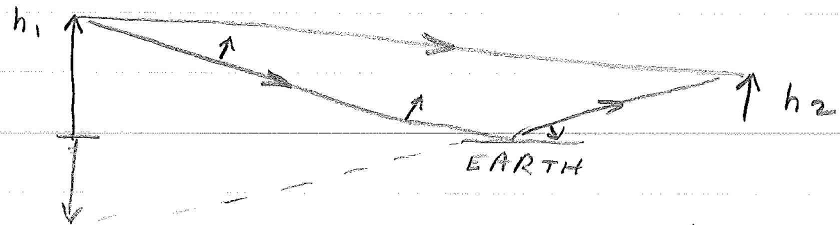
FOR SHORT ANTENNA $= \frac{2\pi\eta l^2}{3\lambda^2}$

$= 789\left(\frac{l_e}{\lambda}\right)^2\Omega$

FOR HALF WAVE DIPOLE, TURNS OUT

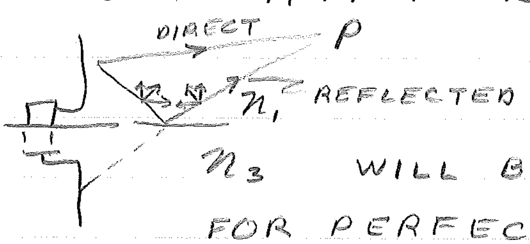
$= 73.1\,\Omega$ (IND. OF ω)

F. ANTENNAS ABOVE GROUND



REFLECTION GIVES 180° PHASE SHIFT
∴ WE WANT $\frac{1}{2}\lambda$ DIFFERENCE IN PATH FOR CONSTRUCTIVE INTERFERENCE
(NOTE ANTENNA IS ESSENTIALLY DOUBLED)

G. GROUNDED ANTENNAS



RADIATION FROM VERTICAL ANTENNAS WILL BE VERTICALLY POLARIZED.
FOR PERFECT REFLECTION (−1), HORIZONTAL COMPONENT WOULD BE REVERSED & VERTICAL COMPONENT THE SAME.
IF EARTH WERE DIELECTRIC, REFL. WOULD BE SMALLER THAN INCIDENT WAVE, AND IS INDEPENDENT OF ANGULAR INCIDENCE.

BREWSTER ANGLE: ALL INCIDENT ENERGY ABSORBED (AS "CRITICAL ANGLE)

RECALL REFLECTION COEFFICIENT

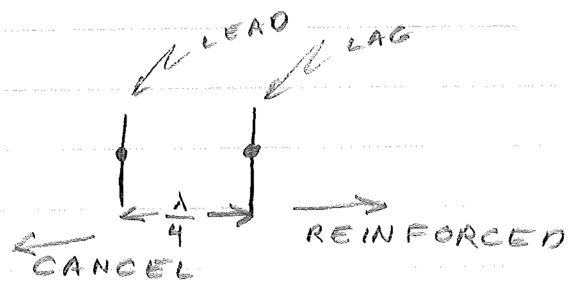
$$E_{t2}/E_{t1} = \frac{n_3 \sin\psi_3 - n_1 \sin\psi_1}{n_3 \sin\psi_3 + n_1 \sin\psi_1}$$

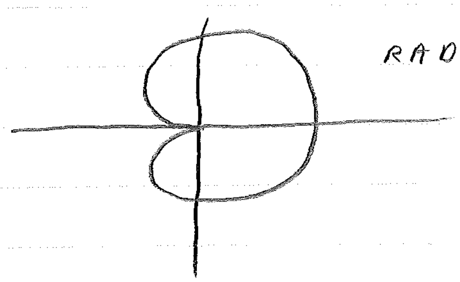
IF ANTENNA IS SHORT WIRE, WE TAKE INTO ACCOUNT REFLECTION:

E_{rms} FIELD STR $= \frac{\eta I_o h_e}{r \lambda} \sin\theta$

MAY INCREASE h_e BY PLACING CAPACITANCE NEAR THE TOP TO FURTHER MAKE CURRENT UNIFORM.

H. ANTENNA ARRAYS





RADIATION PATTERN

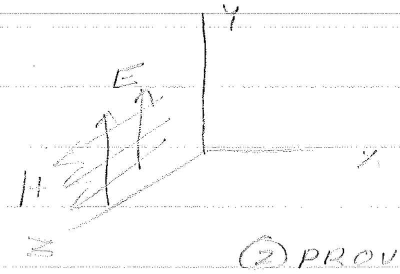
I. RECEIVING

E-M FIELD CAUSES i IN A CONDUCTOR

XIII. WAVE GUIDES

A. GUIDED WAVES

$\begin{cases} \text{E FIELDS CAN TERMINATE ONLY ON ELEC. CHARGE.} \\ \text{H " " " " " CURRENT} \end{cases}$



FOR UNIDIRECTIONAL PLANE WAVE PROPAGATION:

① CURRENT MUST PROVIDE NEC. CURL FOR H FIELD BOUNDRY

② PROVIDE PROPER CHARGE FOR $\vec{E}$ FIELD BOUNDRY

① $\nabla \times H = \frac{\delta D}{\delta t} \Rightarrow -\frac{\delta H_z}{\delta x} = \frac{\delta D_y}{\delta t}$

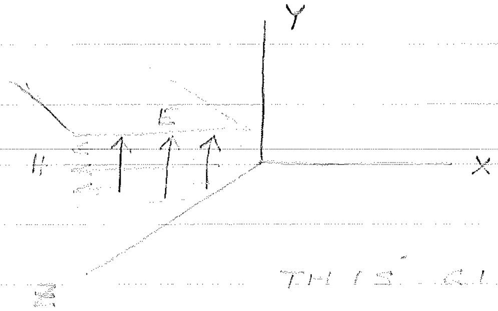
② $\sigma = D_y \Rightarrow -\frac{\delta I_x}{\delta x} = \frac{\delta \sigma}{\delta t}$ ⟸ CURRENT REQUIRED TO TERMINATE E FIELD

③ $H_z = I_x \Rightarrow \frac{\delta H_z}{\delta x} = \frac{\delta I_x}{\delta x}$

$\Rightarrow \frac{\delta H_z}{\delta x} = -\frac{\delta D_y}{\delta t}$ ⟸ CONCLUSION: SPECIAL CASE OF A MAXWELL.

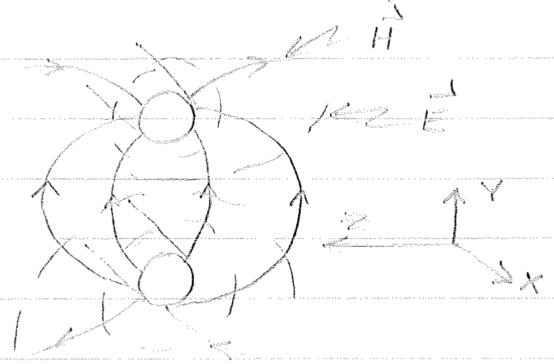
∴ A WAVE CAN BE LIMITED TO THE SPACE TWIXT TWO CONDUCTORS SINCE THE CURRENT THAT FLOWS ON THESE SURFACES IS PROPER TO TERMINATE BOTH E AND H FIELDS.

B. FINITE WAVES

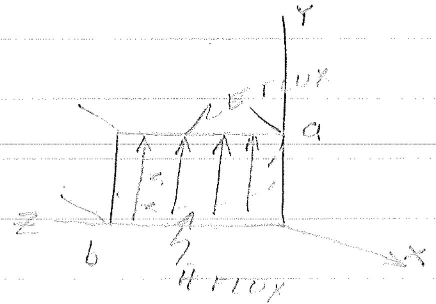


BENDING TOP UP INTO A CYLINDER & BENDING BOTTOM DOWN INTO A CYLINDER.

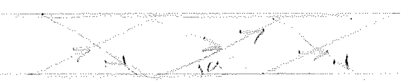
THIS GIVES PARALLEL-WIRE TRANSMISSION LINE:



C. HOLLOW WAVE GUIDES

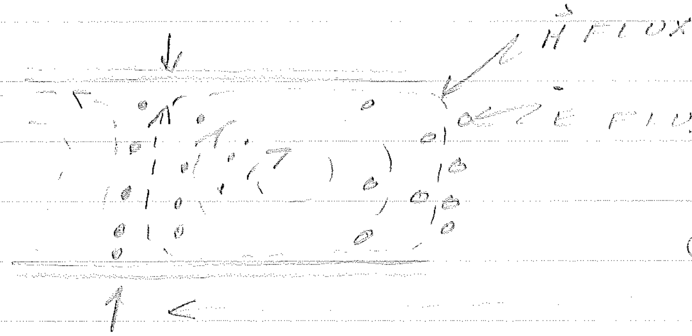


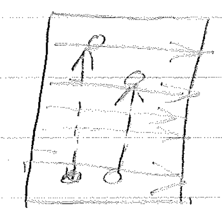
THERE CAN BE NO TANGENTIAL $\vec{E}$ FIELD AT SURFACE SINCE IN CONDUCTS. BUT: TWO SINUSOIDS OF SAME AMPLITUDE & FREQ, WHEN SUPERIMPOSED @ AN ANGLE, WILL ADD TO ZERO.



α DEPENDS ON FREQ. & WAVE GUIDE.

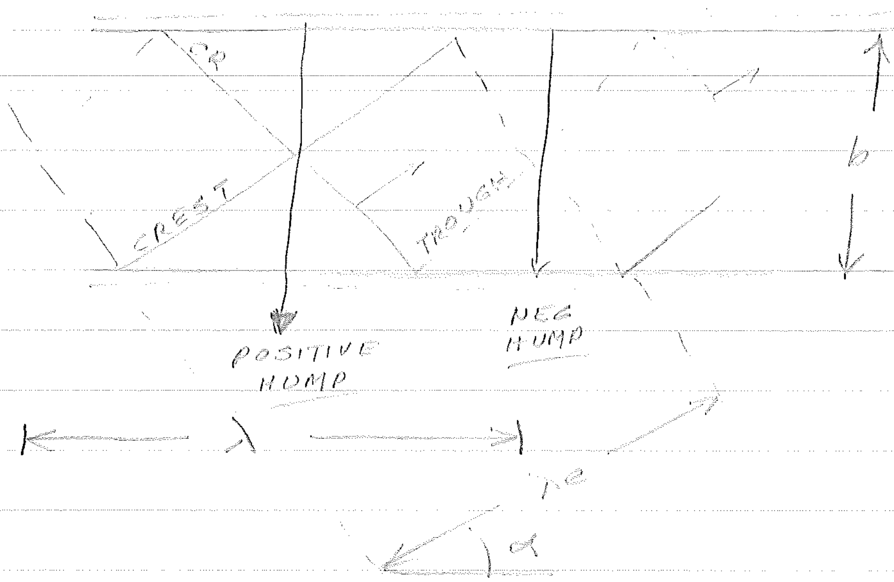
TE_{01} MODE:





D. GROUP VELOCITY

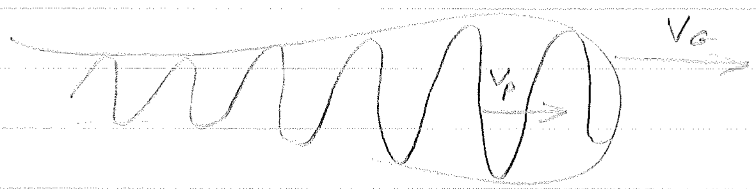
V_g = GROUP VELOCITY < c. (DUE TO ZIG-ZAG PATH)



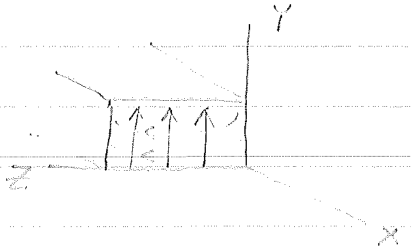
LONGEST WAVELENGH: $\lambda_0 = 2b$

$\Rightarrow$ CUT OFF FREQ: $f_0 = \frac{v_c}{2b} = \frac{1}{2b\sqrt{\mu\epsilon}}$

E. PHASE VELOCITY:



F. DERIVATION



FROM MAXWELL:

$$\nabla^2 E = \mu\epsilon \frac{\delta^2 E}{\delta t^2}, \quad \nabla \cdot E = 0$$

A SOLUTION IS

$$E_Y = A \sin\left(\frac{z\pi}{b}\right) \sin (\beta x - \omega t)$$

$$\beta = \omega / V_\phi, \quad V_\phi^2 = \left[\mu\epsilon - \left(\frac{\pi}{\omega b}\right)^2\right]^{-1}$$

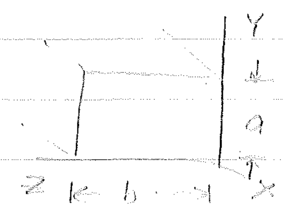
$\vec{H}$ FIELD FROM MAXWELL EQUATION:

$$\nabla \times E = -\mu \frac{\delta H}{\delta t}$$

$$\Rightarrow H_z = \frac{A}{\mu V_\phi} \sin \frac{z\pi}{b} \sin (\beta x - \omega t)$$

$$H_x = \frac{\pi}{\mu\omega b} A \cos\left(\frac{z\pi}{b}\right) \cos (\beta x - \omega t)$$

G. GENERAL SOLUTION FOR RECTANGULAR GUIDE



TM → TRANSVERSE MAGNETIC FIELD

TE → " ELECTRIC FIELD.

TEM → NO AXIAL COMP. OF EITHER FIELD.

DIFF. EQ → $\nabla^2 \bar{E} = \mu\epsilon \frac{\delta^2 \bar{E}}{\delta t^2}$; $\nabla \cdot E = 0$

B.C. → $E_X = E_Y = 0$ @ $z = 0, b$

$E_X = E_Z = 0$ @ $Y = 0, a$

ASSUME SINUSOIDAL STUFF:

$$E_x = E_{mx} e^{j\omega t - \Gamma x}$$

$$E_Y = E_{mY} e^{j\omega t - \Gamma x}$$

$$E_z = E_{mz} e^{j\omega t - \Gamma x}$$

THEN

$$\frac{\delta^2 E_x}{\delta x^2} + \frac{\delta^2 E_x}{\delta y^2} + \frac{\delta^2 E_x}{\delta z^2} = \mu\epsilon \frac{\delta^2 E_x}{\delta t^2}$$

BECOMES:

$$\frac{\delta^2 E_{mx}}{\delta y^2} + \frac{\delta^2 E_{mx}}{\delta z^2} = -(\omega^2 \mu\epsilon + \Gamma^2) E_{mx}$$

LET $E_{mx} = Y(y)\,Z(z)$

GIVES $\frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = -(\omega^2\mu\epsilon + \Gamma^2)$

LET $\frac{1}{Y}\frac{d^2Y}{dy^2} = -M^2$; $\frac{1}{Z}\frac{d^2Z}{dz^2} = -N^2$

$\Rightarrow M^2 + N^2 = \omega^2\mu\epsilon + \Gamma^2$

$\therefore\ Y = A_1 \cos My + B_1 \sin My$

$Z = C_1 \cos Nz + D_1 \sin Nz$

$E_{mx} = (A_1 \cos My + B_1 \sin My)(C_1 \cos Nz + D_1 \sin Nz)$

[SIMILAR EXPRESSIONS FOR E_{my} & E_{mz}]

USING BOUNDRY CONDITIONS:

$E_x = 0 @ z = 0 \Rightarrow C_1 = 0$

$E_x = 0 @ y = 0 \Rightarrow A_1 = 0$

$E_x = 0 @ z = b \Rightarrow N = \frac{n\pi}{b}$; $n = 0, 1, 2, \ldots$

$E_x = 0 @ z = a \Rightarrow M = \frac{m\pi}{a}$; $m = 0, \ldots$

$\therefore E_{mx} = B_1 D_1 \sin \frac{m\pi y}{a} \sin \frac{n\pi z}{b}$

THUS

$E_x = K_1 \sin \frac{m\pi y}{a} \sin \frac{n\pi z}{b}\, e^{j\omega t - \Gamma x}$

MAY OBTAIN E_y & E_z FROM $\nabla \cdot E = 0$

GIVES: $\Gamma^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu\epsilon$

MAY GET H FROM

$\frac{\delta H}{\delta t} = -\frac{1}{\mu}\nabla \times E$

H. TRANSVERSE ELECTRIC WAVES: $E_x = 0$

I. TRANSVERSE MAGNETIC WAVE: $H_x = 0$

J. MODES: EACH n, m GIVE A MODE

3/28/77

ELECTRO & MAGNETO STATIC FIELDS

E • $\vec{F} = Q\vec{E}$

$\oint_{\ell} \vec{E} \cdot d\vec{s} = 0$; $\int_v \epsilon \vec{E} \cdot d\vec{a} = \int_v \bar{D} \cdot d\bar{a} = Q$

$\vec{D} = \epsilon \vec{E}$ = ELECROSTATIC FLUX DENSITY

• VECTORS: DIVERGENCE: $\bar{\nabla} \cdot \bar{A} = \frac{\delta A_x}{\delta x} + \frac{\delta A_y}{\delta y} + \frac{\delta A_z}{\delta z}$

GRAD: $\nabla \phi = \vec{i} \frac{\delta \phi}{\delta x} + \vec{j} \frac{\delta \phi}{\delta y} + \vec{k} \frac{\delta \phi}{\delta z}$

CURL: $\bar{\nabla} \times \bar{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ A_x & A_y & A_z \end{vmatrix}$

$\nabla \times \nabla \phi = \bar{\nabla} \cdot \bar{\nabla} \times \bar{A} = 0$

LAPLACIAN: $\nabla^2 F = \left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2} \right) F$

$\nabla^2 \bar{A} = \nabla^2 (\vec{i} A_x + \vec{j} A_y + \vec{k} A_z)$

• THEOREMS

GAUSS: $\oint_{V_0} \vec{D} \cdot d\vec{a} = \int_v \bar{\nabla} \cdot \bar{D} \, dv$

STOKES: $\oint_{\ell} \vec{E} \cdot d\bar{s} = \int_A \bar{\nabla} \times \vec{E} \cdot d\vec{a}$

• POTENTIALS

SCALAR $\left(\begin{smallmatrix}\text{IRROTATIONAL} \\ \text{LAMELLA}\end{smallmatrix}\right)$ IF $\bar{\nabla} \times F = 0 \Rightarrow \vec{F} = -\bar{\nabla} p$

VECTOR $\left(\begin{smallmatrix}\text{SOLENOIDAL} \\ \text{SOURCELESS}\end{smallmatrix}\right)$ IF $\bar{\nabla} \cdot \bar{B} = 0 \Rightarrow \bar{B} = \bar{\nabla} \times \vec{A}$

• FUNDEMENTAL LAWS OF ELECTROSTATICS

$$\begin{cases} \vec{F} = Q\vec{E} \\ \oint_{\ell} \bar{E} \cdot d\bar{s} = 0 \\ \int_v \bar{D} \cdot d\bar{a} = Q \end{cases}$$

OR $\begin{cases} \bar{E} = -\nabla V \\ \nabla^2 V = -\rho/\epsilon \Leftarrow \text{POISSON'S EQ.} \end{cases}$

CONDUCTORS: $\vec{E} = 0$ $V = \text{CONST}$, $\sigma = D_n$ = SURFACE CHARGE

A CHARGED SPHERE: $\nabla^2 V = 0$ GIVES $V = \frac{Q}{4\pi\epsilon r}$

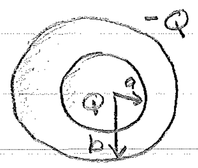
$C = Q/V$ DOUBLE SPHERE: $C = \frac{4\pi\epsilon ab}{b-a}$ (OVER)

COULOMB'S LAW: $\vec{F} = \frac{q_1 q_2}{4\pi\epsilon r^2}$

$V_{12} = \int_1^2 \vec{E} \cdot d\vec{s}$; ELECTROSTATIC $E = \frac{1}{2} \int_v \bar{D} \cdot \bar{E} \, dv$

$V = \frac{1}{4\pi\epsilon} \int_v \rho/r \, dv \Leftarrow$ POTENTIAL INTEGRAL

EXAMPLE CAPACITANCE OF DOUBLE SPHERE



$$V_{12} = \int_1^2 \vec{E} \cdot d\vec{s}$$

FOR INNER SPHERE: $\vec{E} = \dfrac{Q}{4\pi\epsilon r^2}$

$$V_{ab} = \int_a^b \frac{Q}{4\pi\epsilon r^2}\, dr = \frac{Q\,(b-a)}{4\pi\epsilon a b}$$

$$\Rightarrow C = \frac{Q}{V_{ab}} = \frac{4\pi\epsilon a b}{b-a}$$

$$b \to \infty \Rightarrow C = 4\pi\epsilon a$$

- ELECTRIC CURRENT: $I = \dfrac{dQ}{dt}$

γ = CONDUCTIVITY = $\mho/m$; $\dfrac{1}{R} = \gamma \dfrac{\text{AREA}}{\text{LENGTH}}$

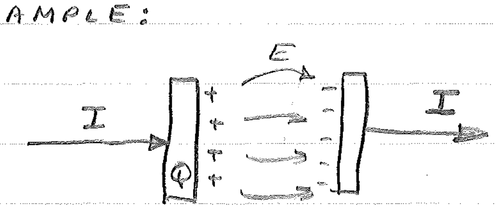
ANALOGY: $V = \int \vec{E} \cdot d\vec{s}$

$I = \int \vec{i} \cdot d\vec{a}$

$\vec{i} = \gamma \vec{E} \Leftarrow$ OHM'S LAW

$\nabla \cdot \vec{i} = 0 \Leftarrow$ KIRCHHOFF'S i LAW (STATIC $\vec{E}$)

EXAMPLE:



$$I = \frac{d}{dt} Q = \frac{d}{dt} \oint \vec{D} \cdot d\vec{a}$$

$$= \oint \frac{d}{dt}\vec{D} \cdot da$$

$$I = \oint \vec{i} \cdot d\vec{a}$$

$$\Rightarrow \oint \left(\vec{i} + \frac{d}{dt}\vec{D}\right) \cdot da = 0$$

$$= \int \nabla \cdot \left(\vec{i} + \frac{d}{dt}\vec{D}\right) dv = 0$$

$$\therefore \nabla \cdot \left(\underset{\text{COND CUR}}{\vec{i}} + \underset{\text{DISPL. CUR}}{\frac{d\vec{D}}{dt}}\right) = 0$$

- EMF

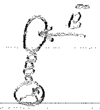
$\vec{E}_s \to$ DUE TO Q $\qquad \vec{E}_m \to$ DUE TO CHANGING H FIELD

$$V = \int E_s \cdot d\vec{s} \qquad emf = \int \vec{E}_m \cdot ds$$

- THE MAGNETIC FIELD

- MAGNETIC FORCE ON A WIRE: $\bar{F} = I\bar{L} \times \bar{B}$

- MAGNETIC FLUX $\Rightarrow \phi = \int \bar{B} \cdot d\bar{a}$



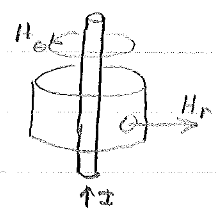
$$EMF = \oint \bar{E} \cdot d\bar{s} = -d\phi/dt$$

$$\int \bar{\nabla} \times \bar{E} \cdot da = \int \frac{d}{dt} \bar{B} \cdot da$$

$$\therefore \bar{\nabla} \times \bar{E} = -\frac{d}{dt}\bar{B} \Leftarrow \text{FARADAY'S LAW}$$

- VOLTAGE FROM MOTION: $\oint \bar{H} \cdot d\bar{s} = I \Rightarrow \bar{\nabla} \times \bar{H} = \bar{J}$

- BIOT-SAVART LAW



$$I = \oint \vec{H} \cdot d\vec{s} = \oint H_\theta ds = H_\theta 2\pi r \Rightarrow H_\theta = \frac{I}{2\pi r}$$

- FORCE BETWEEN WIRES

$I_1 \odot \leftarrow d \rightarrow \odot I_2$, $\vec{B}$

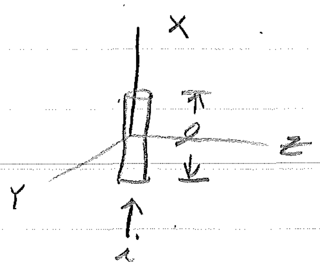
FROM B-S LAW: $B_1 = \mu H_1 = \frac{\mu I_1}{2\pi d}$

$$\Rightarrow \bar{F} = I\bar{L} \times \bar{B} \Rightarrow F/L_2 = \frac{\mu}{2\pi d} I_1 I_2$$

- MAGNETIC POTENTIAL

ELECTRO	MAGNETIC
$\nabla \times E = 0$	$\bar{\nabla} \cdot \bar{H} = 0$
$\bar{E} = -\bar{\nabla} V$	$H = \bar{\nabla} \times \bar{A}$
	$J = \bar{\nabla} \times \bar{H} = \nabla(\cancel{\nabla \cdot A}) - \nabla^2 A$ $= 0$ IN MAGNETOSTA
$\nabla^2 V = -\rho/\epsilon$	$-J = \nabla^2 \bar{A}$
$V = \frac{1}{4\pi\epsilon} \int \frac{\rho dv}{r}$	$A = \frac{1}{4\pi} \int \frac{\mu dv}{r}$

- QUASI STATIONARY MAG. FIELD



$$\vec{A} = \frac{1}{4\pi}\int \bar{J}/r \, dv$$

$$A_x = \frac{1}{4\pi}\int \frac{J_x}{r} dv$$

$$= \frac{1}{4\pi}\int J_x \frac{da\,dx}{r}$$

$$= \frac{1}{4\pi}\int \frac{i\,dx}{r}$$

$$= \frac{1}{4\pi}\int \frac{I \sin\omega t \, dx}{r}$$

$$\simeq \frac{I \sin\omega t}{4\pi r}\int_{-l/2}^{l/2} dl \quad ; r >> l$$

$$= \frac{i\,l}{4\pi r}$$

- MAGNETIC ENERGY $= \frac{1}{2}\int \bar{B}\cdot\bar{H}\,dv$

E & M FIELDS

- MAXWELL'S EQUATIONS;

$$\bar{\nabla} \times \bar{H} = \bar{J} + \frac{\delta \bar{D}}{\delta t}$$

$$\bar{\nabla} \times \bar{E} = -\delta \bar{B}/\delta t \qquad \bar{D} = \epsilon \bar{E}$$

$$\bar{\nabla} \cdot \bar{B} = 0 \qquad \bar{B} = \mu \bar{H}$$

$$\bar{\nabla} \cdot \bar{D} = \rho \qquad \bar{J} = \sigma \vec{E}$$

IN HOMO MATERIAL WITH NO Q OR σ (CONDUCTIVITY)

$$\bar{\nabla} \times H = \epsilon \frac{\delta \bar{E}}{\delta t}$$

$$\bar{\nabla} \times E = -\mu \frac{\delta \bar{H}}{\delta t}$$

$$\bar{\nabla} \cdot \bar{H} = 0$$

$$\bar{\nabla} \cdot \bar{E} = 0$$

- DERIVATION OF WAVE EQUATION

$$\bar{\nabla} \times (\bar{\nabla} \times \bar{E}) = \bar{\nabla}(\overset{0}{\bar{\nabla} \cdot \bar{E}}) - \nabla^2 E \Leftarrow \text{VECTOR IDENTITY}$$

$$= -\nabla^2 E$$

$$\bar{\nabla} \times (\bar{\nabla} \times \bar{E}) = -\mu \nabla \times \frac{\delta \bar{H}}{\delta t} = -\mu \frac{\delta}{\delta t} \nabla \times \bar{H} = \mu \frac{\delta^2}{\delta t^2} \epsilon \bar{E}$$

$$\therefore \nabla^2 E = -\mu\epsilon \frac{\delta^2 \bar{E}}{\delta t^2} \quad (\text{ALSO } \nabla^2 \bar{H} = \mu\epsilon \, \delta^2 \bar{H}/\delta t^2)$$

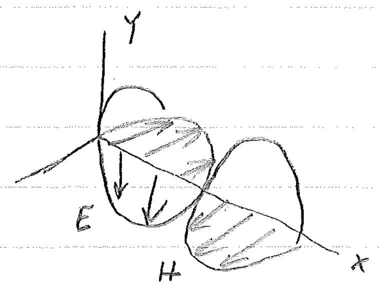
FOR $E_Y = E_Z = 0$, $E_x = f(x - ct) \Rightarrow c = \frac{1}{\sqrt{\mu\epsilon}}$

- PLANE WAVES

$$E_x = E_z = 0 \Rightarrow E_Y = E_M \cos(\omega t - Bx)$$

$$B_z = \frac{B}{\omega} \cos(\omega t - Bx)$$

$$\uparrow \ \eta^2 = \frac{\mu}{\epsilon} = \text{INTRINSIC } Z$$



- PROPAGATION IN A CONDUCTING MEDIA

$$\bar{\nabla} \times \bar{E} = -\mu \frac{\delta \bar{H}}{\delta t} \; ; \quad \bar{\nabla} \times \bar{H} = \gamma \bar{E} + \epsilon \frac{\delta \bar{E}}{\delta t}$$

$$\vec{E} = E_0 e^{j\omega t} \qquad \vec{H} = H_0 e^{j\omega t}$$

$$\Rightarrow \nabla^2 E_0 = \Gamma^2 E_0 \; ; \; \Gamma^2 = j\omega\mu(\gamma + j\omega\epsilon) = (\alpha + j\beta)^2$$

$$E_Y = E_m e^{-\alpha x} e^{j(\omega t - \beta x)} \qquad H_Z = \frac{\Gamma}{j\omega\mu} E_Y$$

$$\frac{E_Y}{H_Z} = \text{INT IMP.}$$

- POYNTING VECTOR

$$\vec{P} = \vec{E} \times \vec{H}$$

- BOUNDRY CONDITIONS

NORMAL $\bar{D}$ FIELD CONTIN.

TAN $\bar{E}$ FIELD SAME

NORMAL $\bar{B}$ FIELD SAME

TAN $\bar{H}$ FIELD SAME

- SKIN EFFECT

$$\Gamma^2 = j\omega\mu(\gamma + j\omega\epsilon) \approx j\omega\mu\gamma$$

$$\left|\frac{I}{e}\right| = i_m e^{-x/\delta} \quad \Rightarrow \quad \delta = 1/\sqrt{\pi f \mu \gamma} = \text{SKIN DEPTH}$$

- DIELECTRIC DIFFRACTION

$$\rho = \frac{\eta_3 - \eta_1}{\eta_3 + \eta_1} = \text{REFLEC. COEFF.}$$

- SWR $= \frac{MAX}{AVE}$

- SNELL'S LAW: $\frac{\cos \psi_1}{\cos \psi_2} = \frac{\lambda_1}{\lambda_3} = \frac{V_1}{V_3}$

ADVANCED FIELDS

- GREEN'S FUNCTIONS: $\nabla^2 \psi(\vec{x},t) - \frac{\delta^2}{\delta t^2}\psi = -4\pi f(\vec{x},t)$

$\Box_x = \nabla_x^2 - \frac{\delta^2}{\delta t^2} \Rightarrow \Box_x G = -4\pi\ \delta^3(\vec{x}-x')\,\delta(t-t')$

G = GREEN'S FUNCTION

- $\vec{E} = \frac{1}{4\pi\epsilon}\sum^n \frac{q_i}{r_i^3}\vec{r}_i$ ⇐ E FIELD INTENSITY

- DIPOLE MOMENT: $\vec{m} = qh$ $\quad \overset{-q}{\circ}\xrightarrow{h}\overset{q}{\circ}\ m$

$Q = \oint_s \epsilon \vec{E}\cdot\vec{n}\,ds$ ⇐ GAUSS' FLUX THEM

$C = \frac{Q}{V}$; $\frac{4\pi\epsilon ab}{b-a}$ ⇒ 2 CONC. SPHERES

$\frac{2\pi\epsilon}{\ln b/a}$ ⇒ 2 CONC. CYL.

$\frac{\epsilon A}{a}$ ⇒ 2 // PLATES

$W = \frac{1}{2}\sum q_i V_i$ ⇐ E IN CHARGED C

$\sum Q_i' V_i = \sum Q V_i'$ ⇐ GREEN'S REC. THEM

• GENERAL THEOREMS

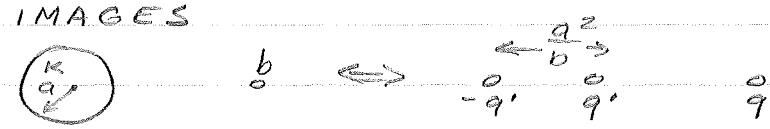
GAUSS: $\oint_s \vec{A}\cdot\vec{n}\,ds = \int_v \vec{\nabla}\cdot\vec{A}\,dv$

STOKE: $\oint_l F\cdot dl = \int_s \vec{n}\cdot\vec{\nabla}\times\vec{F}\,dA$

POISSON: $\vec{\nabla}\cdot\epsilon\vec{\nabla}V = -\rho$

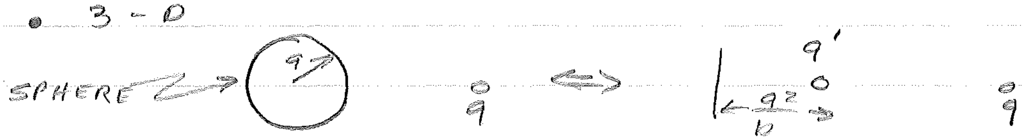
• TWO-D DISTRIBUTIONS: $E = \frac{q}{4\pi\epsilon r}$

2-D IMAGES



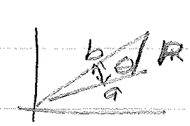
; $q' = \frac{1-K}{1+K}q$

• 3-D



$q' = \frac{-aq}{b}$

$rr' = K^2 \ni K^2$ = RADIUS OF INVERSION.



$\frac{1}{R} = \frac{1}{b}\sum_{n=0}^{\infty}\left(\frac{a}{b}\right)^n P_n(\cos\theta)$

- MUTUAL INDUCTANCE : $M_{12} = \int_{S_1} B_2 \cdot n \, dS$

$$= \oint A_2 \cdot ds_1$$

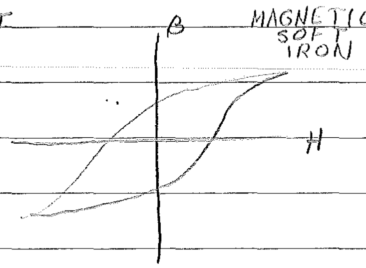
$$= \frac{\mu}{4\pi} \oint \oint \frac{d\bar{s}_1 \, d\bar{s}_2}{r}$$

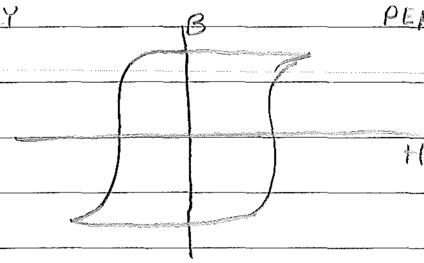
ENERGY CONVERSION

TUES

DUE MONDAY - 2-8, 2-10, 3-6, 3-16

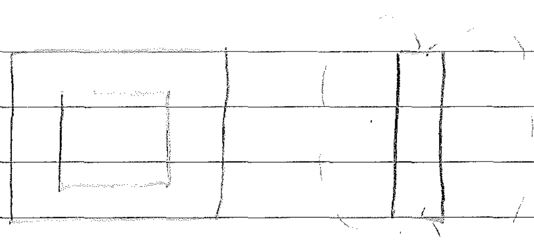
FRT





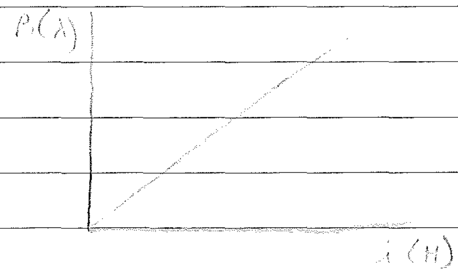
LARGER AREA, LARGER HYSTERSIS (LOSS OF ENERGY)

MON



DIFFERENT H B LOOPS

$L = \lambda / i$

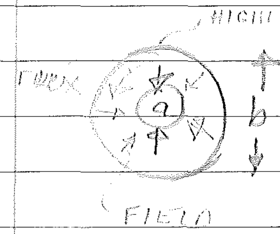


$B = \mu H$

$W_b \sim \phi = BA$

$\frac{AT}{M} \sim H$

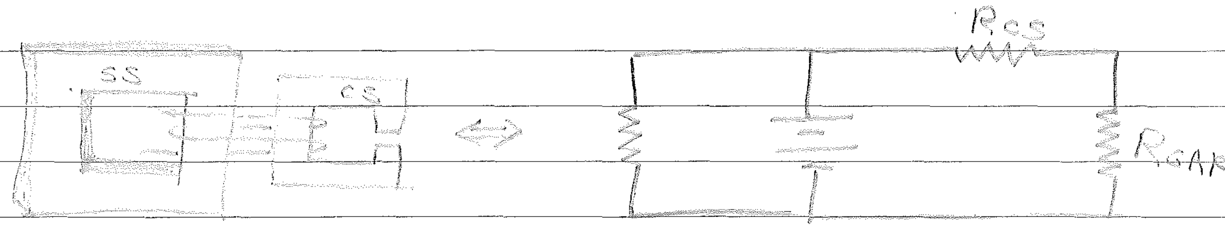
$\frac{W}{m^2} \sim B$ (TESLA $= \frac{W}{m^2}$)



$L = \frac{\mu_0}{2\pi} \ln \frac{b}{a}$

$C = 2\pi\epsilon_0 \ln \frac{a}{b}$

8-16)

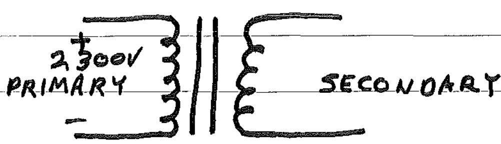


ASSIGNMENT IN CHAPT 8, TO 8-46

(SKIP AUTOTRANSFORMERS AND POLYPHASE CONNECTIONS)

WEDNESDAY

~~8-2)~~ 8-2)



2300/115

NO LOAD ⇒ OPEN SECONDARY (TEST) - TO MEASURE CORE LOSSES

I IN NO LOAD = EXCITING CURRENT

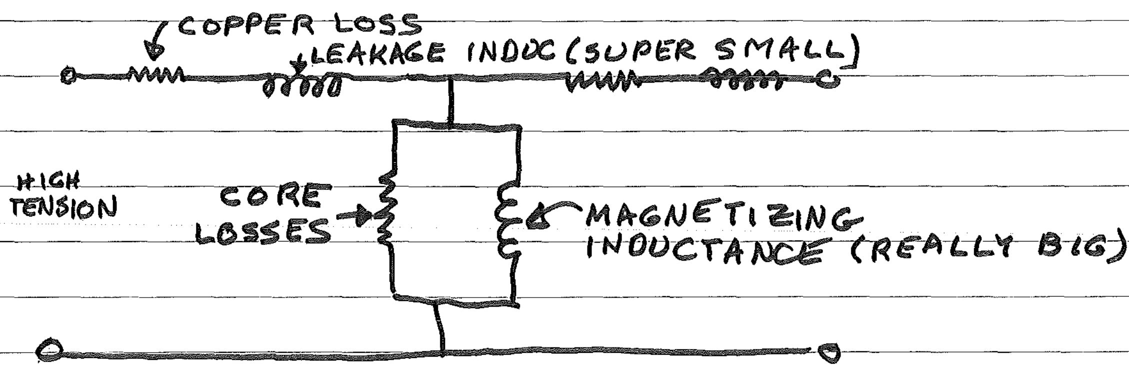
HEAT FROM COPPER LOSSES & HYSTERISIS

SHORT CIRCUIT (TEST)

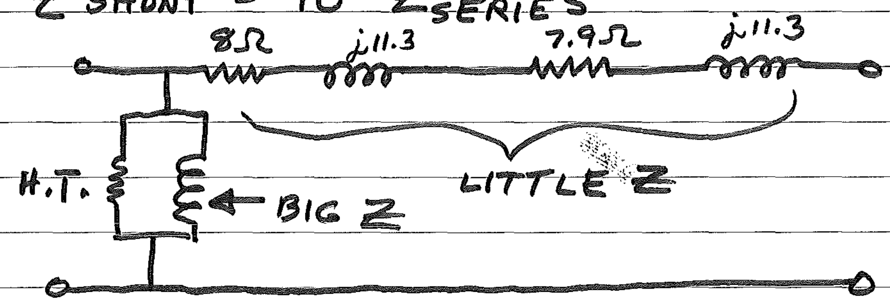
RATED CURRET = $\frac{V\text{-}A}{V}$ (CAN DELIVER PWR. ALL DAY)

- TO MEASURE COPPER LOSS -

TRANSFORMERS RATED IN V-A (NOT WATTS)



$Z_{SHUNT} \cong 10^3 Z_{SERIES}$



APPROXIMATE EQUIV. CIRCUIT

SEE: OPEN CIRCUIT GIVES CORE LOSSES

SHORT CIRCUIT GIVES COPPER LOSSES

8-2) a) $|Z| = \frac{60}{2.17} = 27.6$

$P.F. = \frac{75}{60 \times 2.17} = .576 \Rightarrow \varphi = \cos^{-1}(.576) = 54.8^\circ$

$\Rightarrow Z = 27.6 \angle 54.8^\circ$

$R_{eq} = 27.6 \times 0.576 = 15.9\,\Omega$

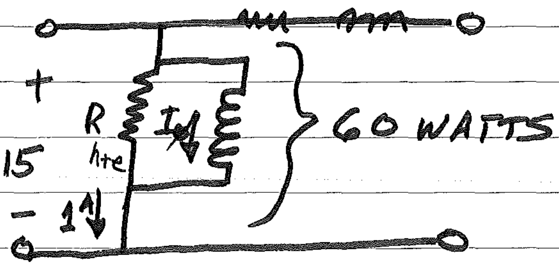
$X_{eq} = 27.6 \times 0.818 = 22.6\,\Omega$

b) $\cancel{R_{eqp} = 2.17A}$;

$R_{eqp} = 8\,\Omega$; $R_{eqs} = 7.9\,\Omega$ (TO BIG)

$R_L = 7.9/400 = .0198\,\Omega$

COMPUTING CORE PARAMETERS



$R_{h+e} = \frac{V^2}{P_{h+e}} = \frac{(115)^2}{60} = 220\,\Omega$

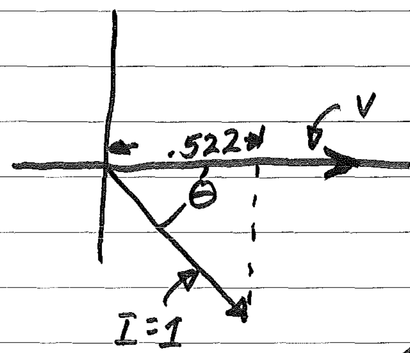
FROM OTHER SIDE

$R_{h+e} = (220)(20)^2$

$P.F. = 60/(115)(1) = .522$

$I_{h+e} = (1)(0.522)$

$\left(\frac{L_{1REF}}{L_2} = \frac{N_1^2}{N_2^2}\cdot\frac{N_2^2}{N_1^2} \Rightarrow L_{1REF} = L_2\right)$



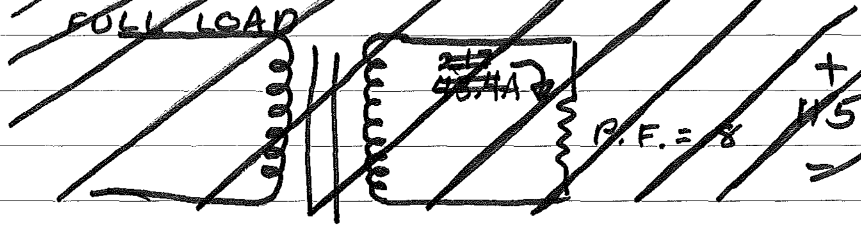
I_ϕ = MAGNETIZING

$= (1)(.885) = 0.885\ A$

$X_\phi = 115/0.885 = 135\,\Omega$

~~COMPUTE VOLTAGE REGULATION~~

~~FULL LOAD~~



THURS

ASSIGNMENT (FOR GIGGLES)

4-1) ANS. $E = 8000\,J$ COE = 200

4-2) $E = 38.4$ COE = 57.6

4-3) $E = COE = 250\,J$

4-10) $e_a = 0.0123\,M$

ASSIGNMENT (FOR REAL)

(4-11) 13.6 A

(4-14)(a) $W = \lambda_f^3 + \lambda_f^2/2\,(5 - .01x)^2$ JOULES

~~(4-15)~~(b) $W' = 2\lambda_f^3 + \lambda_f^2/2\,(5 - .01x)^2$ JOULES

~~(4-21)~~(c) $f = -.01\,\lambda_f^2\,(5 - 0.01x)$ NEWTONS

4-17)(a) $W = 10x^2 + 2942.5\,x + 311.25\,j$

(b) $20x + 2942.5$ NEWTONS

(c) 147.15 NEWTON METERS

(d) 289.5 JOULES; 5.5 JOULES

4-21)(a) 9000 JOULES

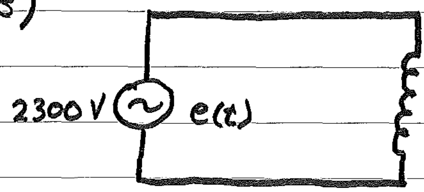
(b) 1.125×10^6 NEWTONS

(c) 4.5×10^6 NEWTONS

(d) 9000 JOULES

(23, 24, 25, 26, 27 DUE IN WED WEEK)

8-8)



$$e(t) = \sqrt{2}\,2300 \sin 2\pi \cdot 60 t = 1150 \frac{d\phi}{dt}$$

$$\Rightarrow \phi(t) = \frac{-\sqrt{2}\,2300}{2\pi \cdot 60 \cdot 1150} \cos 2\pi \cdot 60 t$$

$$\therefore \phi_{MAX} = \frac{\sqrt{2} \cdot 2300}{2\pi \cdot 60 \cdot 1150} = 0.0075 \text{ W}$$

$$.0075 \text{ W} \times 10^8 \frac{\ell}{W} = 7.5 \times 10^5 \text{ lines}$$

$$A_{(\text{AREA})} = \phi / B = \frac{7.5 \times 10^5}{0.8 \times 10^5} = 9.375 \,''^2 \times \frac{(6.45 \times 10^{-4} \text{ m}^2)}{\text{IN}^2}$$

$$= .00605 \text{ m}^2$$

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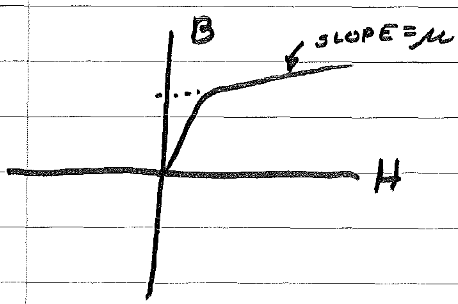


MON

$$f = -\frac{\delta W_m}{\delta x} = \frac{\delta W_m'}{\delta x}$$

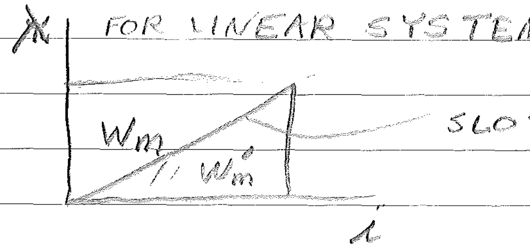
$$W_m' = \frac{1}{2} L i^2$$

$$W_m = \frac{\lambda^2}{2L}$$

TUES

λ - FLUX LINKAGES

FOR LINEAR SYSTEM

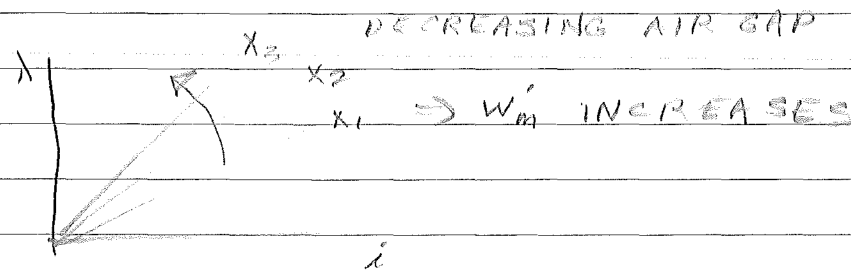


$W_m = W_m'$ FOR LINEAR SYSTEM

$W_m'(i)$ $\qquad W_m(\lambda)$

$$f = \frac{-\delta W_m}{\delta x} = \frac{\delta W_m'}{\delta x}$$

$W_m(\lambda, x)$ $\quad$ $W_m'(i, x)$ $\quad\quad\quad$ $(\lambda = L i)$



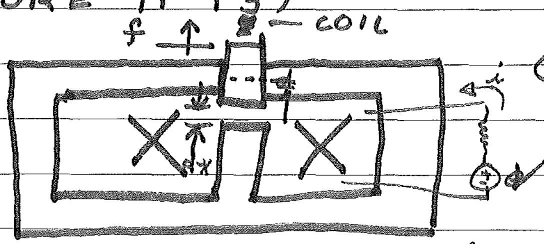
$e = \frac{d\lambda}{dt}$ (FARADAY'S LAW)

THURS

11.1 READ 11-9, 11-10, 11-11, 11-14

$f = \delta W_m' / \delta x = \frac{i^2}{2} \frac{\delta L}{\delta x}$ ⇒ FORCE OF ELECTROMECHANICAL ENERGY

FIGURE 11-13)



① $L = \frac{N^2 \mu_0 A}{(d-x)}$

② $V(t) = iR + \frac{d(Li)}{dt}$

$= iR + L\frac{di}{dt} + i\frac{dL}{dx}\underbrace{\frac{dx}{dt}}_{\text{SPEED VELOCITY}}$

③ $f_a(t) = M\frac{d^2x}{dt^2} + \overset{(\mu_k)}{D}\frac{dx}{dt} + \overset{\text{SPRING CONSTANT}}{K}X = \frac{i^2}{2}\frac{dL}{dx}$

($\frac{dL}{dx}$ COMMON TO 2, 3. WITHOUT $\frac{dL}{dx}$, NO EQUATION COUPLING

WANT TO FIND x AND i

(HIGHLY NON-LINEAR DIFFERENTIAL)

FOR LINEARIZATION; ASSUME A D.C. COMPONENT "0"

AND "SMALL" A.C. COMPONENT SIGNAL "1"

(CONT)

$f_a(t) = f_0 + f_1(t)$
$V(t) = V_0 + V_1(t)$
$i(t) = i_0 + i_1(t)$
$X(t) = X_0 + x_1(t)$

NEGLECT PRODUCTS OF SMALL SIGNALS WITH EACH OTHER

$\frac{d\,i(t)}{dt} = \frac{d\,i_1(t)}{dt}$, ETC.
$\frac{d^2 x(t)}{dt^2} = \frac{d^2 x_1(t)}{dt^2}$, ECT.

NOW $x_1 < d - x_0$

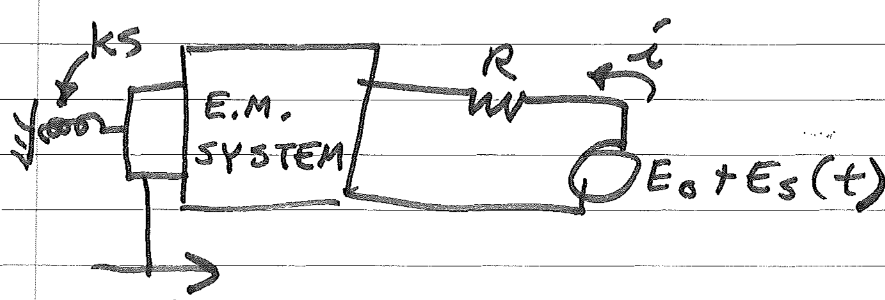
. . . .

$V_0 + V_1 = i_0 R + i_1 R_1 + L_0 \frac{di_1}{dt} + \frac{i_0 L_0}{d - X_0} \cdot \frac{dx_1}{dt}$
$f_1 = M \frac{d^2 x_1}{dt^2} + D \frac{dx_1}{dt} + K x_1 + K X_0 - \frac{1}{2} i_0^2 \frac{L_0}{d-X}$
$- i_0^2 \frac{L_0 x_1}{(d-X)} - \frac{i_0 L_0}{d - X_0} i_1$

SEPARATING INTO A.C. AND D.C.

$V_0 = i_0 R$
$0 = k X_0 - \frac{1}{2} i_0 \frac{L_0}{d - x_0}$ ∋ L_0 EVALUATED @ X_0
$\Longrightarrow L_0 = \frac{N^2 \mu_0 A}{d - x_0}$

11-1 DUE MON



WRITE D.C. & A.C. EQUATIONS SOLVE FOR QUIESCENCE

$f_a(t) = f_0 + f_1 t$

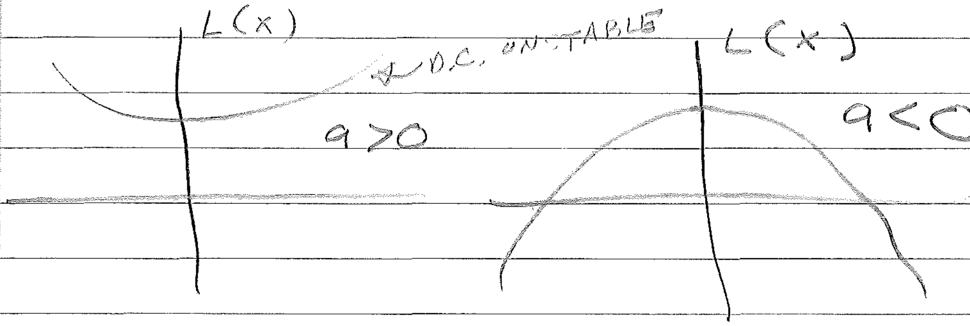
$L_0 = L(1 + a x^2)$
$a > 0;\ a < 0 \quad |a| \ll 1$

11-13, 12-7

A:

11-6 SHOULD BE 11-13
11-5 " " 11-11
11-7 " " 11-15

11-6 DUE THURS

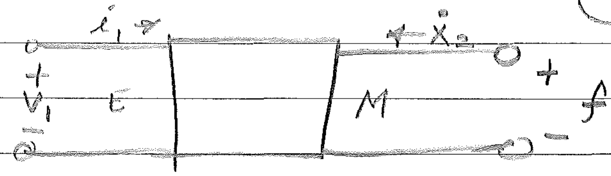


SYSTEM ~~GOE~~ ALWAYS GOES TO STATE OF MAXIMUM INDUCTANCE

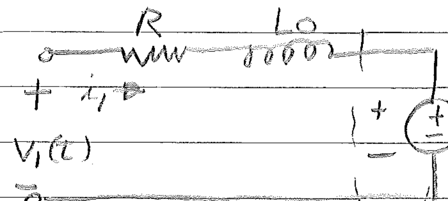
ALL ELECTRICAL CIRCUIT IN E-M SYSTEM

(1) $v_1(t) = i_1 R + L \frac{di_1}{dt} + L_o \frac{i_o}{d - x_o} \cdot \frac{dx_1}{dt}$

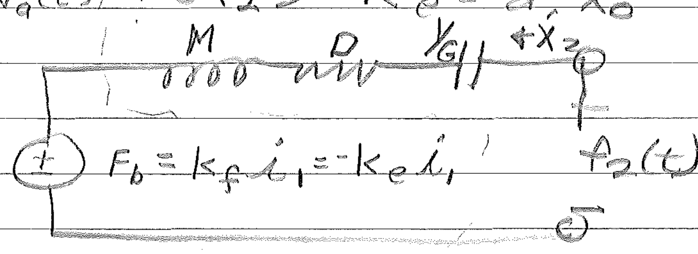
$f_1(t) = M \frac{d^2 x_1}{dt^2} + D \frac{dx_1}{dt} + \underbrace{\left(K - \frac{i_o L_o}{(d - x_o)^2}\right)}_{G} x_1 - \underbrace{\underbrace{\frac{i_o L_o}{d - x_o}}_{K_f} i_1}_{K_e}$



(MECHANICAL PARTS SUBSCRIPT OF 2)

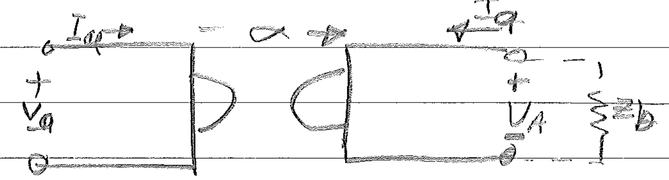


(1) $V_a(t) = K_e \dot{x}_2 \quad K_e = \frac{L_o i_o}{d - x_o}$



$F_b = K_f i_1 = -K_e i_1$

GYRATOR

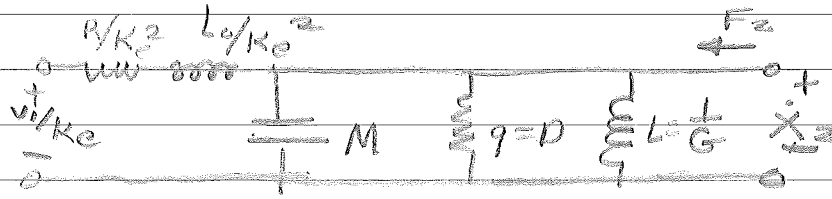


$V_a = \alpha I_b$

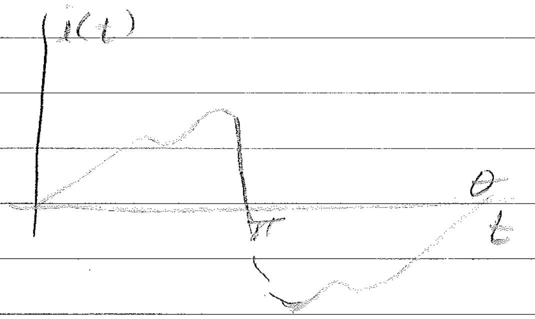
$V_b = -\alpha I_a$

$$\left(\begin{array}{l} V_{11} = \cancel{I_1 Z_{11}}^{\,0} + I_2 Z_{12} \\ V_{12} = I_1 Z_{21} + \cancel{I_2 Z_{22}}^{\,0} \end{array}\right)$$

$$\frac{V_A}{I_a} = \frac{-\alpha^2}{V_b/I_b} = \frac{-\alpha^2}{Z_B} = Z_A$$



THURS



$f(\theta) = -f(\pi + \theta)$

NO EVEN HARMONICS

TO COMPUTE CO-EFFICIENTS

$$\int_0^{2\pi} f(\theta)\begin{pmatrix}\sin n\theta \\ \cos \theta\end{pmatrix} d\theta$$

CONSIDER EVEN HARMONICS

$$b_{2n} = \int_0^{2\pi} f(\theta) \cos 2n\theta \, d\theta$$

$$= \int_0^{\pi} f(\theta) \cos 2n\theta \, d\theta + \int_{\pi}^{2\pi} f(\theta) \cos 2n\theta \, d\theta$$

LET $\alpha = \theta - \pi \Rightarrow d\alpha = d\theta$

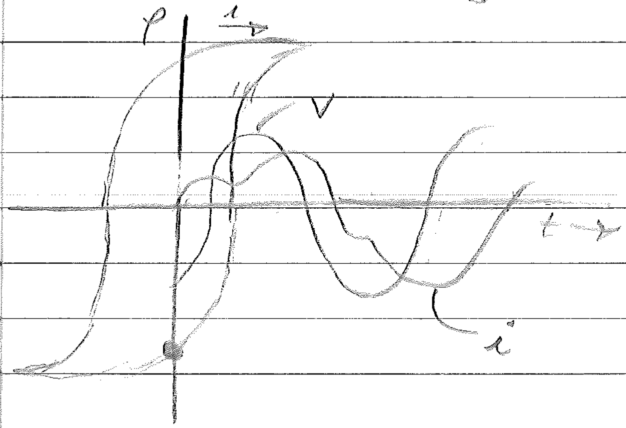
$$\Rightarrow b_{2n} = \int_0^{\pi} f(\theta) \cos 2n\theta \, d\theta + \int_0^{\pi} f(\alpha + \pi) \cos 2n(\alpha + \pi) \, d\alpha$$

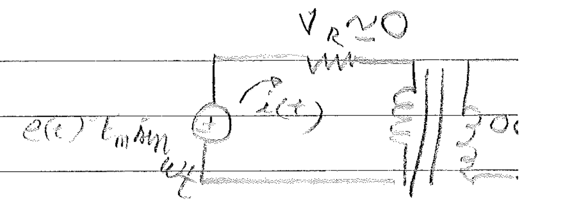
$$= \quad " \quad \int_0^{\pi} f(\alpha + \pi) \cos 2n\alpha \, d\alpha$$

$$= \quad " \quad \int_0^{\pi} -f(\alpha) \cos 2\pi\alpha \, d\alpha$$

$\Rightarrow$ NO CO-EFFICIENTS FOR ODD HARMONICS

FIG 8-25 ON Pg 8-56





$\varphi(t) = \frac{1}{N}\int e(t)\,d$

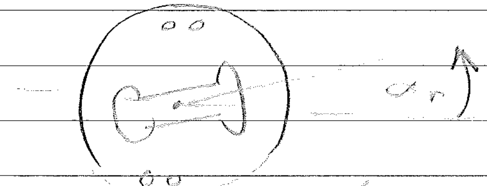
$= -\frac{E_m}{WN} \cos$

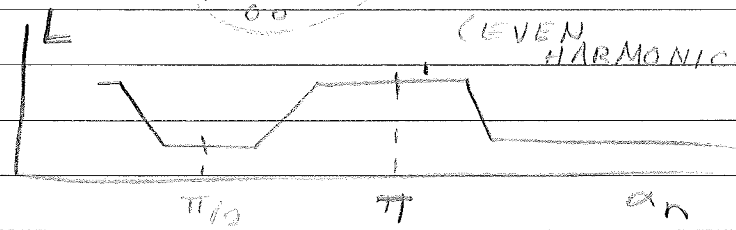
MON

WED

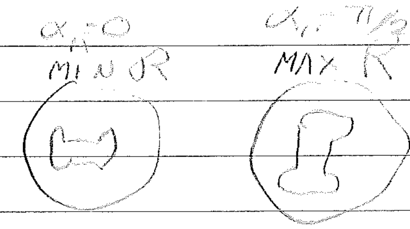
SALIENT POLES

SALIENCY ON ROTOR





(EVEN HARMONICS)



$L_{11} = L_K + L_V \cos 2\alpha_r$ (1ST 2 FOURIER TERMS)

$L_{22} = L_K + L_V \cos 2(\alpha_r - 90°)$

$= L_K - L_V \cos 2\alpha_r$

$L_{12} = L_V \cos 2(\alpha_r - 45)$

$= L_V \sin 2\alpha_r$

(CONTABAMION)

(CHAPT 12, 12-9
Eq 12-20
L SHOULD BE 2
IN 3RD TERM
IN 4TH LINE BELOW
cos ωt FOLLOWS
sin ωt)

$L_{aa} = L_K + L_V \cos 2P\alpha_m$

$L_{bb} = L_K + L_V \cos(P\alpha_m - 120°)$

$L_{cc} = L_K + L_V \cos(P\alpha_m - 240°)$

$L_{ab} = -[M_K - L_V \cos(2P\alpha_m - 120°)]$

$L_{bc} = -[M_K - L_V \cos 2P\alpha_m]$

ETC

MON
(DITTO)
(ON CHAPT. 5)

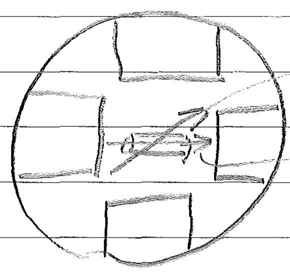
TUES

HOMEWORK: PROB 16.4 - 16.5 (DUE THURS) Pg 16.33, 34
Y CONNECTED

Eg 16-1

2 POLE PAIRS, Y CONNECTED

SALIENT POLE MACHINE



QUADRATURE AXIS (SMALL L) L_q

BIG L_d (DIRECT AXIS L)

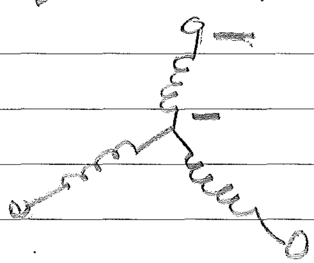
(a) $L_d = L_K + M_K + \frac{3}{2} L_V = 0.0425\,h$

$L_q = L_K + M_K - \frac{3}{2} L_V = 0.0275\,h$

(b) $E_g = E_{NO\ LOAD} = P\omega_m M_{af_m} I_f$

$= (377)(0.2)(200) = 15{,}080\ V$

(c) TERMINAL VOLTAGE PER PHASE



(Eq 5.48) $= E_g - I_q R_{aa} - I_d X_d + j(R_{aa} I_d - X_q I_q)$

I_q = QUAD. AXIS i = $I_m \cos\theta$

I_d = DIRECT AXIS i = $I_m \sin\theta$

(CONT)

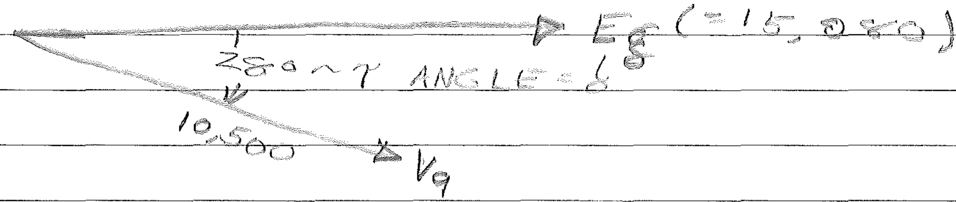
X_d = DIRECT AXIS REACTANCE $= \omega L_d$, $X_q = \omega L_q$

$V_a = 15{,}080 - (600 \cos 37^\circ)(0.1) - (600 \sin 37^\circ)\underbrace{(16)}_{X_d}$

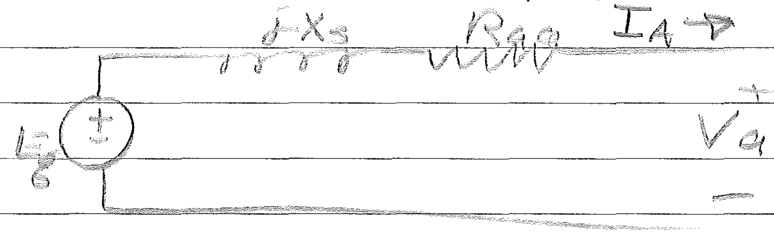
$\quad + j\left[(600 \sin 37^\circ)(0.1) - (10.4)(600 \cos 37^\circ)\right]$

$\quad = 10{,}500\, e^{-j28^\circ}$

LINE $V = \sqrt{3}\, V_a$



EQ CIRCUIT FOR CONSTANT GAP MACHINE



d) % REGULATION $= \dfrac{|E_g| - |V_a|}{V_a} \times 100\%$

$= \dfrac{4580}{10{,}500} \times 100\% = 43.5\%$

Pg 203

e) $P_{MECH\,(IN)} = 3\left[\left(\dfrac{E_g V_A}{X_d} \sin\delta\right) + \underbrace{(X_d - X_q) V_a^2 \sin 2\delta / 2X_d X_q}_{\text{RELUCTANCE TORQUE}}\right]$

CONSTANT GAP

SECOND TERM = 0 FOR CONSTANT GAP MACHINE

$\Rightarrow P = 9.3$ MW

f) $P_{CONVERTED\ ELEC} = P_{MEC} = 9.3$ MW

g) $P_{\cancel{OUTPUT(E)}} = I^2 R_{aa}/2$

$\cancel{P_{TOTAL} = 3P_{OP(E)} = \frac{3}{2}}$

$P_{Ee} = (9300)\,KW - (\frac{3}{2})(600)^2(.1)\,R = 3(\frac{I_a^2}{2} R_{aa})$

$\eta = 9240/9300$ $= 9246$

$VA = \frac{3}{2}(V_a I_a) = 9450 \Rightarrow PF = 0.98$

j) $\tau = P_{MECH}/\omega_m$ $\quad P = 2$

$\omega_m = \frac{377}{2}$

$\Rightarrow \tau = 9.3\times10^6/188 = 50{,}000$ nt-m

R TORE =

THURS

LET $P = 4$ (POLE PAIRS) (CHAPT. 5)

MON

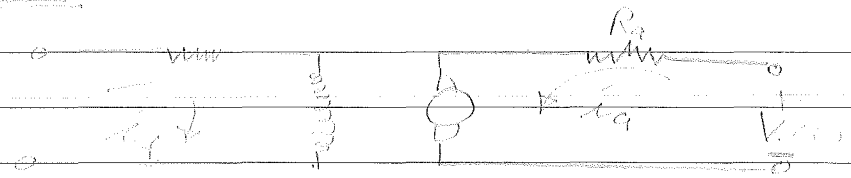
D.C. MACHINES

DRUM WINDING

232-8001

TUE'S

WED



$\alpha = \int$

MON)

A SHUNT MOTOR WITH THE FOLLOWING CONSTANTS IS CONNECTED TO AN A.C. VOLTAGE: 340 sin 377t. FIND THE AVERAGE TORQUE. REPEAT, ASSUMING D.C. EXITATION @ 240 V (RMS OF 340 V)

$R_f = 50, L_{ff} = 25, K_f = 0.9, R_a = 0.1, L_{aa} = 0.01, J_m = 10 kg\text{-}m^2$

"R_a

(NOT TRANTIENT)

$T = K_f \, i_a(t) \, i_f(t)$

WED:

15-2; 15-3 (ON FINAL · IN BOOK)
NOT DUE BEFORE 28TH

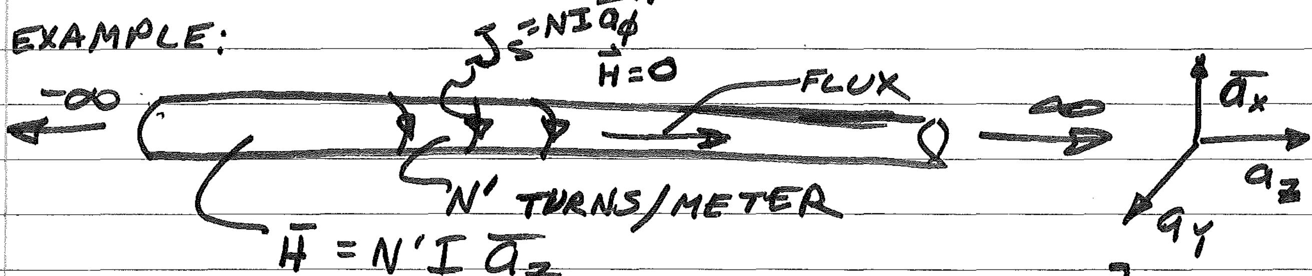
MAXWELL STRESS TENSOR

$$\bar{\bar{\sigma}}_M = \begin{bmatrix} \dfrac{B_x^2 - B_y^2 - B_z^2}{2\mu_0} & \dfrac{B_x B_y}{\mu_0} & \dfrac{B_x B_z}{\mu_0} \\[2ex] \dfrac{B_x B_y}{\mu_0} & \dfrac{B_y^2 - B_z^2 - B_y^2}{2\mu_0} & \dfrac{B_y B_z}{\mu_0} \\[2ex] \dfrac{B_x B_z}{\mu_0} & \dfrac{B_y B_z}{\mu_0} & \dfrac{B_z^2 - B_y^2 - B_y^2}{2\mu_0} \end{bmatrix}$$

SIGNIFICANCE:

$\bar{\bar{\sigma}}_m \cdot \bar{a}_n = \bar{p} = \dfrac{\text{FORCE}}{\text{AREA}}$ ACTING ON SURFACE WHOSE NORMAL IS $\bar{a}_n$

EXAMPLE:



$$\bar{\bar{\sigma}}_{INSIDE} = \begin{bmatrix} -B_z^2/2\mu_0 & 0 & 0 \\ 0 & -\dfrac{B_z^2}{2\mu_0} & 0 \\ 0 & 0 & \dfrac{B_z^2}{2\mu_0} \end{bmatrix}$$

$$\bar{\bar{\sigma}}_{OUTSIDE} = \bar{0}$$

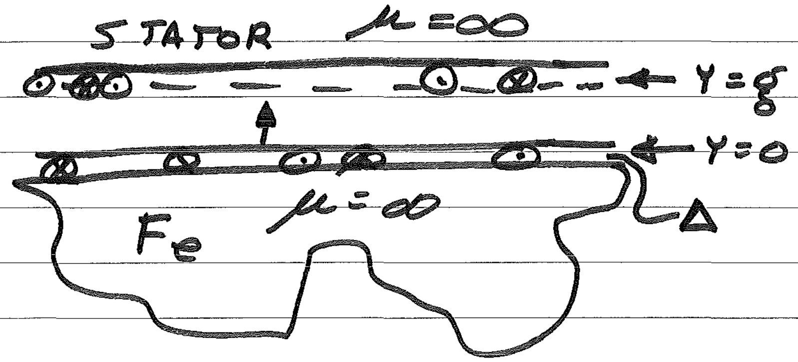
$$\bar{p} = \left(\bar{\bar{\sigma}}_{OUT} - \bar{\bar{\sigma}}_{IN}\right) \cdot \bar{a}_r$$ (CONT.)

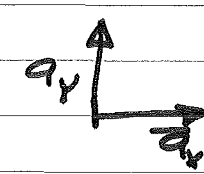
$$\Rightarrow \bar{p} = B_z^2/2\mu_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cancel{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \begin{array}{c} \bar{a}_r \\ \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix} \end{array}$$

$$= \frac{B_z^2}{2\mu_0} \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix} = \frac{B_z^2}{2\mu_0} \bar{a}_r$$

$$= \frac{\mu_0}{2} (N'I)^2 \bar{a}_R$$

LINEAR INDUCTION MACHINE





$$\bar{p} = \bar{\bar{\sigma}}_m \cdot \bar{a}_Y$$

$$= \begin{bmatrix} B_x^2 - B_Y^2/2\mu_0 & \frac{B_x B_Y}{\mu_0} & 0 \\ B_x B_Y/\mu_0 & B_Y^2 - B_x^2/2\mu_0 & 0 \\ 0 & 0 & -\frac{B_x^2 + B_Y^2}{2\mu_0} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} B_x B_Y/\mu_0 \\ B_Y^2 - B_x^2/2\mu_0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \bar{p} = \frac{B_x B_Y}{\mu_0} \bar{a}_x + \frac{B_Y^2 - B_x^2}{2\mu_0} \bar{a}_Y \qquad \text{FOR } Y = 0$$

FOR $Y = g$, LET $Y = -Y$ (CHANGE SIGN ON $\bar{a}_n$)

D.C. ρ

$$\langle \bar{p} \rangle = \bar{a}_x \, Re\left[\frac{B_x B_y^*}{\mu_o}\right] + \bar{a}_y \, Re\left[\frac{B_y B_y^* - B_x B_x^*}{2\mu_o}\right]$$

(PLUG AND CHUG AND VWALA!)

$$\Rightarrow \langle \bar{p} \rangle = \frac{\mu_o (N' I_m)^2}{4}\left[\frac{2 s R_m' \bar{a}_x + [1-(sR_m')^2]\bar{a}_y}{\sinh^2 kg + (s R_m' \cosh kg)^2}\right]$$

$$s = \frac{V_{SYN} - V_{IN}}{V_{SYN}} \quad \text{(SLIP)}$$

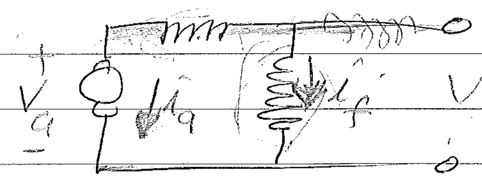
INTEGRATING OVER 1 λ FOR X, & Z TO FIND F

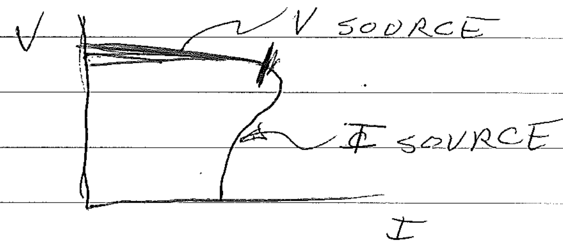
$\Rightarrow$ MULTIPLY $\langle \bar{p} \rangle$ BY λl

MONDAY

(15-2, 15-3)

LAST LAB

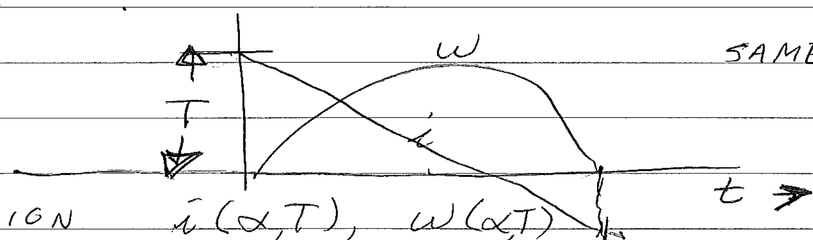




DIFFERENTIAL $\Rightarrow$ OPPOSING FLUXES.

TAKE HOME

(PROB. I)

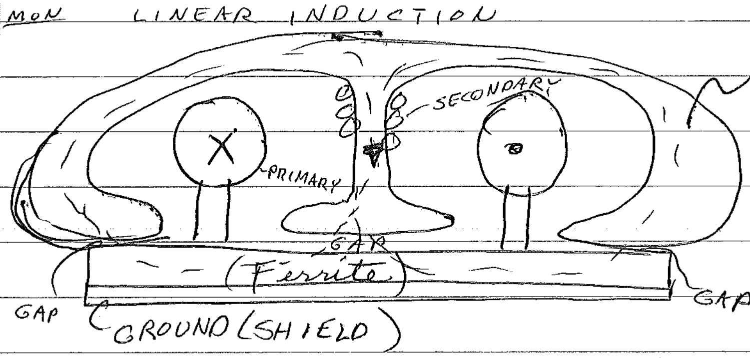


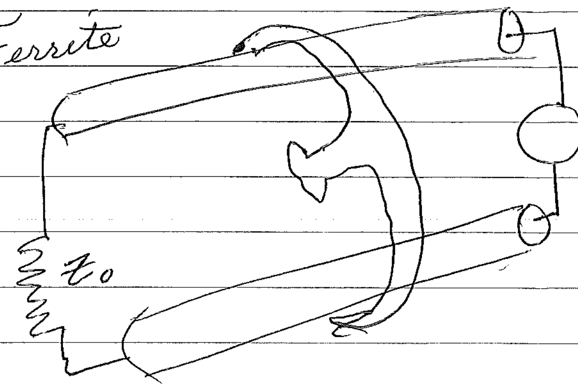
SAME as Before

SOLUTION $i(\alpha, T)$, $\omega(\alpha, T)$

USE Q TO SOLVE $\Rightarrow i(\alpha, Q), \omega(\alpha, Q)$

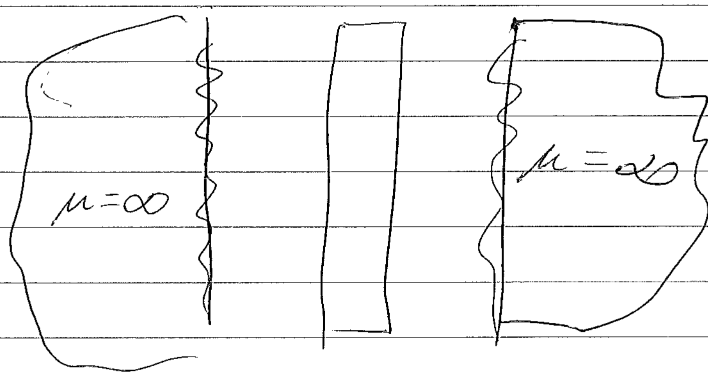
MON LINEAR INDUCTION

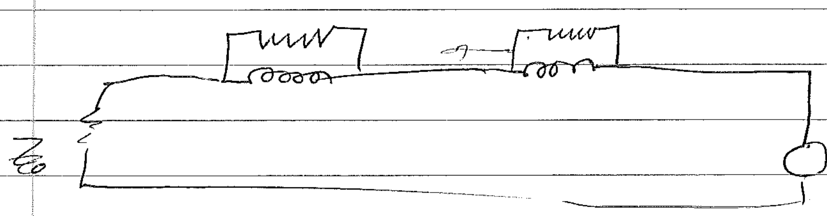




$\underline{18\,KHz} \Rightarrow \lambda = \frac{3\cdot10^8}{18{,}000} \simeq$ 10 MILES ~ 15 KM

(POP SCIENCE)





$$\left(\frac{F_x}{\frac{2}{\lambda_0 \mu l}\left(\frac{|V|}{N' V_{SYN}}\right)^2} \right) = \frac{S R_m'}{(\cosh k\bar{g})^2 + (S R_m' \sinh k\bar{g})^2}$$

← NORMALIZED

Pg 74 - 1-226

PER UNIT SYNC-SPEED = $\frac{V_m}{V_{SYNC}}$

$$S = \frac{V_{SYN} - V_M}{V_{SYN}} = 1 - \text{PER. UNIT. SYN. SPEED}$$

$R_m' = \mu_0 \sigma \Delta V_{SYN}$ (MANNETIC RENYOLDS #)

(NORMALIZED THICKNESS)

Δ ROTOR THICKNESS

$$\left(\frac{F_x}{\frac{2}{\lambda_o \mu l}\left(\frac{|V|}{N' V_{SYN}}\right)^2} \right) = \frac{S R_m'}{(\cosh kg)^2 + (S R_m' \sinh kg)^2}$$

NORMALIZED

Pg 74 - 1-226

PER UNIT SYNC-SPEED = $\frac{V_m}{V_{SYNC}}$

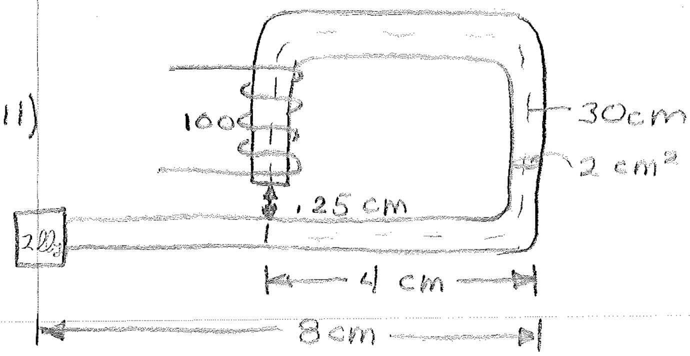
$S = \frac{V_{SYN} - V_M}{V_{SYN}}$ = 1 - PER. UNIT. SYN. SPEED

$R_m' = \mu_0 \sigma \Delta V_{SYN}$ (MANNETIC RENYOLDS #) (NORMALIZED THICKNESS)

ROTOR THICKNESS

BOB MARKS

26/40

4-11) 

$W = 2\ lbs \times \frac{4.45\ nt}{lb} = 8.90\ nt$

$F_g = W \frac{l_c}{l_B} = (8.90)\left(\frac{4}{8}\right) = 4.45\ nt$

$F_g = -\frac{\lambda^2}{2} \frac{1}{N^2 \mu_0 A}$

$= -\frac{(Li)^2}{2} \frac{1}{N^2 \mu_0 A}$

$L = \frac{N^2 \mu_0 A}{l_g}$

$\Rightarrow F_g = -\frac{N^4 \mu_0^2 A^2}{l_g^2} i^2 \frac{1}{N^2 \mu_0 A}$

$= -\frac{N^2 \mu_0 A i^2}{l_g^2} \Rightarrow i^2 = -\frac{F_g}{\mu_0 A}\left(\frac{l_g}{N}\right)^2$

$\therefore |i| = \sqrt{\frac{F_g}{\mu_0 A}} \frac{l_g}{N}$

$= \sqrt{\frac{4.45\ nt\ m}{4\pi \times 10^{-7} H \times 2 \times 10^{-4}\ m^2}} \frac{.25 \times 10^{-2}\ m}{100} = 3.3266\ A$

3

4-14) $i = 3\lambda^2 + \lambda(5-x)^2$

a) $W_m = \int_0^\lambda i(\lambda, x)\, d\lambda$

$= \int_0^\lambda \left(3\lambda^2 + \lambda(5-x)^2\right) d\lambda$

$= \lambda^3 + \frac{\lambda^2}{2}(5-x)^2$ ERG (CGS)

b) $W_m' = i_{MAX}\, \lambda'_{MAX} - W_m$

$= \left(3\lambda^2 + \lambda(5-x)^2\right)\lambda - \left(\lambda^3 + \frac{\lambda^2}{2}(5-x)^2\right)$

$= (3\lambda^3 - \lambda^3) + \left(\lambda^2(5-x)^2 - \frac{\lambda^2}{2}(5-x)^2\right)$

$= 2\lambda^3 + \frac{\lambda^2}{2}(5-x)^2$ ERG

c) $f = \frac{-\delta W_m(\lambda, x)}{\delta x}$

$= \frac{-\delta}{\delta x}\left(\lambda^3 + \frac{\lambda^2}{2}(5-x)^2\right)$

$= +2\frac{\lambda^2}{2}(5-x)$

$= \lambda^2(5-x)$ DYNE

4-17) $L_1 = .5 + 25X$; $L_2 = 1 + .2(1-X)^2$; $L_{12} = .3 + (1+X)$

$i_1 = 15A$; $i_2 = 10$ AMPS

a) $W_m = \frac{L_{11} i^2}{2} + L_{12} i_1 i_2 + \frac{L_{22} i_2^2}{2}$

$= \frac{(.5+25X)225}{2} + (1.3+X)150 + 50(1+.2(1-X)^2)$

$= 10X^2 + 2942.5X + 311.5$ J ✓

b) $f = \frac{\delta W_m}{\delta X}$

$= 20X + 2942.5$ nt ✓

c) $W = \int_{X_1}^{X_2} f(X)\,dX$

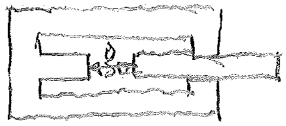
$= \int_0^{.05} (20X + 2942.5)\,dX$

$= (10X^2 + 2942.5X)\Big|_0^{.05}$

$= 147.15$ J ✓

d?

6

4·21) 

$g = 8\text{ cm}$

$L = 1.8\text{ h}$

$i = 100\text{A}$

a) $W_m = \frac{Li^2}{2} = \frac{1.8\times10^4}{2} = 9000 j$

b) $f = \frac{\lambda^2}{2N^2\mu_0 A}$, $\lambda = Li = \frac{\mu_0 N^2 A i}{x}$

$\Rightarrow f = \frac{\mu_0^2 N^4 A^2 i^2}{x^2}\frac{1}{2N^2\mu_0 A} = \frac{\mu_0 N^2 A i^2}{2x^2}$

$= \frac{Li}{2x}$

$= \frac{1.8\times10^4}{1.6\times10^{-2}}$

$= 1.125\times10^6\text{ nt}$

c) $L \propto \frac{1}{x} \Rightarrow LX = \text{CONSTANT}$

$L_1X_1 = L_2X_2$

$(1.8)(.8) = L_2(.4) \Rightarrow L_2 = 3.6\text{ H}$

$f_2 = \frac{L_2 i^2}{2x} = \frac{(3.6)10^4}{2(.4)\times10^{-2}} = 4.5\times10^6\text{ nt}$

d) $W_m = \int_{x_1}^{x_2} \frac{L_2 i^2}{2}\frac{1}{x}\,dx$

$= \int_{x_1}^{x_2} \cancel{L_2}\frac{i^2 \mu_0 N^2 A}{2}\frac{1}{x^2}\,dx$

$= \frac{i^2}{2}L \Big|_{x=.4\times10^{-2}\ (L=1.8)}^{x=.8\times10^{-2}\ (L=3.6)}$

$= \frac{10^4}{2}(1.8)$

$= 9000 j$

9

BOB MARKS

10

5 January 1972

EE 353, ENERGY CONVERSION

LAB PROJECT NO. 2

1. Sketch the circuit used and describe the purpose of each instrument.

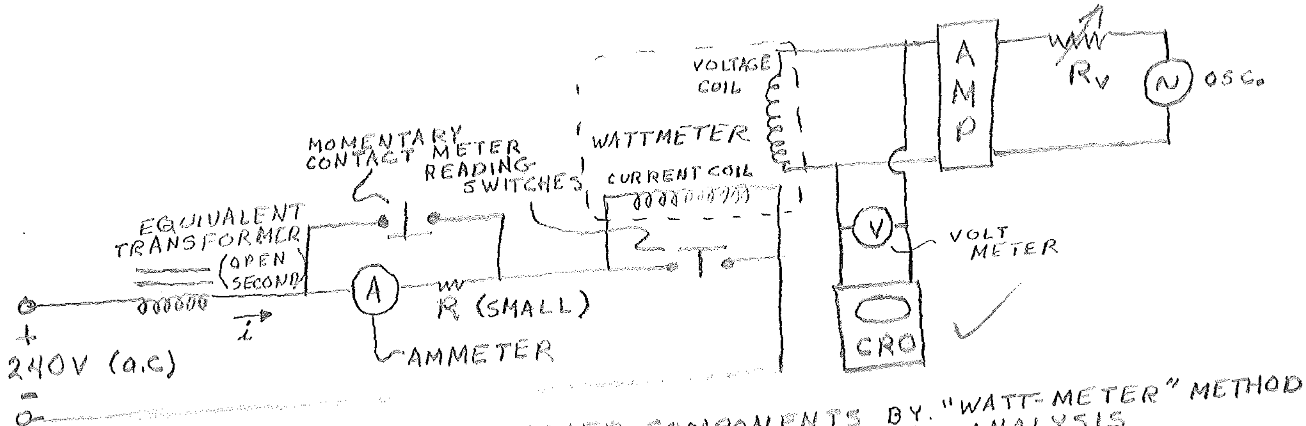
2. In the "watt-meter" method describe the procedures used.

3. Explain how the watt-meter is able to measure Fourier components. Be clear, not too verbose, and accurate. Use enough analysis to explain the theory.

4. Contrast the digital method, and its results, with the watt-meter method. Which is easier for use? Which is more accurate?

5. What are your results? Are they consistent with the analysis of Chapter 8?

1)



WATTMETER USED TO COMPUTE FOURIER COMPONENTS BY "WATT-METER" METHOD

a) CRO USED TO DISPLAY "CURRENT" WAVEFORM FOR DIGITAL ANALYSIS

b) AMP USED TO AMPLIFY VOLTAGE AMPLITUDE FROM OSC.

c) AMMETER USED TO KEEP i CONSTANT, THUS KEEPING RELATIVE FREQUENCY COMPONENTS OF i CONSTANT

d) OSCILLATOR USED TO VARY FREQUENCY OF VOLTAGE COMPONENT IN THE WATT-METER TO VALUES MATCHING THE HARMONICS OF THE CURRENT

e) VOLT METER USED TO DETERMINE THE VALUES OF THE "CURRENT" HARMONIC COMPONENTS

(NOTE: 120 VOLTS USED WHEN PHOTOGRAPH OF WAVEFORM WAS MADE FOR DIGITAL ANALYSIS)

2) THE PURPOSE OF THE WATT-METER METHOD IS TO DETERMINE THE FREQUENCY COMPONENTS OF THE CURRENT WAVEFORM, WHICH WAS DISTORTED FROM A SINUSOID BY THE NON-LINEAR CHARACTERISTICS OF THE TRANSFORMER, AS PICTORALLY DESCRIBED FROM A HYSTERISIS CURVE ON PAGE 8-56 OF THE TEXT. THE SINUSOIDAL VOLTAGE $v(t) = V_m \cos(\omega_1 t + \phi_1)$, AND THE PERIODIC CURRENT $i(t)$ EXPRESSED IN A FOURIER SERIES ($i(t) = \sum C_n \cos(n\omega_2 t + \theta_2)$) ARE FED INTO THE VOLTAGE AND CURRENT COILS OF THE WATTMETER. THESE EXPRESSIONS ALONG WITH THE DIFFERENTIAL EQUATION FOR TORQUE OF THE WATTMETER NEEDLE: $\tau(t) = A\,v(t)\,i(t) = J\ddot{\theta} + D\dot{\theta} + K_s\theta$, YIELD FOR A SINGLE COMPONENT, DISREGUARDING TRANSIENTS:

$$\theta(t) = \frac{AEI}{2D}\left[\frac{\cos(\omega_a t + \psi_1)}{\left[\left(\frac{K}{D}\right)^2 + \omega_a^2\right]^{\frac{1}{2}}} + \frac{\cos(\omega_b t + \psi_2)}{\left[\left(\frac{K}{D}\right)^2 + \omega_b^2\right]^{\frac{1}{2}}}\right]$$

$\omega_a = \omega_1 - \omega_2$

$\omega_b = \omega_1 + \omega_2$

ψ_1 AND ψ_2 ARE PHASE ANGLES, DETERMINED BY PARAMATERS

(THIS SOLUTION FROM "SKETCH OF THEORY WATTMETER" HANDOUT)

NOW $\theta(t)$ IS OF COURSE PROPORTIONAL TO WATTAGE. ANYWAY, IT CAN BE SEEN THAT ONLY WHEN $\omega_a = 0$ ($\Rightarrow \omega_1 = \omega_2$) WILL THE WATTAGE HAVE AN NON-ZERO AVERAGE POWER IN TIME. THUS, BY ADJUSTING THE VOLTAGE FREQUENCY TO THE VARIOUS HARMONICS OF THE CURRENT WAVEFORM, (TO WHERE THE "BEAT FREQUENCY" $\omega_a = 0$), THE CONTRIBUTION OF THAT FREQUENCY, ie THE FOURIER COEFFICIENT, MAY BE COMPUTED. THE WATTMETER, ACTING AS A LOW PASS FILTER DUE TO NEEDLE INERTIA AND FRICTION, WILL OF COURSE NOT RESPOND TO HIGH FREQUENCIES.

2) OSCILLATOR SET TO VARIOUS HARMONICS OF 60 HZ LINE VOLTAGE TO SETTING ON WHICH A READING OCCURED ON WATTMETER. VALUES OF FREQUENY, CURRENT, VOLTAGE AND POWER WERE THEN RECORDED FROM THE CORRESPONDING METERS DEPECTED IN FIGURE IN PART(1).

DATA:

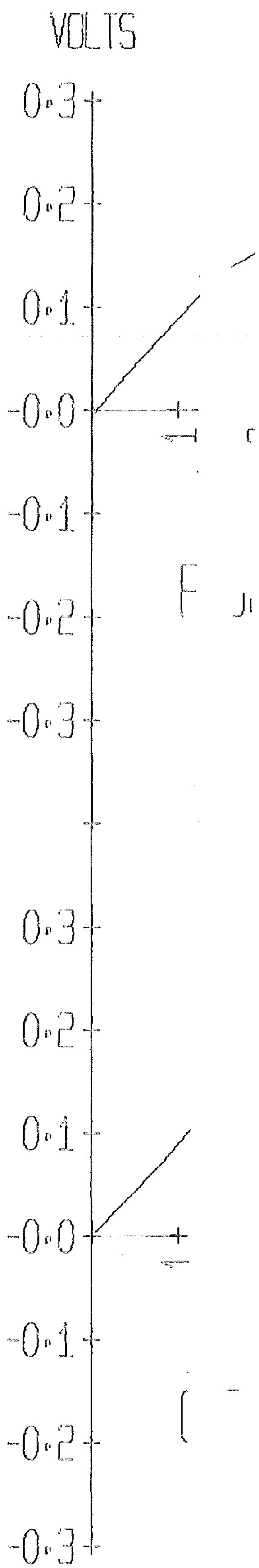
n	f OF $V_{(HZ)}$	V_{RMS} (V)	$P_{(WATTS)}$	I_{RMS} (AMPS)	$I_{n(PP)} = (P/V_{RMS})\sqrt{2}$
1	60	27.0	19.0	0.64	0.995
2	120	11.1	0.3	0.64	0.0382
3	180	9.0	4.5	0.64	0.707
4	240	8.0	0.0	0.64	0.0
5	300	7.0	1.0	0.64	0.202
6	360	6.8	0.0	0.64	0.0
7	420	6.0	0.2	0.63	0.0472
8	480	5.5	0.0	0.63	0.0
9	540	5.0	0.0	0.63	0.0
10	600	5.1	0.0	0.63	0.0

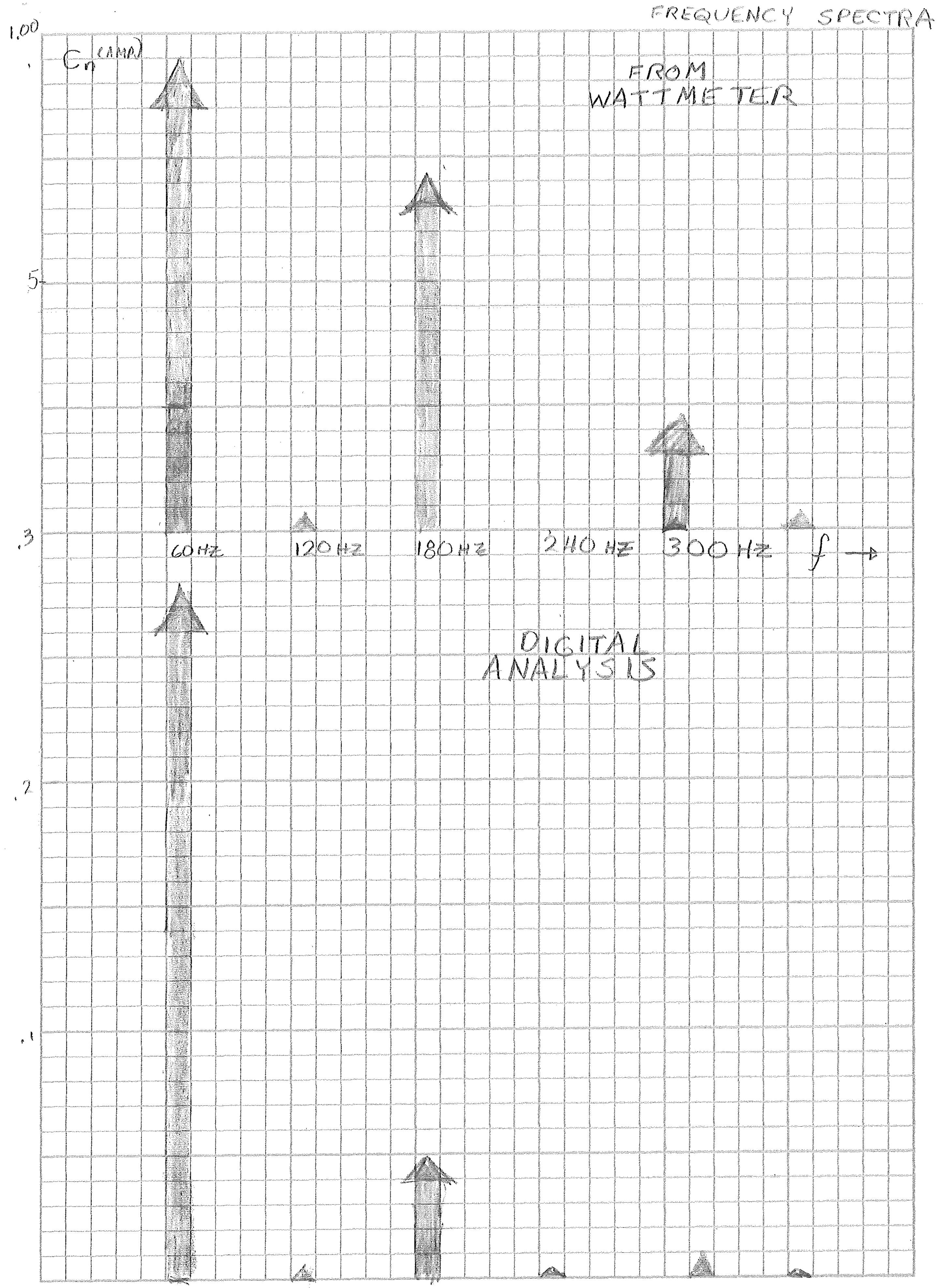
```
NF   = 43
NW   = 10

DC =   -0.000
FO =   59.880
```

TERM	CFA
1	-0.487336
2	0.439136
3	0.329058
4	-0.288442
5	0.750955
6	-0.230704
7	0.866322
8	0.520541
9	0.137384
10	0.180613

EXECUTION TIME 0283


VOLTS
0·3
0·2
0·1
-0·0
-0·1
-0·2
-0·3
0·3
0·2
0·1
-0·0
-0·1
-0·2
-0·3


FREQUENCY SPECTRA
1.00
Cn (AMP)
FROM WATTMETER
.5
.3
60 HZ
120 HZ
180 HZ
240 HZ
300 HZ
f →
DIGITAL ANALYSIS
.2
.1

4) THE WATTMETER METHOD, ALTHOUGH EASIER TO USE, DOES NOT YIELD AS GOOD RESULTS AS THE DIGITAL METHOD. THE RECONSTRUCTION OF THE CURVE, AS REPRESENTED BY THE COMPUTER PLOT, BY THE FOURIER COMPONENTS COMPUTED DIGITALLY, IS NEARLY IDENTICAL TO THAT OF THE ORIGINAL CURVE, IMPLYING GOOD REPRESENTATION OF THE DIGITAL COMPONENTS IN DESCRIBING THE HARMONICS OF THE CURRENT WAVEFORM. THE FREQUENCY SPECTRUM PLOT OF THE TWO METHODS SHOWS THAT THE WATTMETER RESULTS ARE NOT AS FAVORABLE.

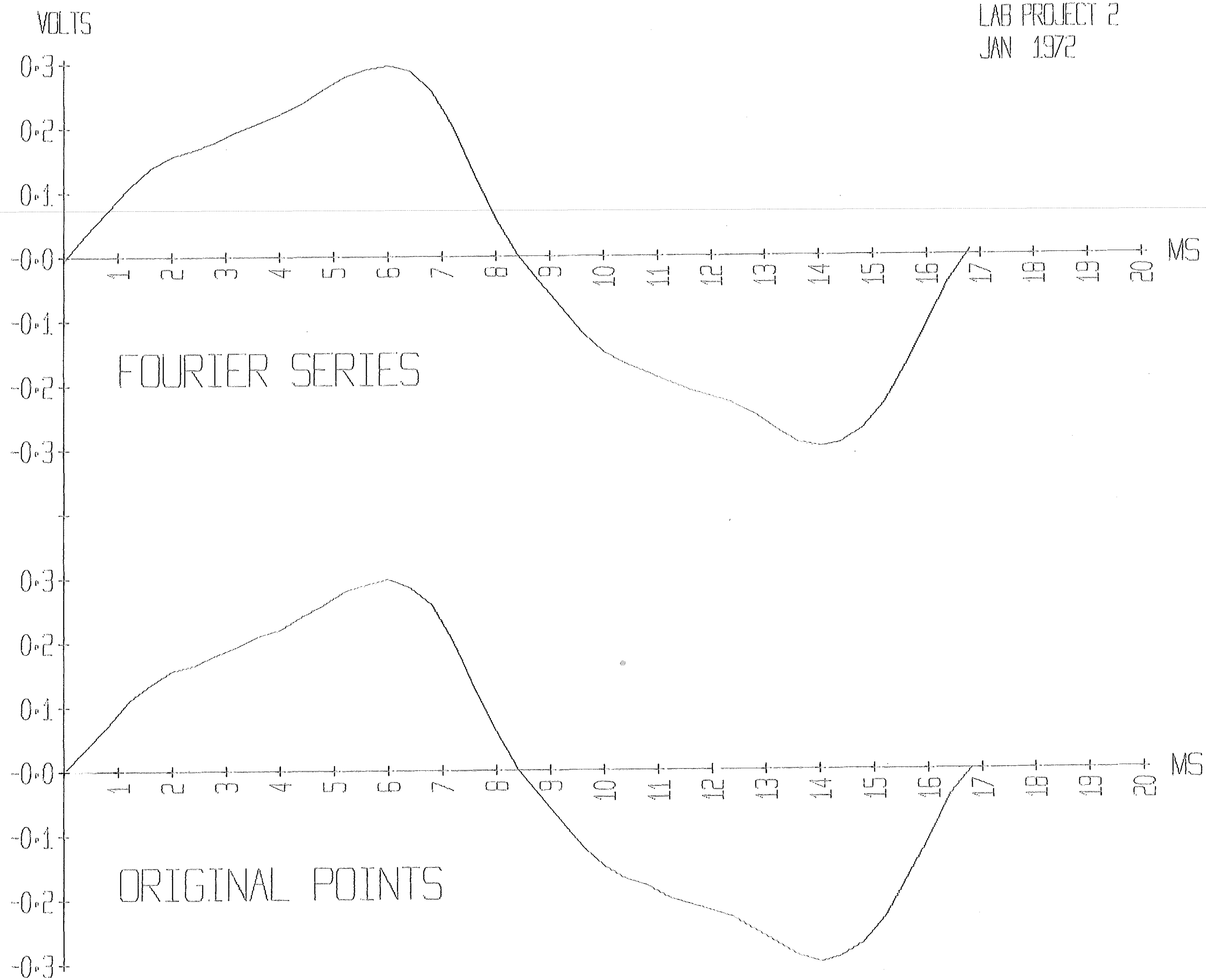
5) THE CURRENT WAVEFORM CONTAINS ONLY ODD HARMONICS, WHICH CAN BE SEEN FROM IT'S SYMMETRY: $i(t) = -i(t+\pi)$. THIS SAME WAVEFORM IS SHOWN, DERIVED FROM THE HYSTERSIS CURVE ON Pg 8-56 OF TEXT.

LAB PARTNERS:
RANDY MILLER
BOB SHAW

TO

DR. SABBAGH

ENERGY CONVERSION
LAB PROJECT 2
JAN 1972



ID=8538

ENERGY CONVERSION
LAB PROJ. 2
FOURIER ANALYSIS

NF = 43
NW = 10

TDEL = 0.400 MS
START = 0.000 MS
PERIOD = 16.700 MS

DC = -0.000
FO = 59.880

TERM	CFA	CFB	MAG	ANGLE	FREQ/HZ	PERCENT
1	-0.487336E-01	0.275391E 00	0.279670E 00	100.03	59.88	93.223
2	0.439136E-02	-0.537465E-02	0.694053E-02	-50.74	119.76	2.313
3	0.329058E-01	0.376387E-01	0.499946E-01	48.83	179.64	16.664
4	-0.288442E-03	-0.240293E-02	0.242018E-02	-96.84	239.52	0.806
5	0.750955E-02	-0.593965E-02	0.957459E-02	-38.34	299.40	3.191
6	-0.230704E-02	0.266643E-03	0.232240E-02	173.40	359.28	0.774
7	0.866322E-03	-0.338009E-02	0.348934E-02	-75.62	419.16	1.163
8	0.520541E-03	0.668234E-03	0.847054E-03	52.08	479.04	0.282
9	0.137384E-02	-0.317104E-02	0.345585E-02	-66.57	538.92	1.151
10	0.180613E-03	0.106478E-03	0.209664E-03	30.52	598.80	0.069

EXECUTION TIME 0283

PROGRAM WRITTEN
BY RANDY MILLER

BOB MARKS

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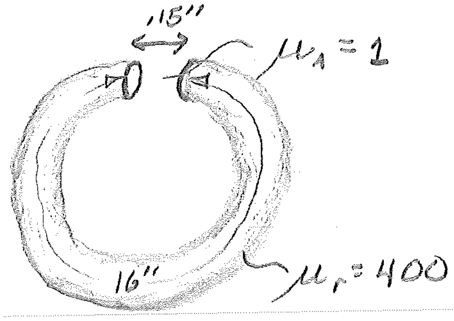
2-9)



$\phi = 5 \times 10^{-4}$ WEB

$A = 4\,s''$

$$\mathcal{R} = \frac{\ell}{\mu A} \Rightarrow \mathcal{R}_A = \frac{(.15'')\ \text{A-T-m}\ \ \ ''}{(1)(4\pi \times 10^{-7})\text{w}\ (4s'')(2.54 \times 10^{-2}\text{m})} = 1.175 \times 10^{5}\ \text{A/W}$$

$$\mathcal{R}_r = \frac{(16'')\ \text{A-T-m}\ \ \ ''}{(4 \times 10^{2})(4\pi \times 10^{-7})\text{w}\ (4s'')(2.54 \times 10^{-2}\text{m})} = 3.133 \times 10^{5}\ \text{A/W}$$

$$\mathcal{R}_{eq} = \mathcal{R}_A + \mathcal{R}_r = 1.488 \times 10^{6}\ \text{A/W}$$

$$\therefore\ \mathcal{F}_{eq} = \phi \mathcal{R}_{eq}$$

$$= (5 \times 10^{-4})(1.488 \times 10^{6}) = 744.08\ \text{A-T}$$

$$\mathcal{F}_A = \phi \mathcal{R}_A$$

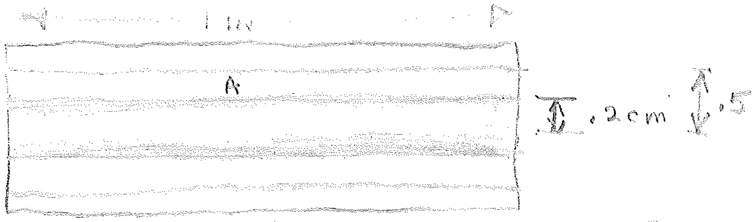
$$= \phi \mathcal{R}_A$$

$$= (5 \times 10^{-4})(1.488 \times 10^{6}) = 587.43\ \text{A-T}$$

$$\frac{\mathcal{F}_A}{\mathcal{F}_{eq}} = .7895\ (= 78.95\%)$$

10

2-10)



~~$A = (.3 \times 10^{-2}) \times (1) = 3 \times 10^{-3}\ m^2$~~

~~$L = \frac{N^2 \mu A}{\ell}$~~

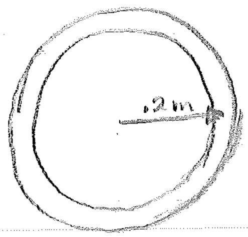
~~$= (1)^2 (4\pi \times 10^{-7})(3 \times 10^{-3}) = 3.7699\ nH/m$~~

$L = \mu \ln\left(\frac{.5}{.2}\right) = \mu_r\, 2 \times 10^{-7} \ln 2.5 = 1.9 \times 10^{-7} \mu_r\ H/m$

X

2

3-6)



$\mathcal{F} = 200$ A-T

$B = .1$ w/m^2

a) $B = \frac{\mu_0 NI}{2\pi R} + \frac{\mu_0 I_A}{2\pi R} \Rightarrow I_A = \left(B - \frac{\mu_0 NI}{2\pi R}\right)\frac{2\pi R}{\mu_0} = \frac{2\pi RB}{\mu_0} - NI$

$\Rightarrow I_A = \frac{2\pi(.20)(.1)}{4\pi \times 10^{-7}} - 200$

$= 10^5 - 200$

$= 1.00 \times 10^5 A$

b) $\mathcal{M} = \frac{I_A}{2\pi R} = \frac{10^5}{2\pi(.20)} = 8.05 \times 10^4$ A/m

c) $\chi = \mathcal{M}/H = \mathcal{M}\,2\pi R/NI$

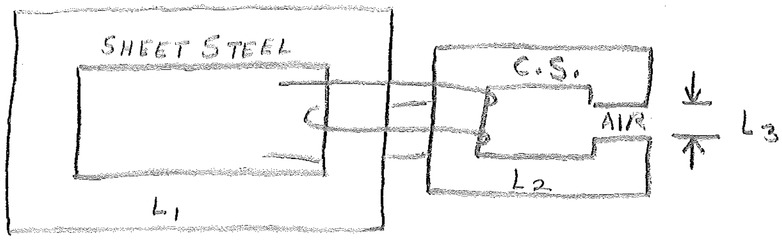
$\Rightarrow \chi = \frac{(8.05 \times 10^4)\,2\pi(.2)}{200} = 5.05 \times 10^2$

d) $\mu = \mu_0(\chi + 1)$

$= 4\pi \times 10^{-7}(506) = 6.36 \times 10^{-4}$ $\frac{H}{m}$

10

3-16)



$L_1 =$ 100 cms SHEET STEEL

$L_2 =$ 50 cms CAST STEEL

$L_3 =$ 0.2 cms AIR GAP

$A_1 =$ 5 sq cm

$A_2 = A_3 =$ 4 sq. cm.

$\phi =$ 60 KLINES

a) $B_3 = B_2 = \frac{\phi}{A}$

$= \frac{6 \times 10^{-4}}{4 \times 10^{-4}} = 1.5 \ W/m^2$

$H_g = B/\mu_0 = \frac{1.5}{4\pi \times 10^{-7}} = 1.19 \times 10^6 \ \frac{A\text{-}T}{m}$

$\mathcal{F}_g = H_g \ell_g = (1.19 \times 10^6)(2 \times 10^{-3}) = 2.38 \times 10^3 \ A\text{-}T$

$H_{cs} = 4 \times 10^{-3} \ A\text{-}T/m$

$\mathcal{F}_{cs} = H_{cs} L_2 = 2 \times 10^3 \ A\text{-}T$

$\mathcal{F}_T = 4.38 \times 10^3 AT$

b) $\mathcal{F}_{ss} = 4.38 \times 10^3 \ A\text{-}T$

$H_{ss} = 4.38 \times 10^3 \ A\text{-}T/m$

$B_{ss} = 1.55 \ W/m^2$

$\phi_{ss} = (1.55)(5 \times 10^{-4}) = 7.75 \times 10^{-4} W$

c) $\phi_{ss} + \phi_{cs} = (7.75 + 6) \times 10^{-4} = 13.75 \times 10^{-4} w$

d) $N\phi = 300(13.75 \times 10^{-4}) = .4125 \ w\text{-}T$

e) $L = N\phi/i = \frac{.4125}{14.6} = 28.3 \ mh$

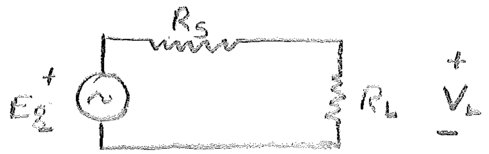
10

BOB MARKS

7

8-4) WITHOUT TRANSFORMER.

(a)



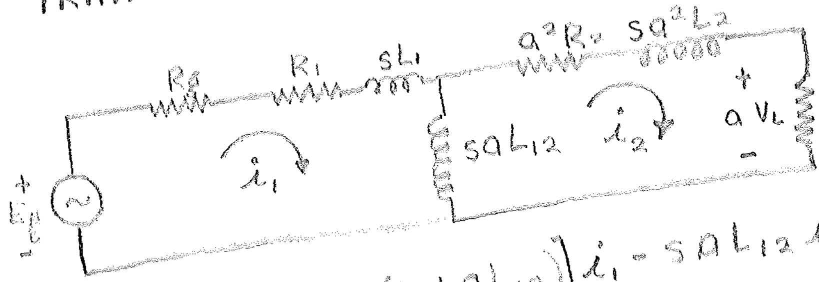
$$P_L = V_L^2/R_L = \left(\frac{R_L}{R_L+R_S}E_g\right)^2 \frac{1}{R_L}$$

$$= R_L\left(\frac{E_g}{R_L+R_S}\right)^2$$

$$= (5\,\Omega)\left(\frac{100\,V}{2005\,\Omega}\right)^2$$

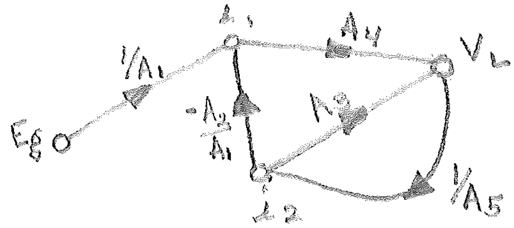
$$= .2494 \text{ WATTS}$$

WITH TRANSFORMER



$$E_g = [R_g + R_1 + s(L_1 + aL_{12})]\,i_1 - saL_{12}\,i_2$$
$$= A_1 i_1 + A_2 i_2$$
$$-aV_L = [a^2R_2 + sa(L_{12} + aL_2)]\,i_2 - saL_{12}\,i_1$$
$$V_L = -[aR_2 + s(L_{12} + aL_2)]\,i_2 + sL_{12}\,i_1 = A_3 i_2 + A_4 i_1$$
$$aV_L = a^2R_L\,i_2 \Rightarrow V_L = aR_L\,i_2 = A_5\,i_2$$

$$\begin{cases} i_1 = \frac{1}{A_1}E_g - \frac{A_2}{A_1} i_2 \\ V_L = A_3 i_2 + A_4 i_1 \\ i_2 = \frac{1}{A_5} V_L \end{cases}$$



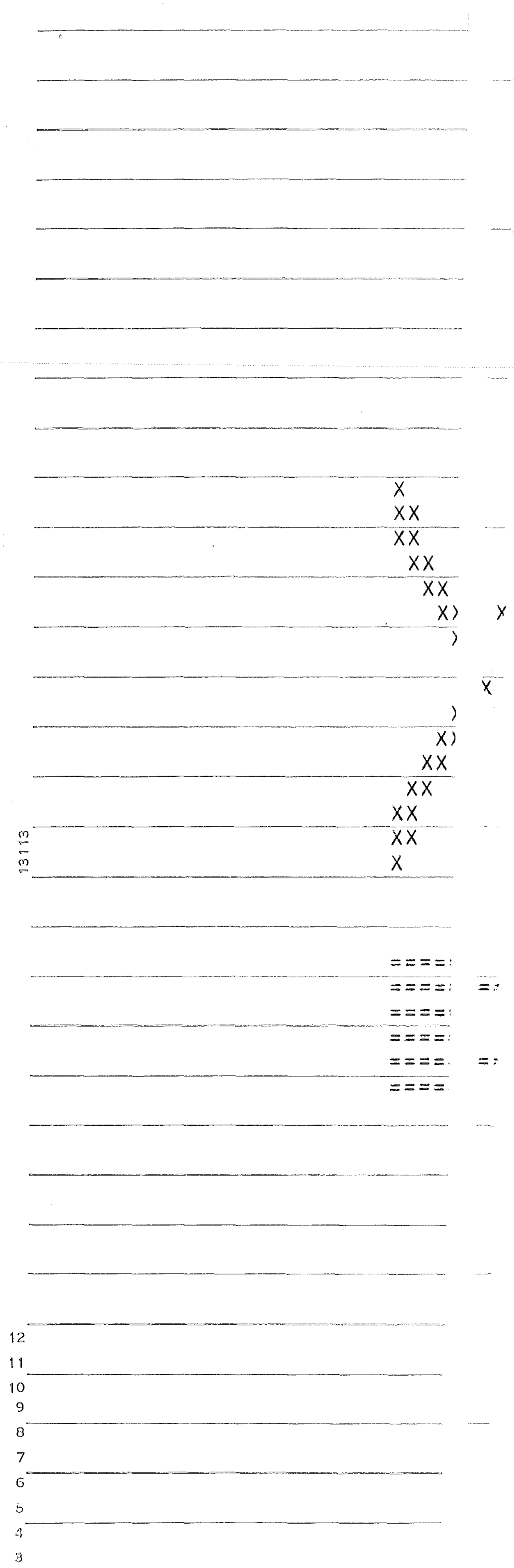
$$\nabla = 1 - \left[\frac{-A_4A_2}{A_1A_5}\right] = 1 + \frac{A_4A_2}{A_1A_5}$$
$$T_1 = A_4/A_1 \qquad \Delta_1 = 1$$
$$\Rightarrow \frac{V_L}{E_g} = \frac{A_4}{A_1\left[1 + \frac{A_4A_2}{A_1A_5}\right]}$$

POWER AT LOAD $= \left(\frac{V_L}{E_g}\right)^2 E_g^2 / R_L$

NOTE: COMPUTER SOLUTION MADE WITH SHUNT INDUCTANCE L_{12} (OPPOSED TO aL_{12}) AS ILLUSTRATED IN FIGURE 8-14 OF TEXT)

~~[NOTE: COMPUTER SOLUTION MADE WITH SHUNT INDUCTIVE REACTANCE OF sL_{12} (AS OPPOSED TO asL_{12}) AS DEPICTED IN FIGURE 8-14 OF TEXT]~~

~~MAXIMUM PWR. DELIVERED TO TRANSFORMER @ 63.1 HZ = 1 WATT, AS OPPOSED TO PREVIOUS PWR. OF .25 WATTS, USING SHUNT REACTANCE OF sL_{12}. SIMILAR RESULTS OBTAINED USING asL_{12} AS SHUNT REACTANCE, BUT AT FREQ. OF 3.3 HZ~~


13113
12
11
10
9
8
7
6
5
4
3

```
                           PROGRAM EE
     000002                DIMENSION    1(
     000002                COMPLEX S,
     000002                WRITE(5,19
     000006                READ(2,20)
     000014                READ(2,20)
     000022                READ(2,20)
     000030                READ(2,20)   I(
     000036                READ(2,20)
     000044                READ(2,20)
     000052                READ(2,20)
     000060                READ(2,20)
     000066                FM=0.0
     000067                PMAX=0.0
     000067                R=10.**(1.
     000075                XL1=XL1SC/
     000077                XL2=XL1
     000077                XL12=XLIOC
     000102                TPI=8.*ATA
     000104                ND=(25.*D)
     000110                DO 90 J=1,
     000114                S=CMPLX(0.
     000117                A1=RG+R1+S   XL
     000130                A2=-S*A*XL
     000134                A3=-(A*R2+
     000150                A4=S*XL12
     000154                A5=CMPLX(A
     000157                AV=A4/(A1*
     000217                G(J)=CABS(   )
     000222                PWR(J)=EG*
     000226                IF(PWR(J)-
     000231             10 PMAX=PWR(J
     000233                FM=F
     000235             15 FR(J)=ALOG
     000241                WRITE(5,21   ,(
     000253                F=F*R
     000255             90 CONTINUE
     000257                WRITE(5,22   MA
     000267                WRITE(5,23
     000273                CALL SETPLT
     000302                WRITE(5,23
     000306                CALL SETPLT
     000315             19 FORMAT('7R(
                          *7X,'FREQUEN   Y
     000315             20 FORMAT(F15.
     000315             21 FORMAT(5X,
     000315             22 FORMAT(//'    MA
     000315             23 FORMAT('7'
     000315                STOP
     000317                END

     PROGRAM LENGTH INCLUDING I
     010455

     UNUSED COMPILER SPACE
     015700


               LOAD MAP.
```

ROBERT J. MARKS II
ENERGY CONVERSION
EE353

FREQUENCY (HZ)	V
1.00000000E-01	7.
1.09647820E-01	8.
1.20226443E-01	9.
1.31825674E-01	1.
1.44543977E-01	1.
1.58489319E-01	1.
1.73780083E-01	1.
1.90546072E-01	1.
2.08929613E-01	1.
2.29086765E-01	1.
2.51188643E-01	1
2.75422870E-01	2
3.01995172E-01	2
3.31131121E-01	2
3.63078055E-01	2
3.98107171E-01	3
4.36515832E-01	3
4.78630092E-01	3
5.24807460E-01	4
5.75439937E-01	4
6.30957344E-01	4
6.91830971E-01	5
7.58577575E-01	5
8.31763771E-01	6
9.12010839E-01	7
1.00000000E+00	7
1.09647820E+00	8
1.20226443E+00	9
1.31825674E+00	
1.44543977E+00	
1.58489319E+00	
1.73780083E+00	
1.90546072E+00	
2.08929613E+00	
2.29086765E+00	
2.51188643E+00	
2.75422870E+00	
3.01995172E+00	
3.31131121E+00	
3.63078055E+00	
3.98107171E+00	
4.36515832E+00	
4.78630092E+00	
5.24807460E+00	
5.75439937E+00	
6.30957344E+00	
6.91830971E+00	
7.58577575E+00	
8.31763771E+00	
9.12010839E+00	
1.00000000E+01	
1.09647820E+01	
1.20226443E+01	
1.31825674E+01	
1.44543977E+01	
1.58489319E+01	
1.73780083E+01	
1.90546072E+01	1.[illegible]

2.08929613E+01	1.	
2.29086765E+01	1.	7
2.51188643E+01	1.	
2.75422870E+01	1.	
3.01995172E+01	1.	5
3.31131121E+01	1.	
3.63078055E+01	1.	
3.98107171E+01	2.	0
4.36515832E+01	2.	
4.78630092E+01	2.	
5.24807460E+01	2.	13
5.75439937E+01	2.	
6.30957344E+01	2.	
6.91830971E+01	2.	29
7.58577575E+01	2.	
8.31763771E+01	2.	
9.12010839E+01	2.	24
1.00000000E+02	2.	
1.09647820E+02	1.	
1.20226443E+02	1.	10
1.31825674E+02	1.	
1.44543977E+02	1.	
1.58489319E+02	1.	34
1.73780083E+02	1.	
1.90546072E+02	1.	
2.08929613E+02	1.	42
2.29086765E+02	1.	
2.51188643E+02	1.	
2.75422870E+02	1.	71
3.01995172E+02	9.	
3.31131121E+02	9.	
3.63078055E+02	8.	85
3.98107171E+02	7.	
4.36515832E+02	7.	
4.78630092E+02	6.	79
5.24807460E+02	5.	
5.75439937E+02	5.	
6.30957344E+02	4.	69
6.91830971E+02	4.	
7.58577575E+02	4.	
8.31763771E+02	3.	93
9.12010839E+02	3.	
1.00000000E+03	3.	

MAXIMUM POWER= 9.99936145E

```
    L           .03 +--------------
    O               I
    G               I
                    I
    F               I
                    I
    V               I
    S               I
                    I
    A               I
    V               I
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                    I
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                .02 +--------------
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                .01 +--------------
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                .00 0000000000000000
                       -1
                     10
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13113

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  L        1.000  +--------
  O               I
  G               I
                  I
  F               I
                  I
  V         .750  +--------
  S               I
                  I
  P               I
  W               I
  R               I
            .500  +--------
                  I
                  I
                  I
                  I
                  I
            .250  +--------
                  I
                  I
                  I
                  I
                  I
           0.000  *********
                -1.000          .5
              10          1(
             XRRJM37.     1

  15.07.46.XRRJM,     41   1
  15.07.46.COMMENT. $RJ
  15.07.46.RDUE
  15.07.46.MAP(PART)
  15.07.46.FUN(E)
  15.07.47. CTIME 000.5
  15.07.47.LGO
  15.07.48.LGO.
  15.07.50.CX      1.921
  15.07.52.STOP
  15.07.53.CP     2.697
  15.07.53.LINES     282
  15.07.53.CM      .121    D-
```

c) $R_P = \dfrac{(R_g + R_1)\, a^2 (R_2 + R_L)}{R_g + R_1 + a^2 (R_2 + R_L)}$

$= \dfrac{(2100)(20)^2(5.25)}{(2100) + (20)^2(5.25)}$

$= 1050\,\Omega$

$L_{eq} = L_1 + a^2 L_2$

$= (.025)(1 + (20)^2)$

$= 10.025$ H

$R_S = R_g + R_1 + a^2 (R_2 + R_L)$

$= (2100) + (20)^2 (5.25)$

$= 4200\,\Omega$

$\omega_L = R_P / a L_{12}$

$= (1050)/(20)(5)$

$= 10.5 \frac{RAD}{SEC}$

$\omega_H = R_S / L_{eq}$

$= (4200)/(10.025)$

$= 418.953 \frac{RAD}{SEC}$

BAND PASS FREQUENCIES

$\Rightarrow \omega_L < \omega < \omega_H$

$10.5 \frac{RAD}{SEC} < \omega < 418.953 \frac{RAD}{SEC}$

$\therefore 1.67113\text{ Hz} < f < 66.6784$ Hz

EE [illegible] Assignment.

1. If the current sheet of Fig 1-5 (my notes, p. 1(?)) is replaced by sinusoidally distributed current sheet $\bar{J}_s(\theta) = -J_{sa} \sin\theta \, \bar{a}_z$ [illegible]

$-\pi \le \theta \le$ [illegible]

show that the armature flux density is

$$\bar{B}_a(r,\theta) = \bar{a}_r \frac{\mu_0 R_a J_{sa}}{r} \left[\frac{\left(\frac{R_f}{r}\right) + \left(r/R_f\right)}{\left(\frac{R_a}{R_f}\right) - \left(\frac{R_f}{R_a}\right)} \right] \cos\theta$$

$$+ \bar{a}_\theta \frac{\mu_0 R_a J_{sa}}{r} \left[\frac{\left(\frac{R_f}{r}\right) - \left(\frac{r}{R_f}\right)}{\left(\frac{R_a}{R_f}\right) - \left(\frac{R_f}{R_a}\right)} \right] \sin\theta .$$

2. Derive an expression for L_{fa}, using the results of (1), and sketch.

3. Derive an expression for torque and sketch.

[illegible]

1\. Since we are concerned with only the equatorial plane, ... from, (1-95) (my notes, p. 20) becomes

(1) $a_m R_f^{(m-1)} - c_m R_f^{-(m+1)} = 0$ (all m).

and (1-97) becomes

(2) $b_m R_f^{(m-1)} - d_m R_f^{-(m+1)} = 0$ (all m).

Equation (1-96) is:

$$(\star)\quad \sum_{n=1}^{\infty} \{ n\left(a_n R_a^{(n-1)} - c_n R_a^{-(n+1)}\right)\cos n\theta + n\left(b_n R_a^{(n-1)} - d_n R_a^{-(n+1)}\right)\sin n\theta \}$$

$$= -\mu_0 J_{sa}(\theta) = \mu_0 J_{sa}\sin\theta .$$

multiplying this equation by $\cos m\theta$, and integrating with respect to θ from $-\pi$ to $+\pi$, we get (remember your integrals of sine and cosine products!)

(3) $\pi m\left(a_m R_a^{(m-1)} - c_m R_a^{-(m+1)}\right) = 0$, for all m.

Equations (1) and (3) imply that $\boxed{a_m = c_m = 0}$, for all m.

Next, multiply $(\star)$ by $\sin m\theta$ and integrate with respect to θ from $-\pi$ to π:

(4) (a) $\pi m\left(b_m R_a^{(m-1)} - d_m R_a^{-(m+1)}\right) = 0$, all $m \neq 1$.

(b) $\pi\left(b_1 - d_1 R_a^{-2}\right) = \mu_0 J_{sa}\pi$

(4)(a) and (2) imply that $b_m = d_m = 0$, all $m \neq 1$. (2)(b) and (4)(b) give us:

$$b_1 - d_1 R_f^{-2} = 0 \qquad \Longrightarrow \quad b_1 = \frac{\begin{vmatrix} 0 & -1/R_f^2 \\ \mu_0 J_{sa} & -1/R_a^2 \end{vmatrix}}{\begin{vmatrix} 1 & -1/R_f^2 \\ 1 & -1/R_a^2 \end{vmatrix}} = \frac{\mu_0 J_{sa}}{R_f^2(\ldots}$$

$$b_1 - d_1 R_a^{-2} = \mu_0 J_{sa}$$

$$A_1 = \frac{\left[\;\mu_0 J_{sa}\right]}{\frac{1}{R_f^2} - \frac{1}{R_a^2}} = \frac{\mu_0 J_{sa}}{\left(\frac{1}{R_f^2} - \frac{1}{R_a^2}\right)}.$$

$$\therefore\ A_z(r,\theta) = \frac{\mu_0 J_{sa}}{\left(\frac{1}{R_f^2} - \frac{1}{R_a^2}\right)}\left[\frac{r}{R_f^2} + \frac{1}{r}\right]\sin\theta.$$

$$\therefore\ \bar{B} = \bar{a}_r \frac{1}{r}\frac{\partial A_z}{\partial\theta} - \bar{a}_\theta \frac{\partial A_z}{\partial r} = \bar{a}_r \frac{\mu_0 J_{sa}}{\left(\frac{1}{R_f^2} - \frac{1}{R_a^2}\right)}\left[\frac{1}{R_f^2} + \frac{1}{r^2}\right]\cos\theta$$

$$+\bar{a}_\theta \frac{\mu_0 J_{sa}}{\left(\frac{1}{R_f^2} - \frac{1}{R_a^2}\right)}\left[\frac{1}{R_f^2} - \frac{1}{r^2}\right]\sin\theta.$$

$$= \boxed{\bar{a}_r \frac{\mu_0 R_a J_{sa}}{r}\left[\frac{\left(\frac{R_f}{r}\right) + \left(r/R_f\right)}{\left(\frac{R_a}{R_f}\right) - \left(\frac{R_f}{R_a}\right)}\right]\cos\theta + \bar{a}_\theta \frac{\mu_0 R_a J_{sa}}{r}\left[\frac{\left(R_f/r\right) - \left(r/R_f\right)}{\left(R_a/R_f\right) - \left(R_f/R_a\right)}\right]\sin\theta.}$$

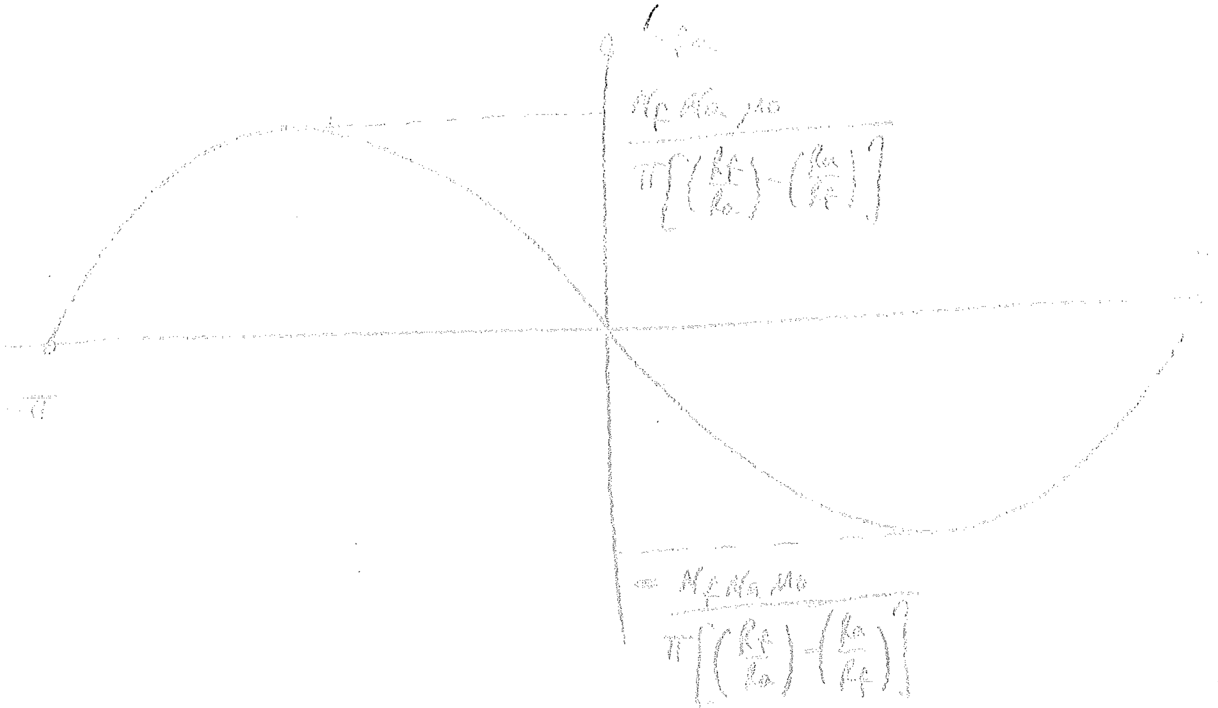
(2) $$\lambda_{fa} = N_f \int_{-\alpha}^{\pi-\alpha} B_r(R_f,\theta)\, R_f\, d\theta = \frac{2 N_f \mu_0 R_a J_{sa}}{R_f\left[\left(\frac{R_a}{R_f}\right) - \left(\frac{R_f}{R_a}\right)\right]}\int_{-\alpha}^{\pi-\alpha} R_f \cos\theta$$

$$= \frac{4 N_f \mu_0 R_a J_{sa}}{\left[\left(\frac{R_a}{R_f}\right) - \left(\frac{R_f}{R_a}\right)\right]}\sin\alpha.$$

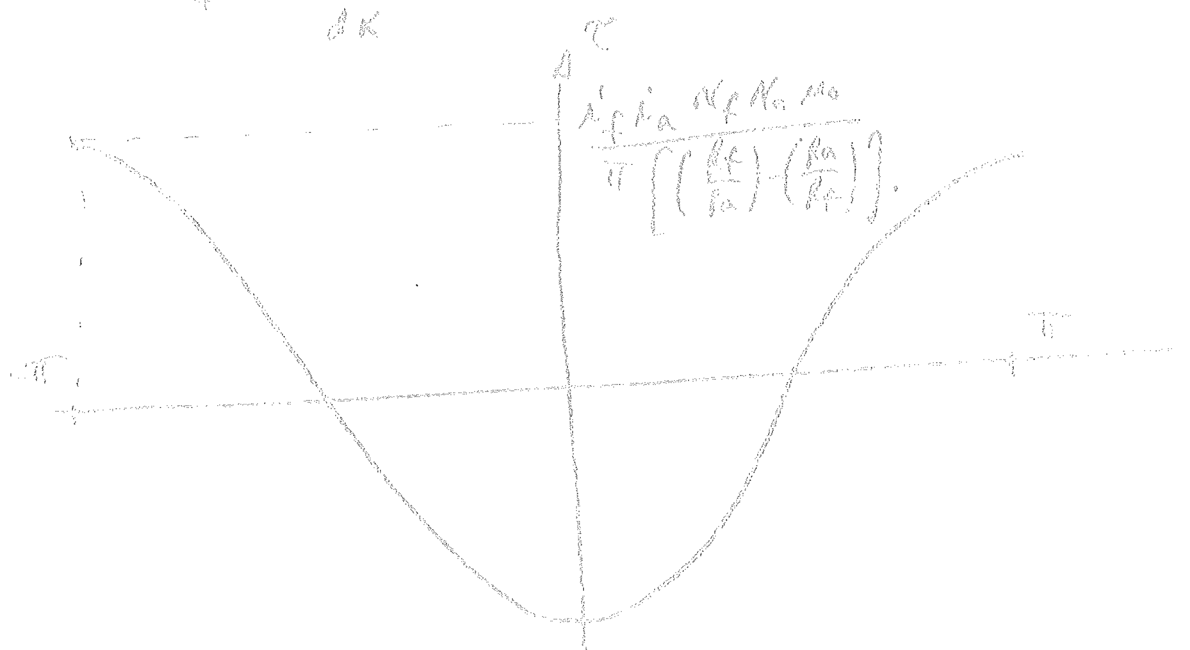
$$= \frac{N_f N_a}{\pi}\,\frac{\mu_0 \sin\alpha}{\left[\left(\frac{R_a}{R_f}\right) - \left(\frac{R_f}{R_a}\right)\right]}\, i_a.$$

$$\therefore\ L_{fa} = \frac{N_f N_a \mu_0 \sin\alpha}{\pi\left[\left(\frac{R_a}{R_f}\right) - \left(\frac{R_f}{R_a}\right)\right]}$$

$$= \frac{N_f N_a \mu_0 \sin\alpha}{\ldots}$$



(3) $\tau = i_f i_a \frac{dL_{fa}}{dK}$



18. Calculate the force in tons that is necessary to hold two charges of 1 coulomb at a separation of 1 meter. $\sim 10^{6}$ tons

19. Calculate the distance of separation of two electrons in a vacuum for which the force between them is equal to the gravitational force on one of them at the earth's surface.

20. If $\phi = 1/\sqrt{x^2+y^2}$ volts in the xy plane, determine the magnitude and direction of the electric field at the point (1,1).

21. A p-n semicoqnductor junction is shown. Because of the diffusion of charge carriers, a layer of positive charge accumulates on one side of the junction, and an equal layeroof negative charge accumulates on the other. If an electron moves into the region between the two charge layers, it is urged to the right by the electric field. For an electron on the right to cross the junction, it must have a large enough velocity to penetrate the retarding effect of the field (called a "poteneial barrier"). Calculate the minimum velocity for this to occur if the density of each charge layer is 4.25×10^{-6} coulombs/m^2, the layers are 10^{-3} cm apart, and the dielectric constant of the material K=2.

N 22. If corona discharge occurs at 30 KV/cm in air, what is the smallest radius of a sphere of a van de Graaf generator if it is to be charged to 5 million volts?

N 23. An electron beam in a cathode ray tube is circular in cross section with a radius of 1 mm. Calculate the force that the charge in the beam exerts on an electron that is located at the surface of the beam. Calculate the distance that such an electron will move outward (defocusing the beam) in moving 0.25 m longitudinally if it has a longitudinal velocity of 22.9×10^6 m/sec and a beam current of 40 microamperes.

24. 3.1

N 25. 3.3

26. 3.4

27. 3.5

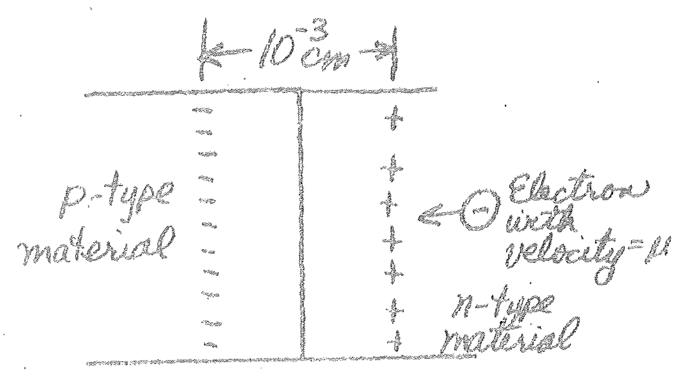
28. 3.6

29. 3.7

30. 3.9

31. 3.16

32. 3.17



prob. 21

(HW #5)

EE 353. Solutions of Some Problems

1. Since we are concerned with only the armature flux, set $J_{sf} = 0$. Thus, (1-95) (my notes, p. 19) becomes

(1) $a_m R_f^{(m-1)} - c_m R_f^{-(m+1)} = 0$ (all m).

and (1-97) becomes

(2) $b_m R_f^{(m-1)} - d_m R_f^{-(m+1)} = 0$ (all m).

Equation (1-96) is:

$$(\star)\quad \sum_{n=1}^{\infty} \{ n(a_n R_a^{(n-1)} - c_n R_a^{-(n+1)}) \cos n\theta + n(b_n R_a^{(n-1)} - d_n R_a^{-(n+1)}) \sin n\theta \}$$

$$= -\mu_0 J_{sa}(\theta) = +\mu_0 J_{sa} \sin\theta.$$

Multiplying this equation by $\cos m\theta$, and integrating with respect to θ from $-\pi$ to $+\pi$, we get (remember your integrals of sine and cosine products),

(3) $\pi m (a_m R_a^{(m-1)} - c_m R_a^{-(m+1)}) = 0$, for all m.

Equations (1) and (3) imply that $\boxed{a_m = c_m = 0}$, for all m.

Next, multiply ($\star$) by $\sin m\theta$ and integrate with respect to θ from $-\pi$ to π:

(4)(a) $\pi m (b_m R_a^{(m-1)} - d_m R_a^{-(m+1)}) = 0$, all $m \neq 1$.

(b) $\pi (b_1 - d_1 R_a^{-2}) = \mu_0 J_{sa} \cdot \pi$

(4)(a) and (2) imply that $b_m = d_m = 0$, all $m \neq 1$. (4)(b) and (2) give us:

$$\begin{aligned} b_1 - d_1 R_f^{-2} &= 0 \\ b_1 - d_1 R_a^{-2} &= \mu_0 J_{sa} \end{aligned} \Longrightarrow b_1 = \frac{\begin{vmatrix} 0 & -1/R_f^2 \\ \mu_0 J_{sa} & -1/R_a^2 \end{vmatrix}}{\begin{vmatrix} 1 & -1/R_f^2 \\ 1 & -1/R_a^2 \end{vmatrix}} = \frac{\mu_0 J_{sa}}{R_f^2\left(\frac{1}{R_f^2} - \frac{1}{R_a^2}\right)}$$

$$a_1 = \frac{\begin{vmatrix} 1 & 0 \\ 1 & \mu_0 J_{sa} \end{vmatrix}}{\frac{1}{R_f^2} - \frac{1}{R_a^2}} = \frac{\mu_0 J_{sa}}{\left(\frac{1}{R_f^2} - \frac{1}{R_a^2}\right)}.$$

$$\therefore \quad A_z(r,\theta) = \frac{\mu_0 J_{sa}}{\left(\frac{1}{R_f^2} - \frac{1}{R_a^2}\right)}\left[\frac{r}{R_f^2} + \frac{1}{r}\right] \sin\theta.$$

$$\therefore \quad \bar{B} = \bar{a}_r \cdot \frac{1}{r}\frac{\partial A_z}{\partial \theta} - \bar{a}_\theta \frac{\partial A_z}{\partial r} = \bar{a}_r \frac{\mu_0 J_{sa}}{\left(\frac{1}{R_f^2} - \frac{1}{R_a^2}\right)}\left[\frac{1}{R_f^2} + \frac{1}{r^2}\right] \cos\theta$$

$$+ \bar{a}_\theta \frac{\mu_0 J_{sa}}{\left(\frac{1}{R_f^2} - \frac{1}{R_a^2}\right)}\left[\frac{1}{R_f^2} - \frac{1}{r^2}\right] \sin\theta.$$

$$= \boxed{\bar{a}_r \frac{\mu_0 R_a J_{sa}}{r}\left[\frac{\left(\frac{R_f}{r}\right) + \left(r/R_f\right)}{\left(\frac{R_a}{R_f}\right) - \left(\frac{R_f}{R_a}\right)}\right] \cos\theta + \bar{a}_\theta \frac{\mu_0 R_a J_{sa}}{r}\left[\frac{(R_f/r) - (r/R_f)}{(R_a/R_f) - (R_f/R_a)}\right] \sin\theta.}$$

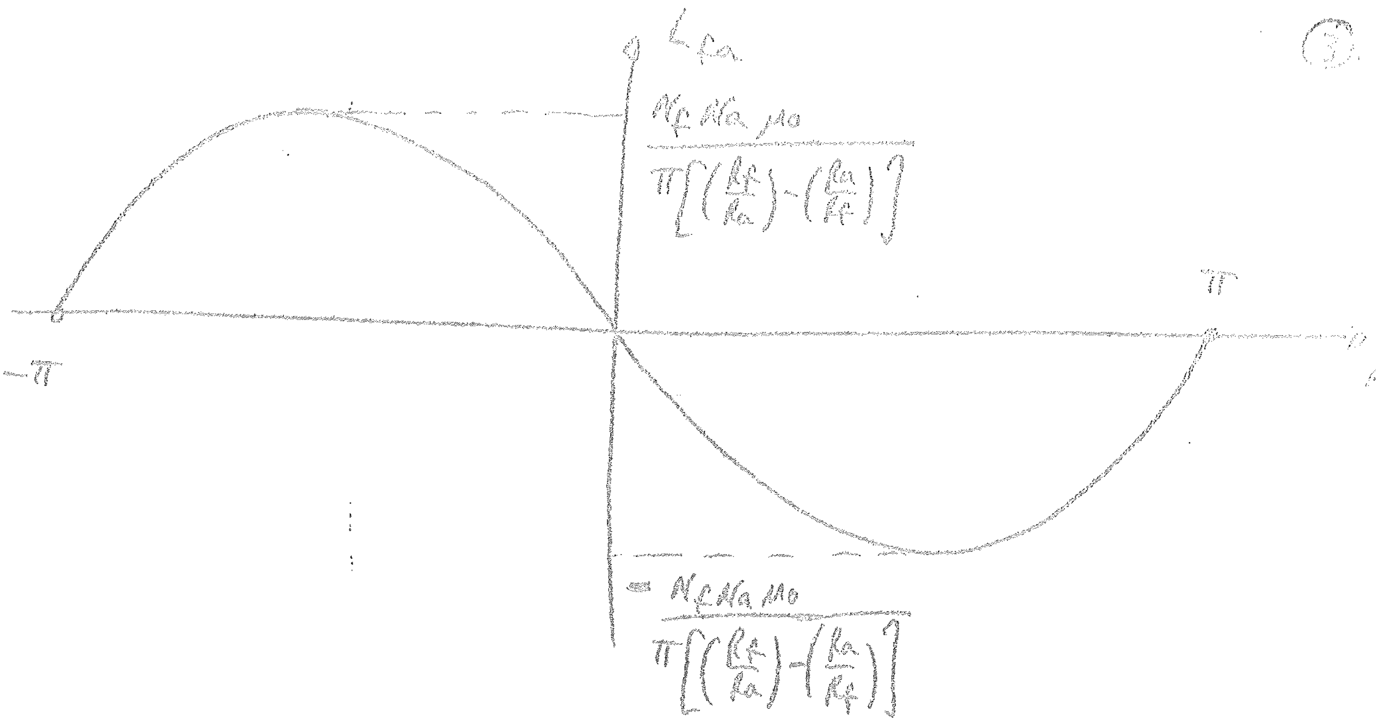
(2)
$$\lambda_{fa} = N_f \int_{-\alpha}^{\pi-\alpha} B_r(R_f,\theta) R_f \, d\theta = \frac{2 N_f \mu_0 R_a J_{sa}}{R_f\left[\left(\frac{R_a}{R_f}\right) - \left(\frac{R_f}{R_a}\right)\right]} \int_{-\alpha}^{\pi-\alpha} R_f \cos\theta$$

$$= \frac{4 N_f \mu_0 R_a J_{sa}}{\left[\left(\frac{R_a}{R_f}\right) - \left(\frac{R_f}{R_a}\right)\right]} \sin\alpha.$$

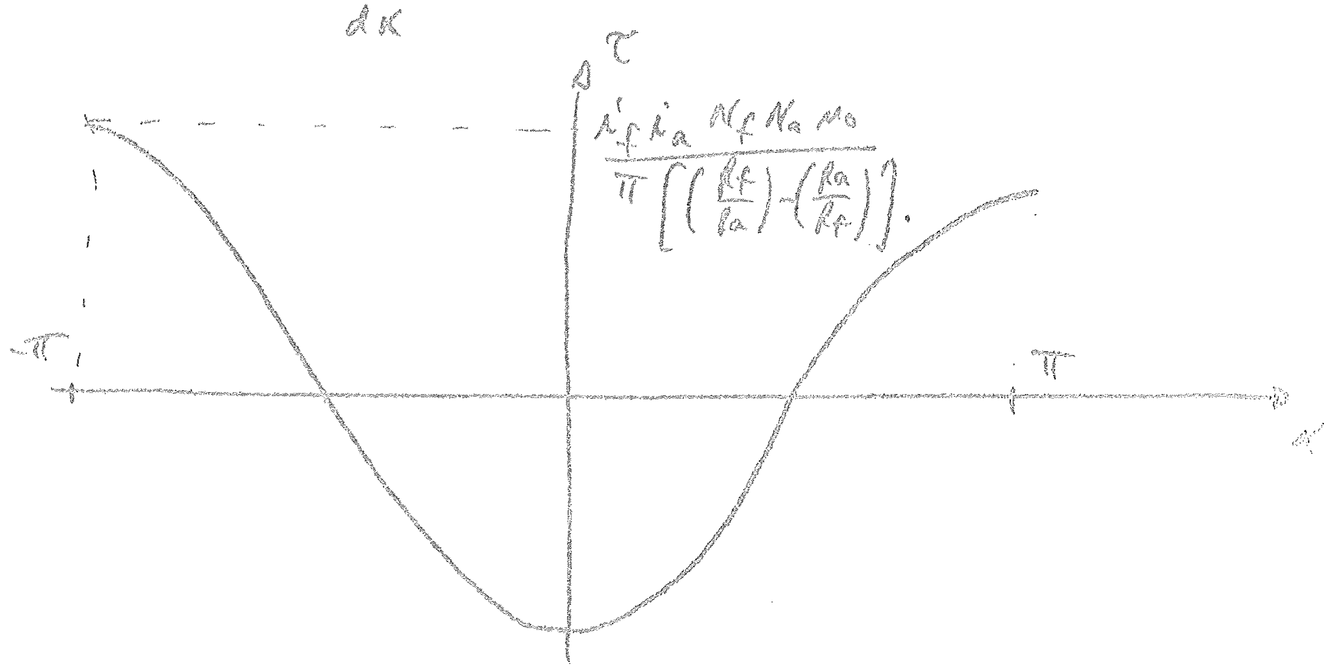
$$= \frac{N_f N_a}{\pi} \frac{\mu_0 \sin\alpha}{\left[\left(\frac{R_a}{R_f}\right) - \left(\frac{R_f}{R_a}\right)\right]} \cdot i_a.$$

$$\therefore \quad L_{fa} = \frac{N_f N_a \mu_0 \sin\alpha}{\pi\left[\left(\frac{R_a}{R_f}\right) - \left(\frac{R_f}{R_a}\right)\right]}$$

$$= \frac{N_f N_a \mu_0 \sin\alpha}{\ldots}$$



(3) $\tau = i_f i_a \frac{dL_{fa}}{d\alpha}$



BOB MARKS

190/200

Excellent paper.

EE-353 ENERGY CONVERSION

MIDTERM EXAM (Winter 1971-72)

Be it jewel or toy,

Not the prize gives the joy,

But the striving to win the prize.

Pisistratus Caxton--
The Boatman

Somebody said that it couldn't be done,
But he with a chuckle replied
That "maybe it could't," but he
would be one
Who wouldn't say so till he'd tried.

So he buckled right in with the trace
of a grin
On his face. If he worried he hid it.
He started to sing as he tackled the
thing
That couldn't be done, and he did it.

Edgar A. Guest--It Couldn't
Be Done

1. 4-23 to 4-27.

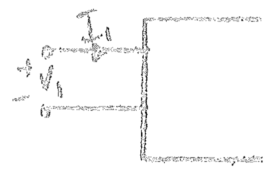
2. 11-13. The origin for x is as depicted in Figure 4-15. The spring is completely relaxed at $x=0$. There is no exte force. Determine the d.c. and a.c. (linearized) equations
$v_a(t) = V_0 + v_{a_1}(t)$, $v_b(t) = V_0 + v_{b_1}(t)$,
where V_0 is the d.c. component and $v_{a_1}(t), v_{b_1}(t)$ are the a. signals. Be sure that you correlate the sign of the mutua inductance with the assumed positive directions of i_a and

3. 12-7.

4. A two-port network may be represented using the hybrid par Z_{11}, G_{12}, H_{21}, Y_{22} in the following way:

$$V_1 = I_1 \cdot Z_{11} + V_2 \cdot G_{12}$$

$$I_2 = I_1 \cdot H_{21} + V_2 \cdot Y_{22}$$



Explain the meaning of these parameters and the physical m measuring them. Let $V_2 = \alpha \dot{X}_2$, $I_2 = F_2/\alpha$, as in the fo current analogy.

5. Determine the hybrid parameters of Prob. 4 for the equival circuit of the hand-out of Chapt. 11. Express your answer terms of R, L_0, C, $\mathscr{G}$, L.

6. A conservative transducer is one for which the average a.c. input into both ports is zero (i.e., there is no dissipate within the transducer). Thus, letting V_1, I_1, V_2, I_2 be pl representations of sinusoids, we have

$$P_{avg} = \frac{1}{2} \, \mathrm{Real} \left[V_1 I_1^* + V_2 I_2^* \right] = 0 \qquad (* = \text{conjug value})$$

(a) Show that for such a transducer the hybrid parameters Problem 4 must satisfy

$$Z_{11} = jX_1, \quad Y_{22} = jB_2, \quad H_{21} = -G_{12}^*,$$

where X_1 and B_2 are real. Explain what these results in terms of the parameters making up the transducer.

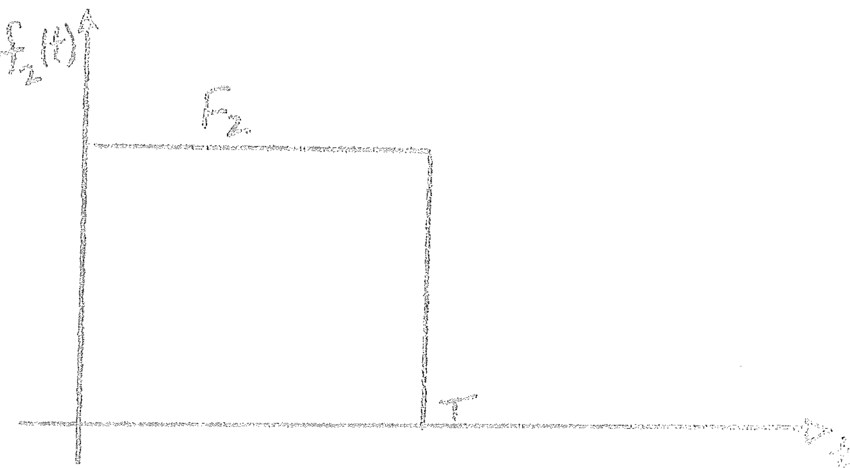
7. A force pulse $f_2(t)$ is applied to the transducer of problem (11-1). If the electrical output (port 1) is open circuited, determine the output voltage, $v_1(s)$. Assume that the system starts from rest at the quiescent point.

$R = 100\,\Omega$, $i_0 = 8/a$

$L_0 = 1\,h$, $b = 0.01\ \frac{N\text{-sec}}{m.}$

$M = 0.1\,Kg$, $a = -0.1/M^2$.

$X_0 = 0.1\,M$, $K_s = 500\ \frac{N}{M}$.



8. Using the same data as in problem 7, this time excite the transducer from the electrical side and calculate the velocity output $\dot{X}_2$. There is no mechanical load applied to port 2. The electrical input, $v_1(t)$, is of the same form as $f(t)$, above, with the constant value being E_1 volts.

"... THE WISDOM OF THIS WORLD
IS FOLLY WITH GOD..."

CORINTHIANS I, 3.19

"Let us pray"

1) TO FIND THE ENERGY AND THE CO-ENERGY, A RELATIONSHIP MUST BE FOUND BETWEEN i AND λ. THEN

$$\text{CO-ENERGY} = \int_0^{i_m} \lambda \, di$$

$$\text{AND ENERGY} = \int_0^{\lambda_{m}} i \, d\lambda$$

CURRENT MAY BE FOUND BY DIVIDING THE SUM OF THE mmf'S OF THE GAP AND THE STEEL BY THE TURNS RATIO. THE mmf OF THE STEEL IN TURN, MAY BE FOUND BY:

$$\mathcal{F}_s = H l_s$$

WHERE l_s IS THE LENGTH OF THE SHEET STEEL. THE mmf OF THE GAP MAY BE FOUND FROM:

$$\mathcal{F}_g = Bx/\mu_0$$

WHERE x IS THE GAP LENGTH. CORRESPONDING VALUES OF B AND H MAY BE READ FROM THE B-H CURVE FOR SHEET STEEL ON PAGE 3-22 OF TEXT.

λ MAY BE SOLVED FOR FROM:

$$\lambda = NAB$$

WHERE N IS THE TURNS RATIO, AND A IS THE CROSS SECTIONAL AREA.

THUS, FOR EACH CO-ORDINATE OF THE B-H CURVE, WITH A GIVEN GAP LENGTH, A CORRESPONDING λ, $\mathcal{F}_g$ AND $\mathcal{F}_s$ MAY BE COMPUTED. $\mathcal{F}_s$ AND $\mathcal{F}_g$ THEN YIELD A CORRESPONDING VALUE OF i.

THE ENERGY AND CO-ENERGY IS THEN FOUND BY NUMERICAL INTEGRATION ON THE RESULTANT λ-i CURVE.

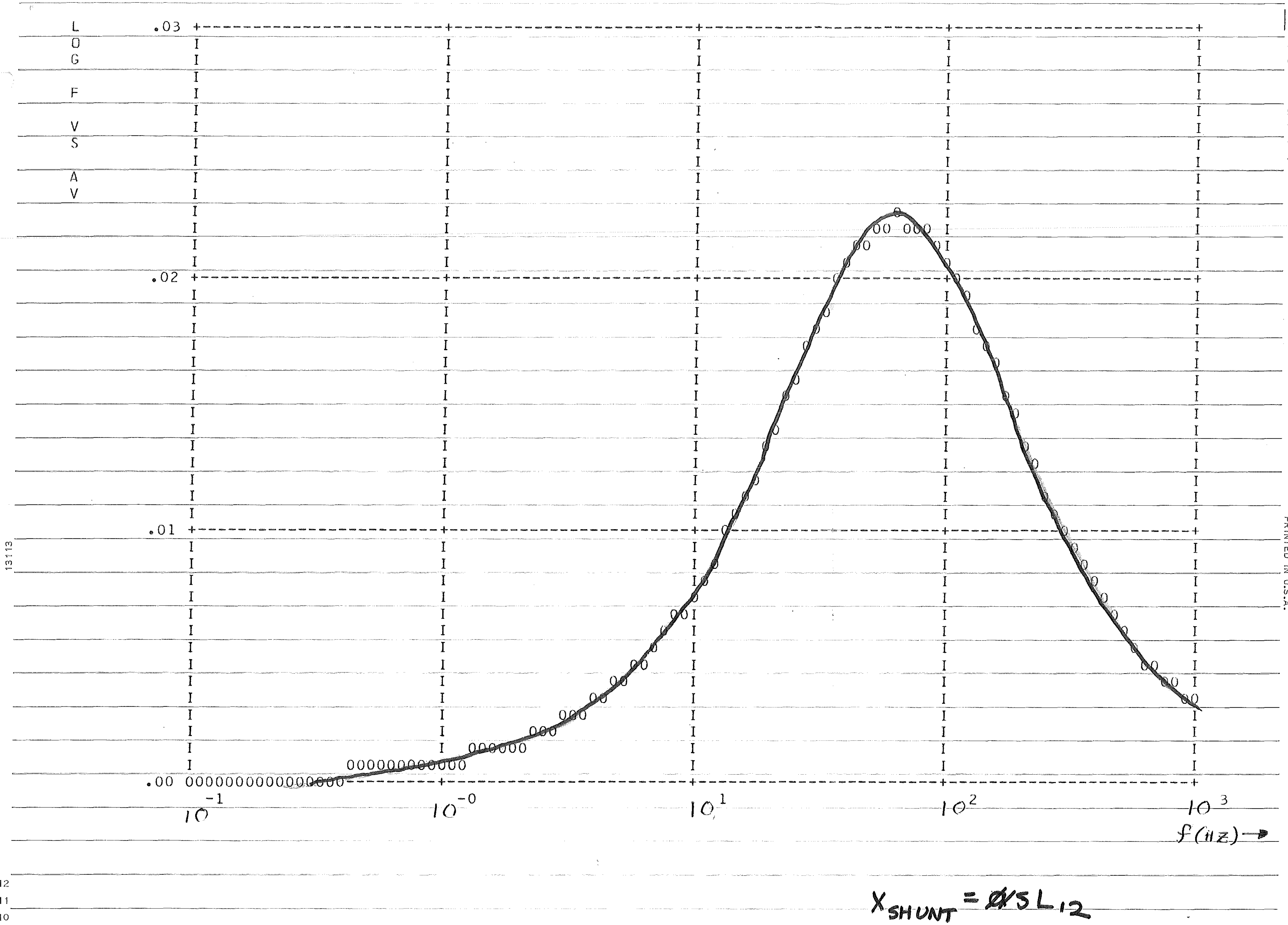
```
                          PROGRAM EE353A(INPUT,OUTPUT,TAPE2=INPUT,TAPE5=OUTPUT)
   000002                 DIMENSION G(1001),FR(1001),PWR(1001)
   000002                 COMPLEX S,A1,A2,A3,A4,A5,AV
   000002                 WRITE(5,19)
   000006                 READ(2,20)RG
   000014                 READ(2,20)RL
   000022                 READ(2,20)XLISC
   000030                 READ(2,20)XLIOC
   000036                 READ(2,20)A
   000044                 READ(2,20)EG
   000052                 READ(2,20)F
   000060                 READ(2,20)D
   000066                 FM=0.0
   000067                 PMAX=0.0
   000067                 R=10.**(1./25.)
   000075                 XL1=XL1SC/2.
   000077                 XL2=XL1
   000077                 XL12=XLIOC/A
   000102                 TPI=8.*ATAN(1.)
   000104                 ND=(25.*D)+1.
   000110                 DO 90 J=1,ND
   000114                 S=CMPLX(0.0,TPI*F)
   000117                 A1=RG+R1+S*(XL1+A*XL12)
   000130                 A2=-S*A*XL12
   000134                 A3=-(A*R2+S*(XL12+A*XL2))
   000150                 A4=S*XL12
   000154                 A5=CMPLX(A*RL,0.0)
   000157                 AV=A4/(A1*(1.+A4*A2/(A1*A5)))
   000217                 G(J)=CABS(AV)
   000222                 PWR(J)=EG*EG*G(J)*G(J)/RL
   000226                 IF(PWR(J)-PMAX)15,15,10
   000231              10 PMAX=PWR(J)
   000233                 FM=F
   000235              15 FR(J)=ALOG10(F)
   000241                 WRITE(5,21)F,G(J),PWR(J)
   000253                 F=F*R
   000255              90 CONTINUE
   000257                 WRITE(5,22)PMAX,FM
   000267                 WRITE(5,23)
   000273                 CALL SETPLT(1,FR,G,ND,1H0,11,11HLOG F VS AV)
   000302                 WRITE(5,23)
   000306                 CALL SETPLT(1,FR,PWR,ND,1H*,12,12HLOG F VS PWR)
   000315              19 FORMAT('7ROBERT J. MARKS II'/' ENERGY CONVERSION'/' EE353'///,
                         *7X,'FREQUENCY (HZ)',6X,'VOLTAGE GAIN',8X,'POWER')
   000315              20 FORMAT(F15.8)
   000315              21 FORMAT(5X,E15.8,5X,E15.8,5X,E15.8)
   000315              22 FORMAT(//'  MAXIMUM POWER=',E15.8,' WATTS AT ',E15.8,' HZ')
   000315              23 FORMAT('7')
   000315                 STOP
   000317                 END

   PROGRAM LENGTH INCLUDING I/O BUFFERS
   010455

   UNUSED COMPILER SPACE
   015700
```

ROBERT J. MARKS II
ENERGY CONVERSION
EE353

FREQUENCY (HZ)	VOLTAGE GAIN	POWER
1.00000000E-01	7.85395257E-05	1.23369142E-05
1.09647820E-01	8.61168130E-05	1.48322109E-05
1.20226443E-01	9.44251227E-05	1.78322076E-05
1.31825674E-01	1.03534976E-04	2.14389826E-05
1.44543977E-01	1.13523696E-04	2.57752592E-05
1.58489319E-01	1.24476063E-04	3.09885805E-05
1.73780083E-01	1.36485032E-04	3.72563281E-05
1.90546072E-01	1.49652524E-04	4.47917558E-05
2.08929613E-01	1.64090283E-04	5.38512422E-05
2.29086765E-01	1.79920830E-04	6.47430101E-05
2.51188643E-01	1.97278492E-04	7.78376069E-05
2.75422870E-01	2.16310543E-04	9.35805024E-05
3.01995172E-01	2.37178448E-04	1.12507232E-04
3.31131121E-01	2.60059221E-04	1.35261597E-04
3.63078055E-01	2.85146925E-04	1.62617538E-04
3.98107171E-01	3.12654301E-04	1.95505424E-04
4.36515832E-01	3.42814557E-04	2.35043641E-04
4.78630092E-01	3.75883326E-04	2.82576550E-04
5.24807460E-01	4.12140804E-04	3.39720085E-04
5.75439937E-01	4.51894090E-04	4.08416537E-04
6.30957344E-01	4.95479737E-04	4.91000339E-04
6.91830971E-01	5.43266541E-04	5.90277070E-04
7.58577575E-01	5.95658581E-04	7.09618289E-04
8.31763771E-01	6.53098522E-04	8.53075358E-04
9.12010839E-01	7.16071219E-04	1.02551598E-03
1.00000000E+00	7.85107617E-04	1.23278794E-03
1.09647820E+00	8.60788983E-04	1.48191535E-03
1.20226443E+00	9.43751473E-04	1.78133369E-03
1.31825674E+00	1.03469105E-03	2.14117114E-03
1.44543977E+00	1.13436876E-03	2.57358495E-03
1.58489319E+00	1.24361634E-03	3.09316321E-03
1.73780083E+00	1.36334222E-03	3.71740404E-03
1.90546072E+00	1.49453775E-03	4.46728620E-03
2.08929613E+00	1.63828374E-03	5.36794720E-03
2.29086765E+00	1.79575711E-03	6.44948718E-03
2.51188643E+00	1.96823766E-03	7.74791899E-03
2.75422870E+00	2.15711462E-03	9.30628696E-03
3.01995172E+00	2.36389280E-03	1.11759784E-02
3.31131121E+00	2.59019807E-03	1.34182520E-02
3.63078055E+00	2.83778151E-03	1.61060077E-02
3.98107171E+00	3.10852182E-03	1.93258158E-02
4.36515832E+00	3.40442499E-03	2.31802191E-02
4.78630092E+00	3.72762027E-03	2.77903057E-02
5.24807460E+00	4.08035095E-03	3.32985277E-02
5.75439937E+00	4.46495844E-03	3.98717077E-02
6.30957344E+00	4.88385719E-03	4.77041220E-02
6.91830971E+00	5.33949809E-03	5.70204798E-02
7.58577575E+00	5.83431711E-03	6.80785123E-02
8.31763771E+00	6.37066551E-03	8.11707581E-02
9.12010839E+00	6.95071786E-03	9.66249575E-02
1.00000000E+01	7.57635353E-03	1.14802266E-01
1.09647820E+01	8.24900801E-03	1.36092266E-01
1.20226443E+01	8.96949112E-03	1.60903542E-01
1.31825674E+01	9.73777128E-03	1.89648379E-01
1.44543977E+01	1.05527281E-02	2.22720142E-01
1.58489319E+01	1.14118810E-02	2.60462055E-01
1.73780083E+01	1.23111066E-02	3.03126694E-01
1.90546072E+01	1.32443689E-02	3.50826615E-01

2.08929613E+01	1.42034892E-02	4.03478213E-01
2.29086765E+01	1.51779970E-02	4.60743185E-01
2.51188643E+01	1.61550993E-02	5.21974465E-01
2.75422870E+01	1.71198096E-02	5.86175760E-01
3.01995172E+01	1.80552620E-02	6.51984973E-01
3.31131121E+01	1.89432187E-02	7.17691070E-01
3.63078055E+01	1.97647512E-02	7.81290783E-01
3.98107171E+01	2.05010502E-02	8.40586118E-01
4.36515832E+01	2.11342952E-02	8.93316865E-01
4.78630092E+01	2.16485090E-02	9.37315882E-01
5.24807460E+01	2.20303256E-02	9.70670488E-01
5.75439937E+01	2.22696192E-02	9.91871883E-01
6.30957344E+01	2.23599658E-02	9.99936145E-01
6.91830971E+01	2.22989254E-02	9.94484145E-01
7.58577575E+01	2.20881472E-02	9.75772490E-01
8.31763771E+01	2.17332990E-02	9.44672573E-01
9.12010839E+01	2.12438191E-02	9.02599697E-01
1.00000000E+02	2.06324889E-02	8.51399196E-01
1.09647820E+02	1.99148355E-02	7.93201345E-01
1.20226443E+02	1.91083866E-02	7.30260879E-01
1.31825674E+02	1.82318285E-02	6.64799140E-01
1.44543977E+02	1.73041326E-02	5.98866008E-01
1.58489319E+02	1.63437288E-02	5.34234945E-01
1.73780083E+02	1.53677949E-02	4.72338239E-01
1.90546072E+02	1.43917124E-02	4.14242773E-01
2.08929613E+02	1.34287155E-02	3.60660799E-01
2.29086765E+02	1.24897273E-02	3.11986578E-01
2.51188643E+02	1.15833626E-02	2.68348579E-01
2.75422870E+02	1.07160571E-02	2.29667762E-01
3.01995172E+02	9.89228490E-03	1.95714601E-01
3.31131121E+02	9.11482318E-03	1.66160003E-01
3.63078055E+02	8.38503445E-03	1.40617606E-01
3.98107171E+02	7.70314130E-03	1.18676772E-01
4.36515832E+02	7.06847922E-03	9.99267969E-02
4.78630092E+02	6.47971884E-03	8.39735124E-02
5.24807460E+02	5.93505427E-03	7.04497384E-02
5.75439937E+02	5.43235808E-03	5.90210285E-02
6.30957344E+02	4.96930530E-03	4.93879903E-02
6.91830971E+02	4.54347032E-03	4.12862451E-02
7.58577575E+02	4.15240062E-03	3.44848618E-02
8.31763771E+02	3.79367139E-03	2.87838852E-02
9.12010839E+02	3.46492478E-03	2.40114075E-02
1.00000000E+03	3.16389696E-03	2.00204879E-02

MAXIMUM POWER= 9.99936145E-01 WATTS AT 6.30957344E+01 HZ



LOG F VS AV
.03
.02
.01
.00
10^{-1}
10^{-0}
10^{1}
10^{2}
10^{3}
f (HZ) →
$X_{SHUNT} = \cancel{\emptyset}/SL_{12}$

```
              PROGRAM ENCON (INPUT,OUTPUT,TAPE2=INPUT,TAPE5=OUTPUT)
000002        DIMENSION B(15),H(15),XL(3,15),CUR(3,15),G(3),E(3),CE(3),XCUR(15),
             1XXL(15)
          C    SLL=LENGTH OF SHEET STEEL
          C    CSA=CROSS SECTIONAL AREA
000002        READ(2,10)SSL,CSA,TURNS
000014        BINC=0.1
000015        UO=16.*ATAN(1.)*(10.**(-7.))
          C    READING IN VALUES OF H, AND CORRESPONDING B READ FROM B-H CURVE
000025        DO 90 N=1,15
000026        READ(2,11)H(N)
000033        XN=(N-1)
000035        B(N)=XN*BINC
000040     90 CONTINUE
          C    COMPUTING  XL-I RELATION FOR DIFFERENT GAPS
000042        DO 93 J=1,3
000043        READ(2,11)G(J)
          C    INDIVIDUAL RELATIONS OF XL-I
000050        DO 92 K=1,15
000064        XL(J,K)=TURNS*CSA*B(K)
000065        FS=H(K)*SSL
000066        FG=B(K)*G(J)/UO
          C    CUR=CURRENT
000070        CUR(J,K)=(FS+FG)/TURNS
000072     92 CONTINUE
          C    FINDING CO-ENERGY (CE) AND ENERGY (E) BY TRAPEZOIDAL INTEGRATION
000075        CE(J)=0.0
000075        E(J)=0.0
000076        DO 93 L=1,14
000106        ACE=(CUR(J,L+1)-CUR(J,L))*(XL(J,L+1)+XL(J,L))/2.
000112        CE(J)=CE(J)+ACE
000113        AE=(XL(J,L+1)-XL(J,L))*(CUR(J,L+1)+CUR(J,L))/2.
000117        E(J)=E(J)+AE
000120     93 CONTINUE
          C    ECHOING DATA
000124        WRITE(5,12)SSL,CSA,TURNS
000135        DO 94 M=1,15
000137        WRITE(5,13)B(M),H(M)
000146     94 CONTINUE
          C    PRINTING RESULTS
000150        DO 95 NN=1,3
000152        WRITE(5,14)NN,G(NN)
000161        DO 96 MM=1,15
000163        WRITE(5,15)CUR(NN,MM),XL(NN,MM)
000200        XCUR(MM)=CUR(NN,MM)
000201        XXL(MM)=XL(NN,MM)
000203     96 CONTINUE
000205        WRITE(5,17)E(NN),CE(NN)
000214        WRITE(5,16)
000220        CALL SETPLT(1,XCUR,XXL,15,1H*,8,8HLAM VS I)
000227     95 CONTINUE
000231        CALL EXIT
000232     10 FORMAT(3F10.4)
000232     11 FORMAT(F10.4)
000232     12 FORMAT('7ROBERT J) MARKS II',/,' ENERGY CONVERSION I',/,' DATA',/,
             1' ALL VALUES MKS UNITS',//,' SHEET STEEL LENGTH=',E15.8,/,
             2' CROSS SECTIONAL AREA=',E15.8,/,' TURNS=',E15.8,//,2X,
             3'B',19X,'H',/)

000232     13 FORMAT(2X,E15.8,5X,E15.8)
000232     14 FORMAT('7GAP(',I2,')=',E15.8,//,2X,'CUR',17X,'LAM')
```

```
000232        15 FORMAT(2X,E15.8,5X,E15.8)
000232        16 FORMAT('7')
000232        17 FORMAT(' ENERGY=',E15.8,5X,'CO-ENERGY=',E15.8)
000232           END
```

PROGRAM LENGTH INCLUDING I/O BUFFERS
002717

UNUSED COMPILER SPACE
006000

LOAD MAP.   01/17/72. 18.10.13.   PAGE   1

FL REQUIRED TO LOAD   27100
FL REQUIRED TO RUN   20600
INITIAL TRANSFER TO ENCON   -   103

BLOCK ASSIGNMENTS.

BLOCK	ADDRESS	LENGTH	FILE
ENCON	102	2717	LGO
SCTPLT	3021	5604	SYSLIB
ORDER	10625	33	SYSLIB
ROUND	10660	57	SYSLIB
REDUCE	10737	40	SYSLIB
FACT	10777	71	SYSLIB
/COMPLOT/	11070	133	
PLOT1	11223	342	SYSLIB
PLOT2	11565	162	SYSLIB
FTNP3	11747	224	SYSLIB
FTNP4	12173	276	SYSLIB
COMPSUB	12471	71	SYSLIB
ACGOER	12562	12	SYSLIB
ALNLOG	12574	67	SYSLIB
ATAN	12663	74	SYSLIB
EXP	12757	57	SYSLIB
INPUTC	13036	102	SYSLIB
KODER	13140	1247	SYSLIB
KRAKER	14407	1174	SYSLIB
/SCOPE2/	15603	0	
SYSTEM	15603	1076	SYSLIB
SIO$	16701	1425	SYSLIB
OUTPTC	20326	72	SYSLIB
RBAIEX	20420	41	SYSLIB
GETBA	20461	17	SYSLIB
RBAREX	20500	57	SYSLIB

ROBERT J) MARKS II
ENERGY CONVERSION I
DATA
ALL VALUES MKS UNITS

SHEET STEEL LENGTH= 5.00000000E-01
CROSS SECTIONAL AREA= 1.00000000E-03
TURNS= 1.00000000E+03

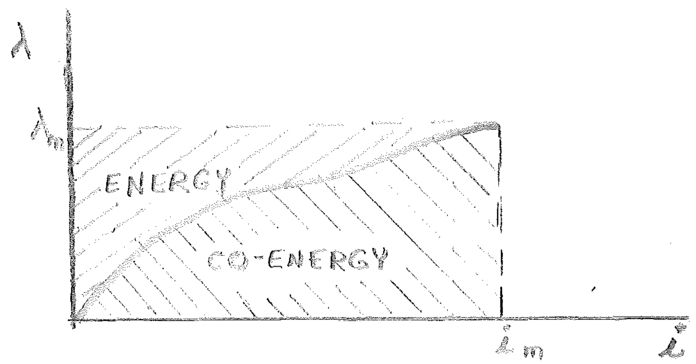
B	H
0.	0.
1.00000000E-01	2.00000000E+01
2.00000000E-01	3.30000000E+01
3.00000000E-01	5.00000000E+01
4.00000000E-01	6.00000000E+01
5.00000000E-01	6.70000000E+01
6.00000000E-01	8.30000000E+01
7.00000000E-01	1.10000000E+02
8.00000000E-01	1.40000000E+02
9.00000000E-01	1.67000000E+02
1.00000000E+00	2.30000000E+02
1.10000000E+00	3.00000000E+02
1.20000000E+00	4.20000000E+02
1.30000000E+00	8.00000000E+02
1.40000000E+00	1.75000000E+03

B-H CO-ORDINATES READ FROM
B-H CURVE FOR SHEET STEEL
ON PAGE 3-22 OF TEXT.

GAP(1)= 5.00000000E-04

CUR	LAM
0.	0.
4.97887358E-02	1.00000000E-01
9.60774715E-02	2.00000000E-01
1.44366207E-01	3.00000000E-01
1.89154943E-01	4.00000000E-01
2.32443679E-01	5.00000000E-01
2.80232415E-01	6.00000000E-01
3.33521150E-01	7.00000000E-01
3.88309886E-01	8.00000000E-01
4.41598622E-01	9.00000000E-01
5.12887358E-01	1.00000000E+00
5.87676094E-01	1.10000000E+00
6.87464829E-01	1.20000000E+00
9.17253565E-01	1.30000000E+00
1.43204230E+00	1.40000000E+00

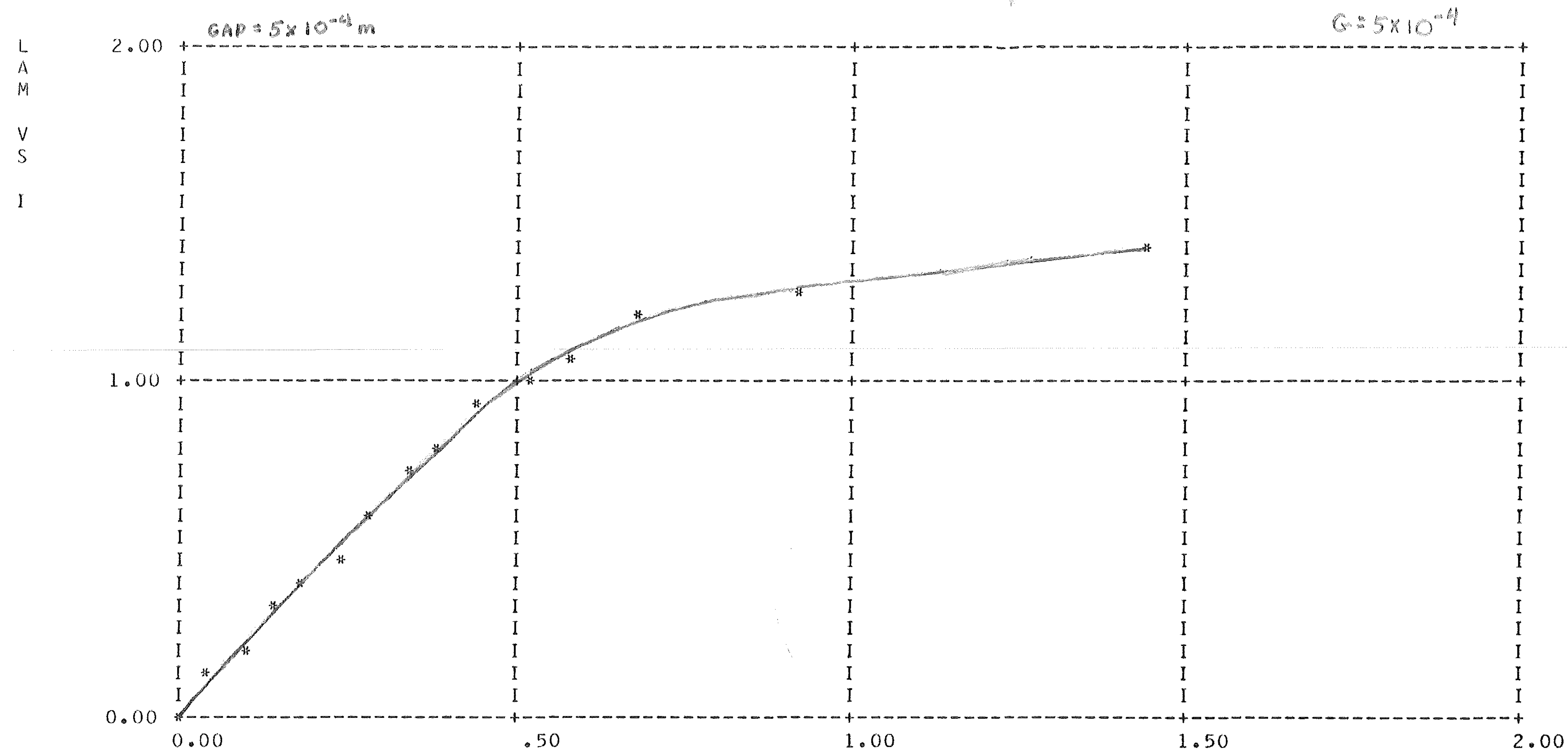
ENERGY= 5.57679611E-01   CO-ENERGY= 1.44717961E+00



$$\text{ENERGY} + \text{CO-ENERGY} = \lambda_m i_m$$

$$(0.58) + (1.45) \stackrel{?}{=} (1.40)(1.43)$$

$$2.03 \cong 2.00$$

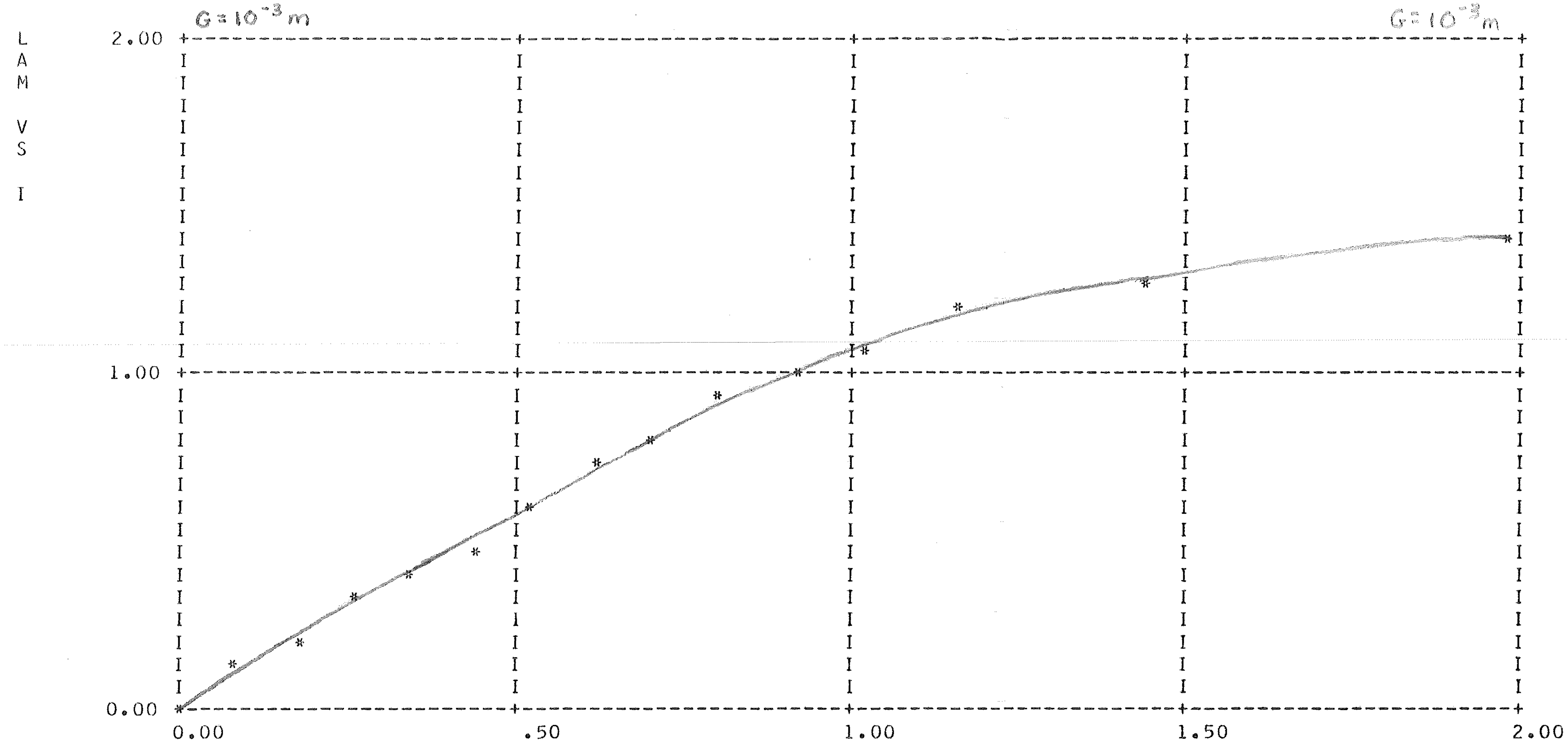

GAP = 5x10^-4 m
G = 5x10^-4
LAM VS I
2.00
1.00
0.00
0.00
.50
1.00
1.50
2.00

```
GAP( 2)= 1.00000000E-03

 CUR                  LAM
  0.                   0.
  8.95774715E-02       1.00000000E-01
  1.75654943E-01       2.00000000E-01
  2.63732415E-01       3.00000000E-01
  3.48309886E-01       4.00000000E-01
  4.31387358E-01       5.00000000E-01
  5.18964829E-01       6.00000000E-01
  6.12042301E-01       7.00000000E-01
  7.06619772E-01       8.00000000E-01
  7.99697244E-01       9.00000000E-01
  9.10774715E-01       1.00000000E+00
  1.02535219E+00       1.10000000E+00
  1.16492966E+00       1.20000000E+00
  1.43450713E+00       1.30000000E+00
  1.98908460E+00       1.40000000E+00
ENERGY= 9.47609221E-01      CO-ENERGY= 1.83710922E+00
```

ENERGY + COENERGY = $\lambda_m i_m$

$(0.95) + (1.84) \stackrel{?}{=} (1.40)(1.99)$

$2.79 = 2.79$



G=10^-3 m
G=10^-3 m
LAM VS I
2.00
1.00
0.00
0.00
.50
1.00
1.50
2.00

GAP(3)= 0.

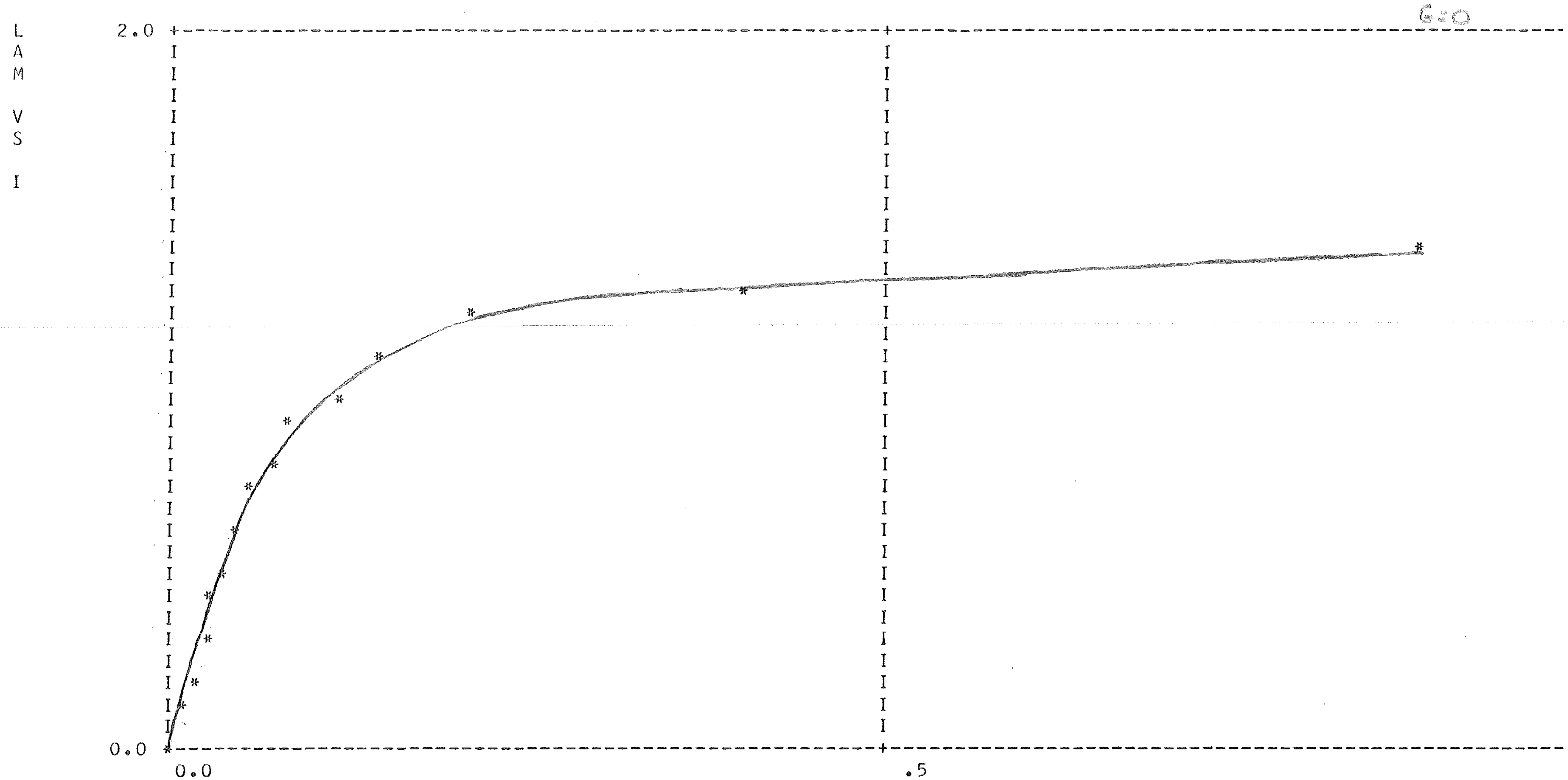
CUR	LAM
0.	0.
1.00000000E-02	1.00000000E-01
1.65000000E-02	2.00000000E-01
2.50000000E-02	3.00000000E-01
3.00000000E-02	4.00000000E-01
3.35000000E-02	5.00000000E-01
4.15000000E-02	6.00000000E-01
5.50000000E-02	7.00000000E-01
7.00000000E-02	8.00000000E-01
8.35000000E-02	9.00000000E-01
1.15000000E-01	1.00000000E+00
1.50000000E-01	1.10000000E+00
2.10000000E-01	1.20000000E+00
4.00000000E-01	1.30000000E+00
8.75000000E-01	1.40000000E+00

ENERGY= 1.67750000E-01 CO-ENERGY= 1.05725000E+00

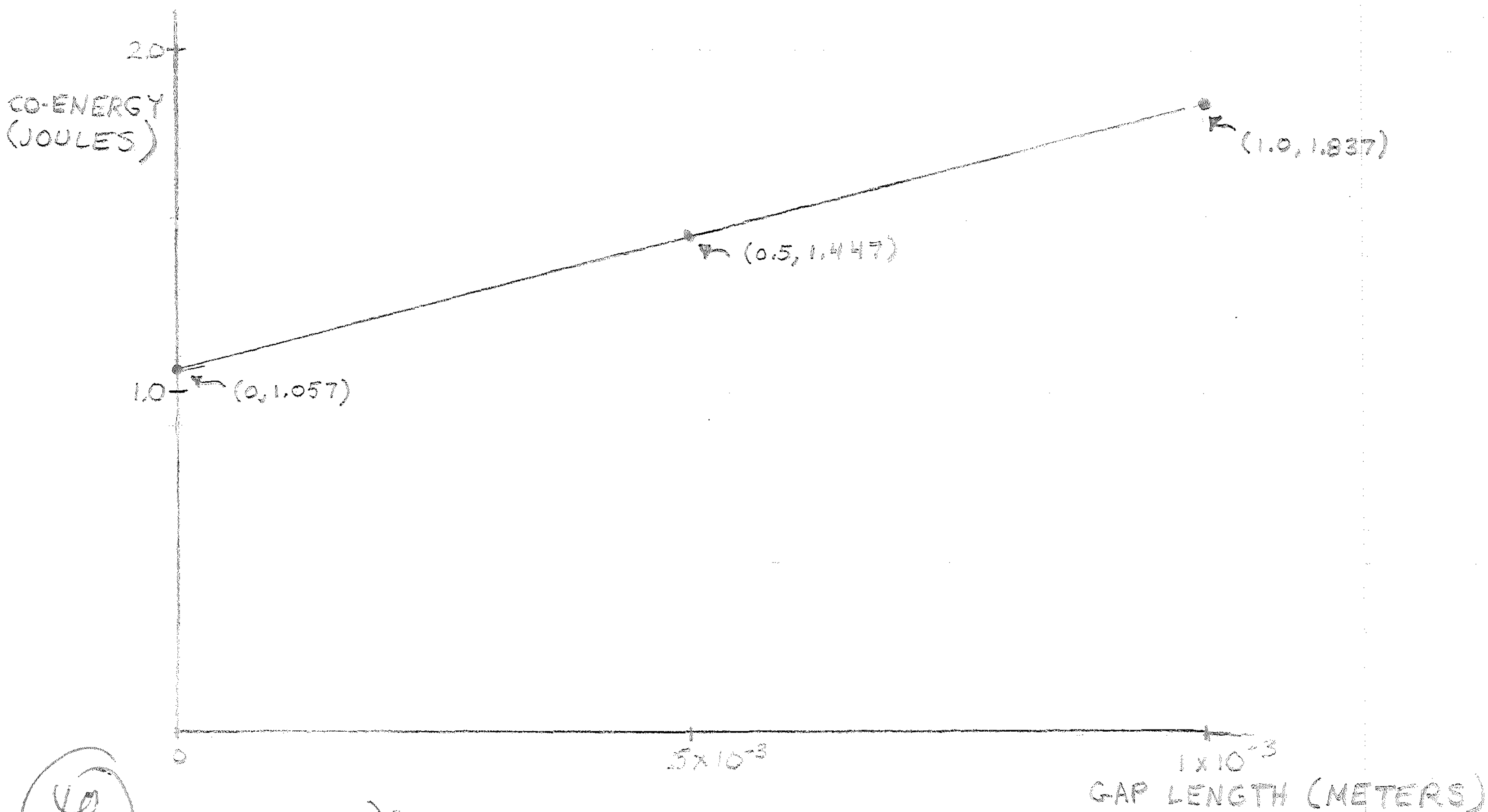
ENERGY + CO-ENERGY $= \lambda_m i_m$

$(0.17) + (1.06) \stackrel{?}{=} (1.40)(.875)$

$1.23 \cong 1.225$



LAM VS I
2.0
0.0
0.0
.5
G=0

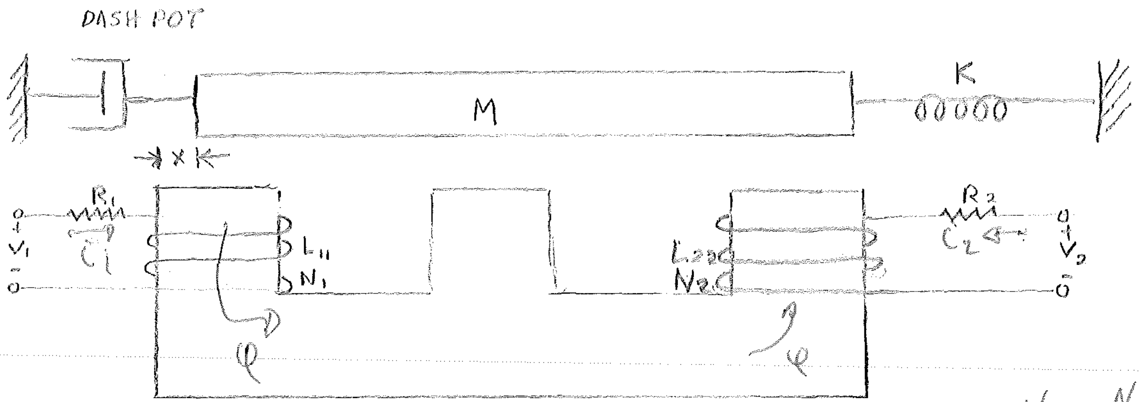


40/50

$$F = \frac{\delta W'}{\delta x} = \cancel{\frac{(1.837-1.057)}{(1.0-0.0)}} \times 10^3 = 780 \text{ nt}$$

Can't use this formula because I initial and I final are not fixed. must use $F \simeq -\frac{\Delta W_m}{\Delta x}$.

2)



$N_1 = N_2$, $R_1 = R_2$

$\therefore a = b = c$.

$$\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} = \begin{bmatrix}L_{11} & -L_{12}\\ -L_{21} & L_{22}\end{bmatrix}\begin{bmatrix}i_1\\ i_2\end{bmatrix} \quad \text{(EQ. 4-91, 4-92)}$$

$$L_{11} = a(\ell^2 - x^2) \ni a = \frac{N_1^2 \mu_0}{2g} \quad \text{(EQ 4-100, 4-101)} \Rightarrow \frac{dL_{11}}{dx} = -2ax$$

$$L_{12} = L_{21} = bx(\ell - x) \ni b = \frac{N_1 N_2 \mu_0}{2g} \quad \text{(EQ 4-104, 4-105)} \Rightarrow \frac{dL_{12}}{dx} = b(\ell - 2x)$$

$$L_{22} = cx(2\ell - x) \ni c = \frac{N_2^2 \mu_0}{2g} \quad \text{(EQ 4-110, 4-111)} \Rightarrow \frac{dL_{22}}{dx} = 2c(\ell - x)$$

$$V_1 = R_1 i_1 + \frac{d\lambda_1}{dt}$$
$$= R_1 i_1 + \frac{d}{dt}\left[L_{11} i_1 - L_{12} i_2\right]$$
$$= R_1 i_1 + \frac{d}{dt}(L_{11} i_1) - \frac{d}{dt}(L_{12} i_2)$$
$$= R_1 i_1 + L_{11}\frac{di_1}{dt} + i_1\frac{dL_{11}}{dt} - L_{12}\frac{di_2}{dt} - i_2\frac{dL_{12}}{dt}$$
$$= R_1 i_1 + L_{11}\frac{di_1}{dt} + i_1\frac{dL_{11}}{dx}\frac{dx}{dt} - L_{12}\frac{di_2}{dt} - i_2\frac{dL_{12}}{dx}\frac{dx}{dt}$$
$$= R_1 i_1 + a(\ell^2 - x^2)\frac{di_1}{dt} - bx(\ell - x)\frac{di_2}{dt} - \frac{dx}{dt}\left[i_2 b(\ell - 2x) + i_1 2ax\right]$$

LET $V_1 = V_{01} + V_{11}$
$i_1 = I_{01} + I_{11}$
$x = X_0 + X_1$
$i_2 = I_{02} + I_{12}$ ETC.

WHERE THE 0 IN THE FIRST DIGIT DESIGNATES A D.C. VALUE, AND 1 AN A-C VALUE

$$V_{01}+V_{11} = R_1 I_{01} + R_1 I_{11} + a\ell^2 \frac{dI_{11}}{dt} - a(x_1+x_0)^2 \frac{dI_{11}}{dt} - b\ell(x_1+x_0)\frac{dI_{12}}{dt} + b(x_1+x_0)^2\frac{dI_{12}}{dt}$$
$$- (I_{02}+I_{12})b\ell\frac{dx_1}{dt} + (I_{02}+I_{12})b\,2(x_0+x_1)\frac{dx_1}{dt} - (I_{01}+I_{11})2a(x_1+x_0)\frac{dx_1}{dt}$$
$$= R_1 I_{01} + R_1 I_{11} + a\ell^2\frac{dI_{11}}{dt} - ax_1^2\frac{dI_{11}}{dt} - 2ax_1x_0\frac{dI_{11}}{dt} - ax_0^2\frac{dI_{11}}{dt}$$
$$- b\ell x_1\frac{dI_{12}}{dt} - b\ell x_0\frac{dI_{12}}{dt} + bx_1^2\frac{dI_{12}}{dt} + 2bx_1x_0\frac{dI_{12}}{dt} + bx_0^2\frac{dI_{12}}{dt}$$
$$- I_{02}b\ell\frac{dx_1}{dt} - I_{12}b\ell\frac{dx_1}{dt} + 2bI_{02}x_0\frac{dx_1}{dt} + 2bI_{02}x_1\frac{dx_1}{dt}$$
$$+ I_{12}2bx_0\frac{dx_1}{dt} + I_{12}2bx_1\frac{dx_1}{dt} - 2aI_{01}x_1\frac{dx_1}{dt} - 2aI_{01}x_0\frac{dx_1}{dt}$$
$$- I_{11}2ax_1\frac{dx_1}{dt} - I_{11}2ax_0\frac{dx_1}{dt}$$

STRIKING OUT TERMS CONTAINING PRODUCTS OF A-C FUNTIONS:

$$V_{01}+V_{11} = R_1 I_{01} + R_{11} I_{11} + a\ell^2\frac{dI_{11}}{dt} - ax_0^2\frac{dI_{11}}{dt} - b\ell x_0\frac{dI_{12}}{dt} + bx_0^2\frac{dI_{12}}{dt}$$
$$- I_{02}b\ell\frac{dx_1}{dt} + 2bI_{02}x_0\frac{dx_1}{dt} - 2aI_{01}x_0\frac{dx_1}{dt}$$

EQUATING A-C AND D-C COMPONENTS:

$$V_{01} = R_1 I_{01}$$
$$V_{11} = R_1 I_{11} + a\ell^2\frac{dI_{11}}{dt} - ax_0^2\frac{dI_{11}}{dt} - b\ell x_0\frac{dI_{12}}{dt} + bx_0^2\frac{dI_{12}}{dt}$$
$$- I_{02}b\ell\frac{dx_1}{dt} + 2bI_{02}x_0\frac{dx_1}{dt} - 2aI_{01}x_0\frac{dx_1}{dt}$$
$$= R_{11}I_{11} + a\left[\ell^2\frac{dI_{11}}{dt} - x_0^2\frac{dI_{11}}{dt} - 2I_{02}\cancel{I_{01}}x_0\frac{dx_1}{dt}\right]$$
$$+ b\left[x_0^2\frac{dI_{12}}{dt} - \ell x_0\frac{dI_{12}}{dt} - I_{02}\ell\frac{dx_1}{dt} + 2I_{02}x_0\frac{dx_1}{dt}\right]$$
$$= R_1 I_{11} + a\left[(\ell^2 - x_0^2)\frac{dI_{11}}{dt} - 2I_{02}x_0\frac{dx_1}{dt}\right]$$
$$+ b\left[(x_0-\ell)x_0\frac{dI_{12}}{dt} + (2x_0-\ell)I_{02}\frac{dx_1}{dt}\right]$$

FROM FIFTH STEP IN DETERMINING V_1, V_2 MAY BE DERIVED BY REPLACING ALL 1 SUB-SCRIPTS WITH 2'S, AND VICE VERSA

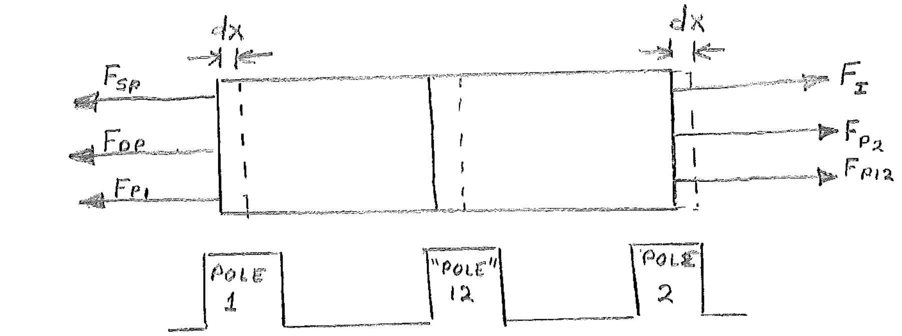
$$V_2 = R_2 i_2 + L_{22}\frac{di_2}{dt} + i_2\frac{dL_{22}}{dx}\frac{dx}{dt} - L_{21}\frac{di_1}{dt} - i_1\frac{dL_{21}}{dx}\frac{dx}{dt}$$
$$= R_2 i_2 + cx(2\ell - x)\frac{di_2}{dt} + i_2 2c(\ell - x)\frac{dx}{dt} - bx(\ell - x)\frac{di_1}{dt} - i_1 b(\ell - 2x)\frac{dx}{dt}$$
$$V_{02}+V_{12} = R_2 I_{02} + R_2 I_{12} + 2\ell c(x_1+x_0)\frac{dI_{12}}{dt} - c(x_1+x_0)^2\frac{dI_{12}}{dt} + 2c\ell(I_{02}+I_{12})\frac{dx_1}{dt}$$
$$- 2c(I_{02}+I_{12})(x_0+x_1)\frac{dx_1}{dt} - b\ell[x+x_0]\frac{dI_{11}}{dt} + b(x+x_0)^2\frac{dI_{11}}{dt}$$
$$- b(I_{01}+I_{11})\ell\frac{dx_1}{dt} + b(I_{01}+I_{11})2(x_0+x_1)\frac{dx_1}{dt}$$

STRIKING OUT TERMS CONTAINING PRODUCTS OF A-C COMPONENTS

$$V_{02}+V_{12} = R_2 I_{02} + R_2 I_{12} + 2\ell c x_0\frac{dI_{12}}{dt} - cx_0^2\frac{dI_{12}}{dt} + 2c\ell I_{02}\frac{dx_1}{dt}$$
$$- 2cI_{02}x_0\frac{dx_1}{dt} - b\ell x_0\frac{dI_{11}}{dt} + bx_0^2\frac{dI_{11}}{dt} - bI_{01}\ell\frac{dx_1}{dt}$$
$$+ 2bI_{01}x_0\frac{dx_1}{dt}$$
$$= R_2 I_{02} + R_2 I_{12} + c\left[2\ell x_0\frac{dI_{12}}{dt} - x_0^2\frac{dI_{12}}{dt} + 2\ell I_{02}\frac{dx_1}{dt} - 2I_{02}x_0\frac{dx_1}{dt}\right]$$
$$+ b\left[x_0^2\frac{dI_{11}}{dt} - \ell x_0\frac{dI_{11}}{dt} - I_{01}\ell\frac{dx_1}{dt} + 2I_{01}x_0\frac{dx_1}{dt}\right]$$
$$= R_2 I_{02} + R_2 I_{12} + c\left[(2\ell - x_0)x_0\frac{dI_{12}}{dt} + (\ell - x_0)2I_{02}\frac{dx_1}{dt}\right]$$
$$+ b\left[(x_0-\ell)x_0\frac{dI_{11}}{dt} + (2x_0-\ell)\frac{dx_1}{dt}\right]$$

$$\therefore V_{02} = R_2 I_{02}$$

FREE BODY OF MASS, (HORIZONAL COMPONENTS)



F_{SP} = FORCE FROM SPRING = kx

F_{DP} = FORCE FROM DASH POT = $b\frac{dx}{dt}$

F_{P1} = FORCE FROM POLE 1, ATTEMPTING TO ALIGN MASS VERTICALLY = $\frac{i_1^2}{2}\frac{dL_1}{dx}$

F_I = INERTIAL FORCE = $M\frac{d^2x}{dt}$

F_{P2} = FORCE FROM POLE 2 = $\frac{i_2^2}{2}\frac{dL_2}{dX}$

F_{P12} = FORCE FROM CENTER POLE = $i_1 i_2\frac{dL_{12}}{dX}$

Can't break up forces in that manner.

$$\Sigma F = 0 = F_I + F_{P2} + F_{P12} - F_{SP} - F_{DP} - F_{P1}$$

$$= M\frac{d^2x}{dt^2} \oplus \frac{i_2^2}{2}\frac{dL_2}{dX} \oplus i_1 i_2\frac{dL_{12}}{dX} - kx - b\frac{dx}{dt} + \frac{i_1^2}{2}\frac{dL_1}{dX}$$

$$= M\frac{d^2x}{dt^2} + \frac{i_2^2}{\cancel{2}}[\cancel{2}c(l-x)] + i_1 i_2 b(l-2x) - kx - b_0\frac{dx}{dt} + \frac{i_1^2}{\cancel{2}}\cancel{2}ax$$

$$= M\frac{d^2x}{dt^2} + i_2^2cl - i_2^2cx + i_1i_2bl - i_1i_2 2x - kx - b_0\frac{dx}{dt} + i_1^2ax$$

DIVIDING INTO COMPONENTS:

$$= M\frac{d^2x_1}{dt^2} + (I_{02}+I_{12})^2cl - (I_{02}+I_{12})^2c(X_0+X_1) + (I_{01}+I_{11})(I_{02}+I_{12})bl$$
$$-(I_{01}+I_{11})2(X_0+X_1) - k(X_0+X_1) - b_0\frac{dX_1}{dt} + (I_{01}+I_{11})^2a(X_1+X_0)$$

STRIKING OUT TERMS CONTAINING PRODUCTS OF A-C COMPONENTS

30/30

$$0 = M\frac{d^2x_1}{dt^2} + clI_{02}^2 + 2cI_{02}I_{12}l - cI_{02}^2X_0 - 2cI_{01}I_{12}X_0 - cI_0^2X_1$$
$$+ blI_{01}I_{02} + blI_{01}I_{12} + I_{11}I_{02}bl - 2I_{01}X_0 - 2I_{01}X_1 + 2I_{11}X_0$$
$$- kX_0 - kX_1 - b_0\frac{dX_1}{dt} + aI_{01}^2X_1 + aI_{01}^2X_0 + 2aI_{01}I_{11}X_0$$

$$= M\frac{d^2x_1}{dt^2} + c\left[lI_{02}^2 + 2lI_{02}I_{12} - I_{02}^2X_0 - 2I_{01}I_{12}X_0 - I_0^2X_1\right]$$
$$+ b\left[lI_{01}I_{02} + lI_{01}I_{12} + I_{11}I_{02}l\right]$$
$$+ a\left[I_{01}^2X_1 + I_{01}^2X_0 + 2I_{01}I_{11}X_0\right] - 2I_{01}X_0 - 2I_{01}X_1 + 2I_{11}X_0$$
$$- kX_0 - kX_1 - b_0\frac{dX_1}{dt}$$

D.C. VALUES:

$$0 = clI_{02}^2 - cI_{02}^2X_0 + blI_{01}I_{02} + aI_{01}^2X_0 - 2I_{01}X_0 - kX_0$$

A.C. VALUES:

$$0 = M\frac{d^2x_1}{dt^2} + c\left[2lI_{02}I_{12} - 2I_{01}I_{12}X_0 - I_0^2X_1\right]$$
$$+ bl\left[I_{01}I_{12} + I_{11}I_{02}\right] + a\left[I_{01}^2X_1 + 2I_{01}I_{11}X_0\right]$$
$$- 2I_{01}X_1 + 2I_{11}X_0 - b_0\frac{dX_1}{dt} - kX_1$$

Right idea, but you made a lot of work for yourself by not using symmetry of structure!

3) $L = L_0 + L_2 \cos 4\alpha + L_6 \cos 12\alpha$

$\alpha = \omega_m t + \delta$

$\Rightarrow L = L_0 + L_2 \cos 4(\omega_m t + \delta) + L_6 \cos 12(\omega_m t + \delta)$

$\frac{dL}{dt} = -4\omega_m [L_2 \sin 4(\omega_m t + \delta) + 3L_6 \sin 12(\omega_m t + \delta)]$

$i(t) = I_m \sin \omega t \Rightarrow i^2(t) = I_m^2 \sin^2 \omega t$

$= \frac{I_m^2}{2}[1 - \cos 2\omega t]$

$$\tau(t) = \frac{-i^2(t)}{2} \frac{dL(t)}{dt}$$

$$= \frac{-1}{2}\left[\frac{I_m^2}{2}(1 - \cos 2\omega t)\right]\left[-4\omega_m \{L_2 \sin 4(\omega_m t + \delta) + 3L_6 \sin 12(\omega_m t + \delta)\right]$$

$$= \omega_m I_m^2 (1 - \cos 2\omega t)[L_2 \sin 4(\omega_m t + \delta) + 3L_6 \sin 12(\omega_m t + \delta)]$$

$$= \omega_m I_m^2 [L_2 \sin 4(\omega_m t + \delta) + 3L_6 \sin 12(\omega_m t + \delta) - L_2 \cos 2\omega t \sin 4(\omega_m t + \delta) - 3L_6 \cos 2\omega t \sin 12(\omega_m t + \delta)]$$

$$= \omega_m I_m^2 [L_2 \sin 4\omega_m t \cos 4\delta + L_2 \cos 4\omega_m t \sin 4\delta + 3L_6 \sin 12\omega_m t \cos 12\delta + 3L_6 \cos 12\omega_m t \sin 12\delta - L_2 \cos 2\omega t \sin 4\omega_m t \cos 4\delta - L_2 \cos 2\omega t \cos 4\omega_m t \sin 4\delta - 3L_6 \cos 2\omega t \sin 12\omega_m t \cos 12\delta - 3L_6 \cos 2\omega t \cos 12\omega_m t \sin 12\delta]$$

10/10

THE FREQUENCIES PRESENT IN THE ABOVE EXPRESSION FOR TORQUE ARE $4\omega_m$, $12\omega_m$, $|4\omega_m - 2\omega|$, $4\omega_m + 2\omega$, 2ω, $|12\omega_m - 2\omega|$, AND $12\omega_m + 2\omega$. ANY FUNCTION WHICH CAN BE EXPRESSED AS A SERIES OF IMPULSES IN THE FREQUENCY DOMAIN, (AS $\tau(t)$ DOES), MUST BE PERIODIC. LET T_τ BE THE PERIOD OF THE TORQUE WAVE. THUS, THE CORRESPONDING PERIOD OF EACH FREQUENCY COMPONENT ($T_c = 2\pi/\omega_c$), IS EVENLY DIVISIBLE INTO T_τ (ie $T_\tau / T_c = I_c \ni I_c \in$ NON-NEGATIVE INTEGER)

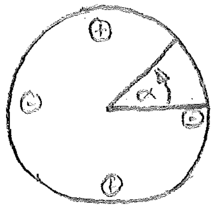
FOR $\tau(t)$ TO HAVE A NON-ZERO VALUE, ITS INTEGRAL OVER T_τ MUST HAVE A NON-ZERO VALUE. ALL PURE SINUSOID TERMS IN $\tau(t)$ WILL YIELD 0 D.C. TORQUE OVER T_τ, IN THAT ANY SINUSOID'S D.C. VALUE OVER A NON-NEGATIVE INTEGER MULTIPLE OF ITS PERIOD IS ZERO. THE TERMS EXPRESSED AS THE PRODUCT OF TWO SINUSOIDS WILL YIELD A D.C. VALUE IFF THEY ARE ORTHOGONAL.

THUS:

$2\omega = 4\omega_m \Rightarrow \omega_m = \frac{1}{2}\omega$

OR $2\omega = 12\omega_m \Rightarrow \omega_m = \frac{1}{6}\omega$

FOR THE MACHINE TO SUPPLY AN AVERAGE TORQUE.



THE RELATIONSHIP BETWEEN ELECTRIC AND MECHANICAL ANGLES IS

$$\psi_e = P\alpha,$$

WHERE P IS THE NUMBER OF POLE PAIRS. DIFFERENTIATING BOTH SIDES:

$$\omega = \frac{d\psi_e}{dt} = P\omega_m$$

IT HAS BEEN ESTABLISHED THAT $\omega_m = \frac{1}{a}\omega$, WHERE $a = 2$ OR 6, IF THE SYSTEM IS EXPECTED TO YEILD ANY AVERAGE TORQUE.

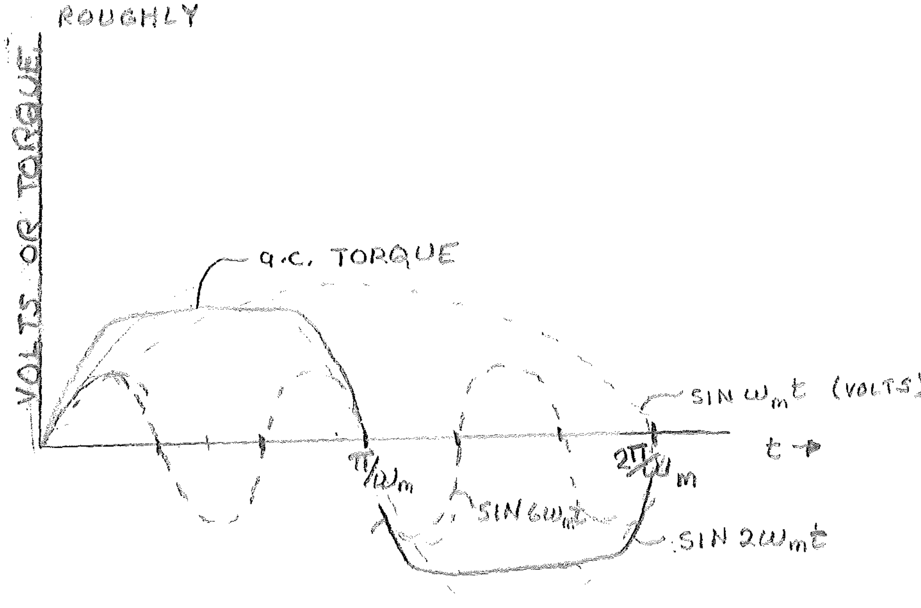
THUS

$$\omega = P\omega_m$$
$$= P\left(\tfrac{1}{a}\omega\right)$$

$$\Rightarrow P = a$$

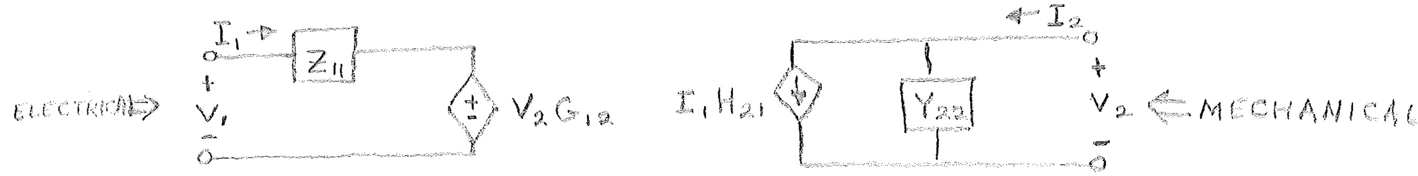
ERGO, THE SYSTEM MUST HAVE EITHER 2 OR 6 POLE PAIRS IN ORDER FOR ANY AVERAGE TORQUE TO BE DELIVERED. AS IT STANDS, THE SYSTEM HAS 2 POLE PAIRS, AVERAGE TORQUE BEING SUPPLIED BY THE $\omega_m = 2\omega$ TERM, AND IT'S THIRD HARMONIC (6ω)

ROUGHLY



4) $V_1 = I_1 Z_{11} + V_2 G_{12}$ $\qquad V_2 = \alpha \dot{x}_2$

$I_2 = I_1 H_{21} + V_2 Y_{22}$ $\qquad I_2 = F_2/\alpha$



THE TWO EQUATIONS REPRESENT THE TWO PORT NETWORK ILLUSTRATED. THE REMAINING TWO EQUATIONS FOR V_2 AND I_2 SUGGEST AN EQUIVALENT ELECTRO-MECHANICAL CIRCUIT, IN ELECTRICAL UNITS (ie, V_2 HAS UNITS OF VOLTS, I_2 UNITS OF AMPS, ETC.).

$I_2 = F_2/\alpha \Rightarrow \alpha$ HAS UNITS OF $\frac{\text{FORCE}}{\text{AMP}} = \text{FORCE}\left(\frac{\text{TIME}}{\text{CHARGE}}\right)$

$$\left[V_2 = \alpha \dot{X} \Rightarrow \text{VOLTAGE} = \left(\frac{\text{FORCE}\cdot\text{TIME}}{\text{CHARGE}}\right)\left(\frac{\text{DISTANCE}}{\text{TIME}}\right)\right.$$
$$= \left(\frac{\text{FORCE}}{\text{CHARGE}}\right)\text{DISTANCE}$$
$$\left. = \text{ENERGY}\cdot\text{DISTANCE} = \text{VOLTAGE}\right]$$

BY SHORTING OUT V_2 TERMINAL (THUS HOLDING $\dot{X}$, THE VELOCITY OF THE MECHANICAL SIDE, ZERO), Z_{11} AND H_{21} MAY BE MEASURED AS FOLLOWS:

$$V_1 = Z_{11} I_1\Big|_{V_2=0} \Rightarrow Z_{11} = \frac{V_1}{I_1}\Big|_{V_2=0} = Z_e$$

$$I_2 = I_1 H_{12}\Big|_{V_2=0} \Rightarrow H_{12} = \frac{I_2}{I_1}\Big|_{V_2=0} = \frac{1}{\alpha}\left(\frac{F_2}{I_1}\right)\Big|_{V_2=0} = \frac{1}{\alpha} H_m$$

BY OPENING THE ELECRICAL TERMINALS, FORCING $I_1 = 0$, G_{12} AND Y_{22} MAY BE MEASURED AS FOLLOWS,

$$V_1 = V_2 G_{12}\Big|_{I_1=0} \Rightarrow G_{12} = \frac{V_1}{V_2}\Big|_{I_1=0} = \frac{1}{\alpha}\left(\frac{V_1}{\dot{X}_2}\right)\Big|_{I_1=0} = \frac{1}{\alpha} G_e$$

$$I_2 = Y_{22} V_2\Big|_{I_1=0} \Rightarrow Y_{22} = \frac{I_2}{V_2}\Big|_{I_1=0} = \frac{1}{\alpha^2}\left(\frac{F_2}{\dot{X}_2}\right)\Big|_{I_1=0} = \frac{1}{\alpha^2} Y_M$$

THUS

$Z_{11} = Z_e \qquad \ni Z_e = \left(\frac{V_1}{I_1}\right)\Big|_{V_2=\dot{x}=0}$

$H_{21} = \frac{1}{\alpha} H_m \qquad \ni H_m = \left(\frac{F_2}{I_1}\right)\Big|_{V_2=\dot{x}=0}$

$G_{12} = \frac{1}{\alpha} G_e \qquad \ni G_e = \left(\frac{V_1}{\dot{X}_2}\right)\Big|_{I_1=0}$

$Y_{22} = \frac{1}{\alpha^2} Y_m \qquad \ni Y_m = \left(\frac{F_2}{\dot{X}_2}\right)\Big|_{I_1=0}$

20/20

CLEARLY Z_e, H_m, G_e, AND Y_M MAY BE COMPUTED EXPERIMENTALLY IN THE SYSTEM DESCRIBED. GENERALLY, EACH WILL CONTAIN RESISTIVE, AS WELL AS REACTIVE COMPONENTS. THUS MEASUREMENTS OF AMPLITUDE AND PHASE RELATIONSHIP SHOULD BE MADE IN A FREQUENCY RESPONSE ANALYSIS OVER THE SPECTRUM OF INTEREST.

EXPRESSING ORIGINAL TWO-PORT EQUATIONS AGAIN:

1) $V_1 = I_1 Z_{11} + V_2 G_{21}$

$= I_1 Z_e + (\alpha \dot{X}_2)(\frac{1}{\alpha} G_e)$

$= I_1 Z_e + \dot{X}_2 G_e$

2) $I_2 = I_1 H_{21} + V_2 Y_{22}$

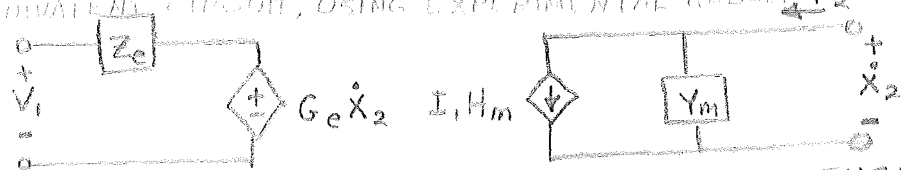
$(F_2/_\alpha) = I_1 (H_m/_\alpha) + (\alpha \dot{X})(\frac{Y_m}{\alpha^2})$

$F_2 = I_1 H_m + \dot{X} Y_m$

AGAIN:

$$\begin{Bmatrix} V_1 = I_1 Z_e + \dot{X}_2 G_e \\ F_2 = I_1 H_m + \dot{X}_2 Y_m \end{Bmatrix}$$

EQUIVALENT CIRCUIT, USING EXPERIMENTAL RESULTS



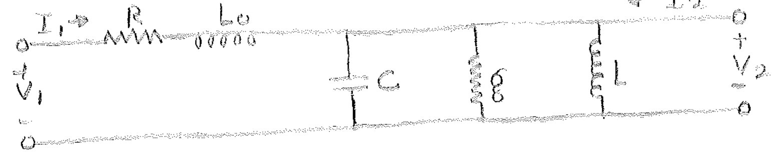
NOW THE LEFT PORT HAS ELECTRICAL DIMENSIONS, AND THE RIGHT PORT MECHANICAL DIMENSIONS, AS OPPOSED TO THE INITIAL CASE WHERE BOTH PORTS BOASTED OF ELECTRICAL DIMENSIONS.

G_e, EXPERIMENTALLY MEASURED, IS A CONVERSION FACTOR OF SORTS. IT CONVERTS THE MECHANICAL UNIT OF VELOCITY INTO THE ELECTRICAL UNIT OF VOLTS. IN THAT $G_{12} = \frac{1}{\alpha} G_e$, AND G_{12} IS UNITLESS, G_e HAS UNITS OF α (= VOLT/VELOCITY). SIMILARLY, THE OTHER "CONVERSION FACTOR", H_m: ELECTRICAL TO MECHANICAL (CURRENT TO VELOCITY) HAS DIMENSIONS OF α (= VELOCITY/AMP = VOLT/VELOCITY)

Z_e IS THE ELECTRICAL IMPEDANCE.

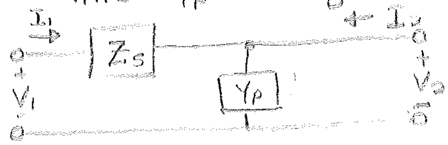
Y_M IS THE MECHANICAL ADMITTANCE. IN THE ELECTROMECHANICAL FORCE-CURRENT ANALOGY, R MIGHT BE ANALAGOUS TO THE (MECHANICAL DAMPING FACTOR)$^{-1}$, C TO MASS, $1/L$ TO SPRING CONSTANT, ECT.

5) CIRCUIT:



LET $Z_S = R + sL_0$

AND $Y_P = sC + g + \frac{1}{sL}$



THUS $V_1 = Z_S I_1 + V_2$

$I_2 = -I_1 + Y_P V_2$

HYBRID EQUATIONS:

$V_1 = Z_{11} I_1 + G_{12} V_2$

$I_2 = H_{21} I_1 + Y_{22} V_2$

$\therefore Z_{11}(s) = Z_s(s) = R + sL_0 = \frac{V_1}{I_1}\Big|_{V_2=0}$

$G_{12}(s) = 1 = \frac{V_1}{V_2}\Big|_{I_1=0}$

$H_{21}(s) = -1 = \frac{I_2}{I_1}\Big|_{V_2=0}$

$Y_{22}(s) = g + sC + \frac{1}{sL} = \frac{I_2}{V_2}\Big|_{I_1=0}$

$\frac{20}{20}$

6) $V_1 = I_1 Z_{11} + V_2 G_{12}$
$I_2 = I_1 H_{21} + V_2 Y_{22}$

FOR A CONSERVATIVE TRANSDUCER:

$$P_{AVE} = \frac{1}{2} Re\left[V_1 I_1^* + V_2 I_2^*\right] = 0$$
$$= \frac{1}{2} Re\left[V_1 I_1^*\right] + \frac{1}{2} Re\left[V_2 I_2^*\right]$$
$$= \frac{1}{2} Re\left[V_1 I_1^*\right] + \frac{1}{2} Re\left[V_2 I_2^*\right]^*$$
$$= \frac{1}{2} Re\left[V_1 I_1^*\right] + \frac{1}{2} Re\left[V_2^* I_2\right] \Leftarrow (XY^*)^* = (X^* Y)$$
$$= \frac{1}{2} Re\left[V_1 I_1^* + V_2^* I_2\right]$$
$$= \frac{1}{2} Re\left[(I_1 Z_{11}) I_1^* + (V_2 G_{12}) I_1^* + (I_1 H_{21}) V_2^* + (V_2 Y_{22}) V_2^*\right] = 0$$

SHORTING SECOND PORT OF CONSERVATIVE TRANSDUCER $\Rightarrow V_2 = V_2^* = 0$

$$\Rightarrow P_{AVE}\Big|_{V_2=0} = \frac{1}{2} Re\left[Z_{11} I_1 I_1^*\right]$$
$$= \frac{1}{2} Re\left[Z_{11} I_1^2\right]$$

THE QUANTITY I_1^2 IS PURE REAL, THUS FOR $P_{AVE}\Big|_{V_2=0} = 0$, Z_{11} MUST BE PURE IMAGINARY, YIELDING A PURE IMAGINARY $Z_{11} I_1^2$ TERM, THE REAL COMPONENT OF WHICH IS OF COURSE ZERO.
IN THAT Z_{11} HAS UNITS OF IMPEDANCE, ITS IMAGINARY COMPONENT WILL BE A REACTANCE jX_1.

$$\Rightarrow Z_{11} = jX_1$$

OPEN CIRCUITING FIRST PORT OF THE CONSERVATIVE TRANSDUCER $\Rightarrow I_1 = I_1^* = 0$

$$\Rightarrow P_{AVE}\Big|_{I_1=0} = \frac{1}{2} Re\left[Y_{22} V_2 V_2^*\right]$$
$$= \frac{1}{2} Re\left[Y_{22} V_2^2\right]$$

BY THE SAME ARGUMENT, ADMITTANCE Y_{22} MUST CONSIST OF A PURELY SUSCEPTIVE COMPONENT:

$$\Rightarrow Y_{22} = jB_2$$

THE POWER EQUATION THUS BECOMES:

$$P_{AVE} = \frac{1}{2} Re\left[G_{12}(V_2 I_1^*) + H_{21}(V_2^* I_1)\right] = 0$$
$$\Rightarrow Re\, H_{21}(V_2^* I_1) = -Re\, G_{12}(V_2 I_1^*)$$
$$= -Re\, G_{12}(V_2^* I_1)^*$$

LET $H_{21} = a + jb$
$G_{12} = c + jd$
$V_2^* I_1 = e + jf$

$$\Rightarrow (a+jb)(e+jf) = -(c+jd)(e-jf)$$
$$(ae-bf) + j(be+fa) = -(ce+df) + j(de-cf)$$

EQUATING REAL AND IMAGINARY:

$$\left.\begin{aligned} ae - bf &= -ce - df \\ be + fa &= de + cf \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} a &= -c \\ b &= d \end{aligned}\right.$$

$$\therefore H_{21} = -G_{12}^*$$

What does this mean in terms of the elements making up the system?

$\frac{20}{20}$

7) THE A-C ELECTRO-MECHANICAL EQUATIONS DESCRING SYSTEM OF PROBLEM (11-1) WERE FOUND TO BE

$$e_1(t) = R\,i(t) + L_0(1 + a x_0^2)\frac{d i_1(t)}{dt} + 2 a L_0 I_0 x_0 \frac{d x_1(t)}{dt}$$

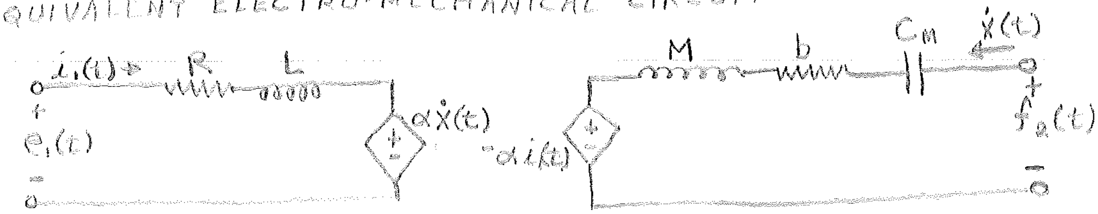
$$f_1(t) = M\frac{d^2 x_1(t)}{dt} + b\frac{d x_1(t)}{dt} + (k_s - a L_0 I_0^2)\,x_1(t) - 2 a L_0 I_0 x_0\, i_1(t)$$

LETTING $\alpha = 2 a L_0 I_0 x_0$

$L = L_0(1 + a x_0^2)$

$C_M = (k_s - a L_0 I_0^2)^{-1}$

THE EQUIVALENT ELECTRO-MECHANICAL CIRCUIT WOULD BE



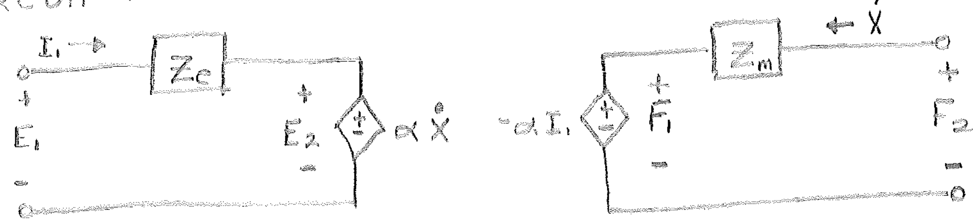
ALSO LET: $Z_e = R + SL$

$Z_M = SM + b + \frac{1}{C_m S}$

AND $e_2 = \alpha \dot{x}$

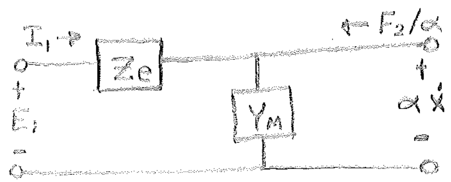
$f_1 = -\alpha i_1$

THE CIRCUIT THUS BECOMES (IN LAPLACE DOMAIN)



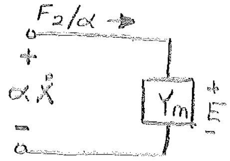
NOW: $\frac{E_2}{I_1} = \frac{-\alpha^2 \dot{X}}{+F_1} = \frac{-\alpha^2}{F_1/\dot{X}} \Rightarrow \frac{I_1}{E_2} = \frac{F_1/\dot{X}}{-\alpha^2} = \frac{-1}{\alpha^2}\left(\frac{F_2 - Z_m \dot{X}}{\dot{X}}\right) = \frac{Z_m}{\alpha^2} - \frac{F_2/\alpha}{\dot{X}\alpha}$

THE CIRCUIT THEN BECOMES:



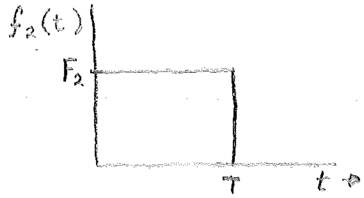
WHERE $Y_m = Z_m/\alpha^2$

OPENING TERMINALS ON ELECTRICAL PORT $\Rightarrow I_1 = 0$.
$\therefore Z_e$ CONTRIBUTES NO DROP. THUS, THE CIRCUIT BECOMES:



$$\Rightarrow E_1 = \frac{F_2}{\alpha Y_m} = \frac{F_2 \alpha^2}{\alpha Z_m} = \frac{F_2 \alpha}{Z_m}$$

GIVEN



$$\Rightarrow \mathcal{F}_2(s) = \mathcal{L}\{f_2(t)\} = \frac{A}{s}(1 - e^{-sT})$$

(COOPER/McGILLEM
METHODS OF SIGNAL AND SYSTEM ANALYSIS
Pg 158)

NOW $$E_1(s) = \frac{\alpha \mathcal{F}_2(s)}{Z_m(s)} = \frac{\alpha F_2}{s}\left[\frac{1 - e^{-sT}}{sM + b + 1/C_m s}\right]$$

$$= \alpha F_2 \left[\frac{1 - e^{-st}}{s^2 M + bs + 1/C_m}\right]$$

$$= \alpha F_2 \left[\frac{1}{s^2 M + bs + 1/C_m} - \frac{e^{-sT}}{s^2 M + bs + 1/C_m}\right]$$

e^{-sT} CORRESPONDS TO A TIME SHIFT OF T IN THE TIME DOMAIN, THUS FINDING THE ROOTS OF THE QUADRATIC $s^2M + bs + 1/C_m$ ARE OF PRESENT CONCERN

```
 88888888    00000000    66666666    99999999
8888888888  0000000000  6666666666  9999999999
88      88  00      00  66      66  99      99
88      88  00      00  66          99      99
88      88  00      00  66          99      99
 88888888   00      00  666666666   9999999999
 88888888   00      00  6666666666   999999999
88      88  00      00  66      66          99
88      88  00      00  66      66          99
88      88  00      00  66      66  99      99
8888888888  0000000000  6666666666  9999999999
 88888888    00000000    66666666    99999999

// JOB T      8069      ROBERT J. MARKS II
*LIMITS T285.
// NOTE PLEASE DO NOT FOLD OJTPUT-ROBERT J. MARKS II-8069-
// FOR
*IOCS(1403 PRINTER,CARD)
*LIST SOURCE PROGRAM
      REAL M,KS,LO,IO
      DIMENSION C(3),COF(3),RR(2),RI(2)
      READ(2,50)R,LO,M,XO,IO,B,A,KS
      ALPHA=2.*A*LO*IO*XO
      WRITE(5,48)
      WRITE(5,49)ALPHA
      CM=1./(KS-A*LO*IO*IO)
      C(1)=1./CM
      C(2)=B
      C(3)=M
      CALL POLRT(C,COF,2,RR,RI,IER)
      DO 20 J=1,3
      WRITE(5,51)J,C(J)
   20 CONTINUE
      DO 21 K=1,2
      WRITE(5,52)K,RR(K),RI(K)
   21 CONTINUE
      STOP
   48 FORMAT('1')
   49 FORMAT(' ALPHA=',E15.8)
   50 FORMAT(8F5.3)
   51 FORMAT(' C(',I2,')=',E15.8)
   52 FORMAT(' RT(',I2,')=',E15.8,' REAL, ',E15.8,' IMAG')
      END

FEATURES SUPPORTED
 IOCS

CORE REQUIREMENTS FOR
 COMMON      0  VARIABLES      50  PROGRAM     216

END OF COMPILATION
// LOAD
```

```
ALPHA=-0.19999998E-02
C( 1)= 0.50000109E 03
C( 2)= 0.10000001E-01
C( 3)= 0.10000000E 00
RT( 1)=-0.50000004E-01 REAL, -0.70710739E 02 IMAG
RT( 2)=-0.50000004E-01 REAL,  0.70710739E 02 IMAG
S 32 STOP 0000

 EXECUTION TIME 0002
```

THUS $s^2M + bs + 1/C_m = (s-R)(s-R^*) (= s^2 + R^2)$

WHERE $R = -0.05 - j70.71 = a + jb_1$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2M + bs + 1/C_m}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-R)(s-R^*)}\right\}$$

$$= \frac{1}{R-R^*}\left(e^{Rt} - e^{R^*t}\right)$$ $\Rightarrow$ CRC, MATH TABLES, 16TH EDITION, Pg 492, #12

$$= \frac{1}{j2b_1}\left[e^{at}\left(e^{jb_1t} - e^{-jb_1t}\right)\right]$$

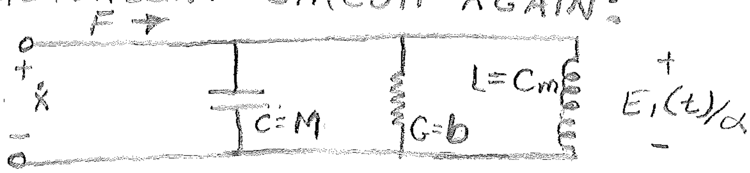
$$= \frac{1}{j2b_1}\left[e^{at}(j2\sin b_1t)\right]$$

$$= \frac{1}{b_1}e^{at}\sin b_1t$$

$$\Rightarrow e_1(t) = \frac{\alpha F_2}{b_1}\left[e^{at}\sin b_1t\,\mu(t) - e^{a(t-T)}\sin b(t-T)\,\mu(t-T)\right]$$

$$\therefore\ e_1(t) = (2.82\times 10^{-6})F_2\left[e^{-0.05t}\sin(+70.7)t\,\mu(t) - e^{-0.05(t-T)}\sin\{(+70.7)(t-T)\}\,\mu(t-T)\right]$$

THE EQUIVALENT CIRCUIT AGAIN:



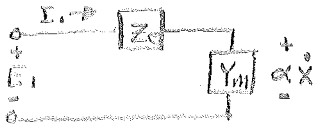
THE RESONANT FREQUENCY OF THE L-C TANK

$$\omega_r = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{MC_m}} = \left(\frac{M}{K_s - aL_0I_0^2}\right)^{\frac{1}{2}} = (5000)^{\frac{1}{2}} = 70.7$$

WHICH IS THE ANGULAR FREQUENY OF THE SINUSOIDS IN THE RESULTANT EXPRESSION FOR $e_1(t)$.

25/25

8) HAVING NO MECHANICAL LOAD WILL REDUCE THE CIRCUIT DERIVED IN PROBLEM 7 TO THE FOLLOWING:



THUS $\alpha\dot{x} = \frac{I_1}{Y_m} = \frac{I_1\alpha^2}{Z_m} \Rightarrow \dot{x} = \frac{\alpha I_1}{Z_m}$

ALSO $E_1 = \left(Z_e + \frac{1}{Y_m}\right) I_1 = \left(Z_e + \frac{\alpha^2}{Z_m}\right) I_1 \Rightarrow I_1 = E_1\left(Z_e + \frac{\alpha^2}{Z_m}\right)^{-1}$

THUS $\dot{x} = \frac{\alpha I_1}{Z_m} = \frac{\alpha E_1}{Z_m(Z_e + \alpha^2/Z_m)} = \frac{\alpha E_1}{Z_m Z_e + \alpha^2}$.

AGAIN $E_1(s) = \frac{E}{s}(1 - e^{-sT})$

$$\Rightarrow \dot{x}(s) = \frac{\alpha E_1}{s}\left[\frac{1 - e^{-sT}}{(sM + b + \frac{1}{C_m s})(R + sL) + \alpha^2}\right]$$

$$= \frac{\alpha E_1}{s}\left[\frac{1 - e^{-st}}{RsM + Rb + R/C_m s + s^2 LM + sLb + L/C_m + \alpha^2}\right]$$

$$= \alpha E_1\left[\frac{1 - e^{-sT}}{s^3(LM) + s^2(Lb + RM) + s(Rb + \frac{L}{C_m} + \alpha^2) + R/C_m}\right]$$

$$= \alpha E_1\left[\frac{1 - e^{-sT}}{As^3 + Bs^2 + Cs + D}\right]$$

$A = LM = ML_0(1 + aX_0^2)$
$B = (Lb + RM) = bL_0(1 + aX_0^2) + RM$
$C = Rb + \frac{L}{C_m} + \alpha^2 = Rb + L_0(1 + aX_0^2)(k_s - aL_0 I_0^2) + (2aL_0 I_0 X_0)^2$
$D = R(k_s - aL_0 I_0^2)$

SOLVING FOR ROOTS OF THE CUBIC $\longrightarrow$

```
 88888888    00000000    66666666    99999999
8888888888  0000000000  6666666666  9999999999
88      88  00      00  66      66  99      99
88      88  00      00  66          99      99
88      88  00      00  66          99      99
 88888888   00      00  666666666   9999999999
 88888888   00      00  6666666666   999999999
88      88  00      00  66      66          99
88      88  00      00  66      66          99
88      88  00      00  66      66  99      99
8888888888  0000000000  6666666666  9999999999
 88888888    00000000    66666666    99999999

// JOB T       8069       ROBERT J. MARKS II
*LIMITS T293.
// NOTE PLEASE DO NOT FOLD OUTPUT-ROBERT J. MARKS II-8069
// FOR
*IOCS(1403 PRINTER,CARD)
*LIST SOURCE PROGRAM
      DIMENSION C(4),COF(4),RR(3),RI(3)
      WRITE (5,12)
      READ(2,14)R,XLO,XM,XO,XIO,B,A,XKS
      XL=XLO*(1.+A*XO*XO)
      C(4)=XM*XL
      C(3)=B*XL+R*XM
      C(2)=R*B+XL*(XKS-A*XLO*XIO*XIO)+((2.*A*XLO*XIO*XO)**2.)
      C(1)=R*(XKS-A*XLO*XIO*XIO)
      CALL POLRT(C,COF,3,RR,RI,IER)
      DO 21 M=1,4
      WRITE(5,13)M,C(M)
   21 CONTINUE
      DO 20 J=1,3
      WRITE(5,15)J,RR(J),RI(J)
   20 CONTINUE
   12 FORMAT('1')
   13 FORMAT(' C(',I2,')=',E15.8)
   14 FORMAT(8F5.3)
   15 FORMAT(' ROOT ',I2,'=',E15.8,' REAL,',E15.8,' IMAG')
      STOP
      END

FEATURES SUPPORTED
 IOCS

CORE REQUIREMENTS FOR
 COMMON       0  VARIABLES      60  PROGRAM     252

END OF COMPILATION
// LOAD
```

```
C( 1)= 0.50000101E 05
C( 2)= 0.50050097E 03
C( 3)= 0.10009990E 02
C( 4)= 0.99899992E-01
ROOT  1=-0.50002090E-01 REAL, 0.70710739E 02 IMAG
ROOT  2=-0.50002090E-01 REAL,-0.70710739E 02 IMAG
ROOT  3=-0.10010011E 03 REAL, 0.00000000E 00 IMAG
S 32 STOP 0000

 EXECUTION TIME 0111
```

THUS $AS^3 + BS^2 + CS + D = (S - R_r)(S - R_c)(S - R_c^*)$

FROM CRC, Pg 492, #14:

$$\mathcal{L}^{-1}\left\{\frac{1}{(S-R_r)(S-R_c)(S-R_c^*)}\right\} = \frac{(R_c - R_c^*)e^{R_r t} + (R_c^* - R_r)e^{R_c t} + (R_r - R_c)e^{R_c^* t}}{(R_c - R_r)(R_c - R_c^*)(R_c^* - R_r)}$$

LET $R_c = a + jb$ AND $R_r = c$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(S-R_r)(S-R_c)(S-R_c^*)}\right\} = \frac{j2be^{ct} + [(a-c) - jb]e^{(a+jb)t} - [(a-c) + jb]e^{(a-jb)t}}{j2b[(a-c)+jb][(a-c)-jb]}$$

$$= \frac{j2be^{ct} + (a-c)e^{(a+jb)t} - jbe^{(a+jb)t} - (a-c)e^{(a-jb)t} - jbe^{(a-jb)t}}{j2b[(a-c)^2 + b^2]}$$

$$= \frac{j2be^{ct} + e^{at}[(a-c)(e^{jbt} - e^{-jbt}) - jb(e^{jbt} + e^{-jbt})]}{j2b[(a-c)^2 + b^2]}$$

$$= \frac{e^{ct} + e^{at}\left[\left(\frac{a-c}{b}\right)\left(\frac{e^{jbt} - e^{-jbt}}{j2}\right) - \left(\frac{e^{jbt} + e^{-jbt}}{2}\right)\right]}{[(a-c)^2 + b^2]}$$

$$= \frac{e^{ct} + e^{at}\left[\left(\frac{a-c}{b}\right)\sin bt - \cos bt\right]}{[(a-c)^2 + b^2]}$$

THUS

$$\dot{x}(t) = \mathcal{L}^{-1}\left\{\alpha E_1\left[\frac{1 - e^{-sT}}{(S-R_r)(S-R_c)(S-R_c^*)}\right]\right\} = \frac{\alpha E_1}{[(a-c)^2+b^2]}\Big[\left\{e^{ct} + e^{at}\left(\frac{a-c}{b}\sin bt - \cos bt\right)\right\}\mu(t)$$
$$- \left\{e^{c(t-T)} + e^{a(t-T)}\left(\frac{a-c}{b}\sin b(t-T) - \cos b(t-T)\right)\right\}\mu(t-T)\Big]$$

$$\dot{X}(t) = (-1.33\times10^{-8})E_1\Big[\{e^{-100t} + e^{-.05t}(+1.41\sin(+70.7)t$$
$$- \cos(70.7)t)\}\mu(t) - \{e^{-100(t-T)}$$
$$+ e^{-0.05(t-T)}(1.41\sin((70.7)(t-T))$$
$$- \cos((70.7)(t-T))\mu(t-T)\Big]$$

O.K.

$\frac{25}{25}$

THUS $AS^3 + BS^2 + CS + D = (S - R_r)(S - R_c)(S - R_c^*)$

FROM CRC, Pg 492, #14:

$$\mathcal{L}^{-1}\left\{\frac{1}{(S-R_r)(S-R_c)(S-R_c^*)}\right\} = \frac{(R_c - R_c^*)e^{R_r t} + (R_c^* - R_r)e^{R_c t} + (R_r - R_c)e^{R_c^* t}}{(R_c - R_r)(R_c - R_c^*)(R_c^* - R_r)}$$

LET $R_c = a + jb$ AND $R_r = c$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(S-R_r)(S-R_c)(S-R_c^*)}\right\} = \frac{j2be^{ct} + [(a-c) - jb]e^{(a+jb)t} - [(a-c) + jb]e^{(a-jb)t}}{j2b[(a-c)+jb][(a-c)-jb]}$$

$$= \frac{j2be^{ct} + (a-c)e^{(a+jb)t} - jbe^{(a+jb)t} - (a-c)e^{(a-jb)t} - jbe^{(a-jb)t}}{j2b[(a-c)^2 + b^2]}$$

$$= \frac{j2be^{ct} + e^{at}\left[(a-c)(e^{jbt} - e^{-jbt}) - jb(e^{jbt} + e^{-jbt})\right]}{j2b[(a-c)^2 + b^2]}$$

$$= \frac{e^{ct} + e^{at}\left[\left(\frac{a-c}{b}\right)\left(\frac{e^{jbt} - e^{-jbt}}{j2}\right) - \left(\frac{e^{jbt} + e^{-jbt}}{2}\right)\right]}{[(a-c)^2 + b^2]}$$

$$= \frac{e^{ct} + e^{at}\left[\left(\frac{a-c}{b}\right)\sin bt - \cos bt\right]}{[(a-c)^2 + b^2]}$$

THUS

$$\dot{X}(t) = \mathcal{L}^{-1}\left\{\alpha E_1\left[\frac{1 - e^{-sT}}{(S-R_r)(S-R_c)(S-R_c^*)}\right]\right\} = \frac{\alpha E_1}{[(a-c)^2 + b^2]}\left[\left\{e^{ct} + e^{at}\left(\frac{a-c}{b}\sin bt - \cos bt\right)\right\}\mu(t) - \left\{e^{c(t-T)} + e^{a(t-T)}\left(\frac{a-c}{b}\sin b(t-T) - \cos b(t-T)\right)\right\}\mu(t-T)\right]$$

$$\dot{X}(t) = (-1.33 \times 10^{-8})E_1\left[\left\{e^{-100t} + e^{-.05t}\left(+1.41\sin(+70.7)t - \cos(70.7)t\right)\right\}\mu(t) - \left\{e^{-100(t-T)} + e^{-0.05(t-T)}\left(1.41\sin((70.7)(t-T)) - \cos((70.7)(t-T)\right)\right\}\mu(t-T)\right]$$

O.K.

25/25

Solutions of Mid-Term Exam, En. Conv.

1. Energy: $\boxed{\begin{matrix}0.61 \text{ joule} \\ (0.05\text{ cm})\end{matrix}}$, $\boxed{\begin{matrix}0.99 \text{ joule} \\ (0.1\text{ cm})\end{matrix}}$, $\boxed{\begin{matrix}0.189 \text{ joule} \\ (0\text{ cm})\end{matrix}}$.

Force: $\text{Force} \simeq -\left.\frac{\Delta W_m}{\Delta x}\right|_{x=0.05} = -\frac{0.99-0.61}{(0.1-0.05)(0.01)} = \boxed{-760 \text{ Newton}}$

$$\text{Force} \simeq -\left.\frac{\Delta W_m}{\Delta x}\right|_{x=0} = -\frac{0.61-0.189}{(0.05-0)(0.01)} = \boxed{-842 \text{ Newton}}$$

2. According to Fig. 8-15 and the corresponding discussion

$L_{aa} = a(l^2 - x^2)$, $l \geq x \geq 0$, $a = N^2 \frac{\mu_0}{2g}$. $(a=b=c)$.

$L_{ab} = -bx(l-x)$, $l \geq x \geq 0$, $b = N^2 \frac{\mu_0}{2g}$ (the # of turns on each coil is equal).

$L_{bb} = cx(2l-x)$, $l \geq x \geq 0$, $c = \frac{N^2 \mu_0}{2g}$.

Note that the negative sign on L_{12} follows because the <u>assumed</u> positive direction for $i_a(t)$, $i_b(t)$ in Fig. P 11-13, p 11-90 produce <u>bucking</u> fluxes.

According to (8-147):

$$f = \frac{\partial W_m'}{\partial x} = \frac{i_a^2}{2}\frac{dL_{aa}}{dx} + i_a i_b \frac{dL_{ab}}{dx} + \frac{i_b^2}{2}\frac{dL_{bb}}{dx}$$

$$= \frac{i_a^2}{2}(-2ax) + i_a i_b(-al + 2ax) + \frac{i_b^2}{2}(2al - 2ax)$$

Equation of dynamics: $f = -i_a^2 ax + a(2x-l)i_a i_b + a i_b^2 (l-x)$.

$$= M\frac{d^2x}{dt^2} + D\frac{dx}{dt} + Kx.$$

$$V_a = i_a R + \frac{d}{dt}(L_{aa} i_a + L_{ab} i_b)$$

$$V_b = i_b R + \frac{d}{dt}(L_{ab} i_a + L_{bb} i_b)$$

$$V_0 = I_{a_0} R \,, \quad V_0 = I_{b_0} R \Rightarrow I_{a_0} = I_{b_0} = I_0 = V_0/R.$$

The electromechanical force term under d.c. conditions, then becomes

$$f = -I_0^2 a X_0 + a(2X_0 - l) I_0^2 + a I_0^2 (l - X_0) \equiv 0.$$

$\therefore \quad 0 = K X_0 \Rightarrow \boxed{X_0 = 0.}$ This result follows from Fig. 11-?, for

because, with bucking fluxes, the armature will not produce more flux per current by moving from the position $X=0$. If, however, the windings were wound as in Fig. 4-15 with a positive mutual inductance (flux aiding for positive currents) then $L_{ab} = +bx(l-x) = (+)ax(l-x)$, and the middle term above in the force expression would have the opposite sign,

$$f = -I_0^2 a X_0 - a(2X_0 - l) I_0^2 + a I_0^2 (l - X_0) = -4aX_0 I_0^2 + 2al I_0^2,$$

which means that the d.c. force balance becomes

$$-4aX_0 I_0^2 + 2al I_0^2 = K X_0 \Rightarrow X_0 = \frac{2al I_0^2}{K + 4aI_0^2},$$

Now, if $K = 0$ (no spring), we see that the quiescent point is $X_0 = l/2$, as we would have anticipated on the basis of the symmetrical system ...

We continue with the negative mutual example. After some algebra, utilizing the facts that $I_{a_0} = I_{b_0} = I_0$, $X_0 = 0$, we find for the a.c. equations

$$v_{a_1}(t) = i_{a_1} R + al^2 \frac{di_{a_1}}{dt} - al I_0 \frac{dx_1}{dt}.$$

$$v_{b_1}(t) = i_{b_1} R + al I_0 \frac{dx_1}{dt}$$ (note no self-inductance term for i_{b_1}, because ...

$$al I_0 \left(i_{b_1}(t) - i_{a_1}(t)\right) = M \frac{d^2 x_1}{dt^2} + D \frac{dx_1}{dt} + K x_1$$

... Note the differential input force. We finally note that $L_{aa} \equiv al^2$ for $x \geq 0$ where l is the length of the bar, $L_{bb} \equiv 0$ for $-l < x < 0$, $L_{ab} \equiv 0$ for $x \leq 0$. These results follow from the idealized magnetic circuit model used in ... the expressions for L_{aa}, L_{bb} and L_{ab} in the first place.

$$= -i^2 \left[2L_2 \sin(\omega_m t + \delta) + 6 L_6 \sin(\omega_m t + \delta) \right].$$

$i = I_m \sin \omega t$, $\quad i^2 = I_m^2 \sin^2 \omega t = \frac{I_m^2}{2}(1 - \cos 2\omega t)$.

$$\therefore \; T_e = -I_m^2 \Big[L_2 \sin^4(\omega_m t + \delta) + 3L_6 \sin^{12}(\omega_m t + \delta)$$

$$- L_2 \cos 2\omega t \sin^4(\omega_m t + \delta) - 3L_6 \cos 2\omega t \cdot \sin^{12}(\omega_m t + \delta) \Big\}.$$

$$= -I_m^2 \Big(L_2 \sin^4(\omega_m t + \delta) + 3L_6 \sin^{12}(\omega_m t + \delta)$$

$$- \frac{L_2}{2} \left\{ \sin\left[(4\omega_m + 2\omega)t + 4\delta\right] + \sin\left[(4\omega_m - 2\omega)t + 4\delta\right] \right\}$$

$$- \frac{3L_6}{2} \left\{ \sin\left[(12\omega_m + 2\omega)t + 12\delta\right] + \sin\left[(12\omega_m - 2\omega)t + 12\delta\right] \right\} \Big]$$

ω_m and ω are both positive. Thus, average torque is developed when

$$\boxed{\omega_m = \frac{\omega}{2}, \; \frac{\omega}{6}}$$

In the first one the machine acts as a 4-pole (2 pairs of poles) machine, while in the second it behaves as an 12-pole (6 pairs of poles) machine.

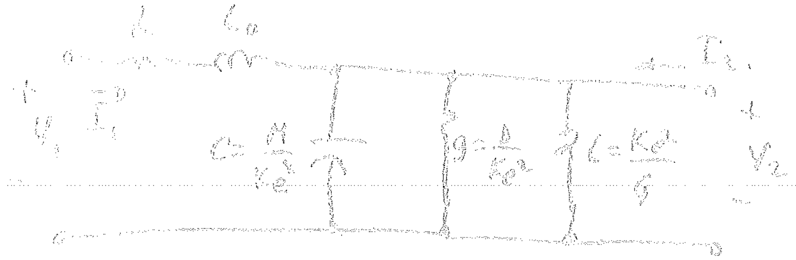
$\underline{z_{11}}$ is the driving point impedance of port-1 measured with port-2 $\underline{\text{short-circuited}}$ ($V_2 = 0$) not open-circuited as is conventional.

$\underline{G_{12}}$ is the voltage transfer ratio between ports two and one and is measured with $I_1 = 0$ (open-circuit at port #1)

$\underline{H_{21}}$ is the current transfer ratio between ports 1 and 2 and is measured with $V_2 = 0$ (short-circuit at port #2).

$\underline{Y_{22}}$ is the driving point admittance of port-2 and is measured with $I_1 = 0$ (open-circuit at port #1).

In measuring "I_2", we actually measure the force f_2 on the element and divide by α.



(a) z_{11}: short out port-2. Then, clearly, $\boxed{z_{11} = R + sL_0}$

(b) G_{12}: open port-1. Then, clearly, $\boxed{G_{12} = 1}$

(c) H_{21}: Short circuit port-2: Then, clearly, $\boxed{H_{21} = 1}$

(d) Y_{22}: open port-1. Then, clearly, $\boxed{Y_{22} = sC + g + \frac{1}{sL}}$

(e) $V_1 = I_1 z_{11} + V_2 G_{12} \Rightarrow V_1 I_1^* = |I_1|^2 z_{11} + I_1^* V_2 G_{12}$

$I_2 = I_1 H_{21} + V_2 Y_{22} \Rightarrow V_2 I_2^* = V_2 I_1^* H_{21}^* + |V_2|^2 Y_{22}^*$

$\frac{1}{2} \mathrm{Re}\{V_1 I_1^* + V_2 I_2^*\} = \frac{1}{2} \mathrm{Re}\{|I_1|^2 z_{11} + |V_2|^2 Y_{22}^* + V_2 I_1^* (G_{12} + H_{21}^*)\}$

This must hold for all V_2 and I_1. Hence, assume that $V_2 = 0$, i.e., that port-2 is short circuited, then $\frac{1}{2}\mathrm{Re}\{|I_1|^2 z_{11}\} = \frac{|I_1|^2}{2} \mathrm{Re}\{z_{11}\} = 0$

Hence, $\boxed{z_{11} = jX_1}$ is purely reactive (X_1 is real, either positive or negative).

Next, assume that $I_1 = 0$, i.e., that port-1 is open circuited. Then $\frac{1}{2} \mathrm{Re}\{|V_2|^2 Y_{22}^*\} = \frac{|V_2|^2}{2} \mathrm{Re}\{Y_{22}^*\} = 0$ or $\boxed{Y_{22} = jB_2}$ is purely reactive (B_2 is real, either positive or negative). Finally, assume that $V_2 = 1$ volt (phase angle $= 0$) and $I_1 = 1$ ampere (phase angle $= 0$). Then
[illegible] $= \frac{1}{2} \mathrm{Re}\{G_{12} + H_{21}^*\} = 0$. (the real part of [illegible] are already [illegible])

[illegible]

Then $\frac{1}{2}\,\mathrm{Re}\left\{|I_1|^2 Z_{11} + |V_2|^2 Y_{22}^* + V_2 I_1^* (G_{12} + H_{21}^*)\right\}$

$= \frac{1}{2}\,\mathrm{Re}\left\{1 \cdot jX_1 + 1 \cdot (-jB_2) - j\,(G_{12} + H_{21}^*)\right\}$.

$= \frac{1}{2}\,\mathrm{Re}\left\{-j\,(G_{12} + H_{21}^*)\right\}$

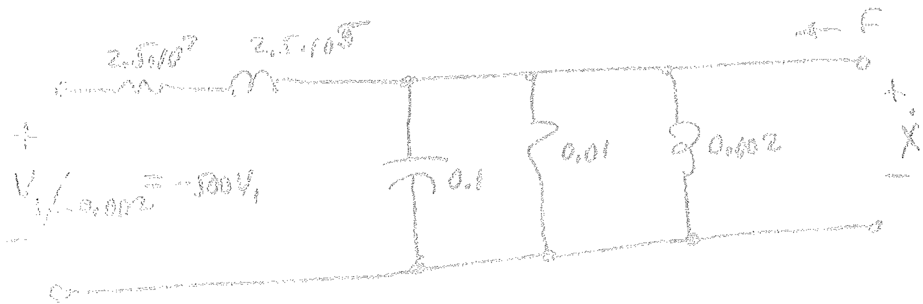
$= \frac{1}{2}\,\mathrm{Im}\left[(G_{12} + H_{21}^*)\right] = 0$.

Hence, we have $\mathrm{Re}\left[G_{12} + H_{21}^*\right] = \mathrm{Im}\left[G_{12} + H_{21}^*\right] = 0$. Thus,

$G_{12} + H_{21}^* = 0$ (because both real and imaginary parts are

or $\boxed{G_{12}^* = -H_{21}}$ G_{12} and H_{21} cannot have real parts, which implies an entirely reactive network.

(7). Equivalent circuit (viewed from mechanical side).



$K_e = \pi a\, C_0\, I_0 X_0 = -0.002$

By simple circuit theory

$-500V_1 = \dot{X} = \dfrac{F(s)}{0.1s + 0.01 + \dfrac{1}{0.002s}}$

$F(s) = \dfrac{F_0}{s}\left(1 - e^{-sT}\right)$.

$= \dfrac{10s\,F(s)}{s^2 + 0.1s + 5000}$

Now $s^2 + 0.1s + 5000 = (s + 0.05 + j\,70.7)(s + 0.05 - j\,70.7)$

$= \dfrac{10F_0\left(1 - e^{-sT}\right)}{s^2 + 0.1s + 5000}$

Thus $\quad -500\, V_1(S) = \dot{X}_2(S) = \dfrac{10\, F_2 (1 - e^{-ST})}{(S+0.05+j70.7)(S+0.05-j70.7)}$

$$= j\frac{F_2(0.0707)}{S+0.05+j70.7} - j\frac{F_2(0.0707)}{S+0.05-j70.7}$$

$$- j\frac{F_2(0.0707)e^{-ST}}{S+0.05+j70.7} + j\frac{F_2(0.0707)e^{-ST}}{S+0.05-j70.7}$$

$\therefore \quad -500\, v_1(t) = -(0.1414) F_2\, e^{-0.05t} \sin 70.7t + (0.1414) F_2\, e^{-0.05(t-T)} \sin \ldots\, u_{-1} \ldots$

where $u_{-1}(t-T)$ is the unit step-function at $t = T$.

and $\boxed{i_1(t) = 0.283 \cdot 10^{-3} F_2 \left[e^{-0.05t} \sin 70.7t - e^{-0.05(t-T)} \sin 70.7(t-T) \cdot \ldots \right.}$

We have a slightly decaying oscillation with $\omega = 70.7$.

(8). Using the same circuit as in (7):

$$\dot{X}(S) = \frac{\dfrac{1}{0.1S + 0.01 + \dfrac{1}{0.0025}} \cdot (-500\, V_1(S))}{2.5 \cdot 10^7 + 2.5 \cdot 10^5 S + \dfrac{1}{0.1S + 0.01 + \dfrac{1}{0.0025}}}$$

$$= \frac{-V_1(S)}{50S^3 + 5000S^2 + 2.5\cdot 10^5 S + 2.5 \cdot 10^7}$$

$$= \frac{-0.02\, V_1(S)}{S^3 + 100S^2 + 5000S + 500{,}000}$$

Now we must find roots of $S^3 + 100S^2 + 5000S + 500{,}000$.

[illegible]

Hence

$$
\begin{array}{r}
s^2 \qquad + 5000 \qquad\qquad \\
s+100 \,\overline{\big)\, s^3 + 100s^2 + 5000s + 500{,}000} \\
\underline{s^3 + 100s^2 \qquad\qquad\qquad\qquad} \\
5000s + 500{,}000 \\
\underline{5000s + 500{,}000} \\
0
\end{array}
$$

Hence

$$s^3 + 100s^2 + 5000s + 500{,}000$$

$$= (s+100)(s^2+5000) = (s+100)(\ldots$$

and

$$\dot{X}(s) = \frac{-0.02\, E_1 (1 - e^{-sT})}{s(s+100)(s+j70.7)(s-j70.7)}$$

So that

$$\dot{x}(t) = -2\cdot 10^{-8} E_1 \Big\{ \Big[2 - 0.67 e^{-100t} - 2\sqrt{\tfrac{2}{3}} \cos(70.7t - \tan^{-1} \ldots$$

$$- \Big[2 - 0.67 e^{-100(t-T)} - 2\sqrt{\tfrac{2}{3}} \cos(70.7(t-T) - \tan^{-1} \ldots$$

$$\cdot\, u_{-1}(t-T)$$

It is a simple matter to verify that $\dot{x}(0) = 0$, and this makes sense because if we excite the circuit on p. 7 of these solutions with a step voltage input at port #1 (the left-hand port) the entire voltage drop, initially, will be across the $2.5\cdot 10^5$ henry inductor, thus reducing to zero the "voltage" at port #2 at the initial instant.

An important observation to make regarding probs. (7) and (8) is that [illegible] can be used as a force to voltage (generator) transducer or, reciprocally, as a voltage to motion (motor) transducer.

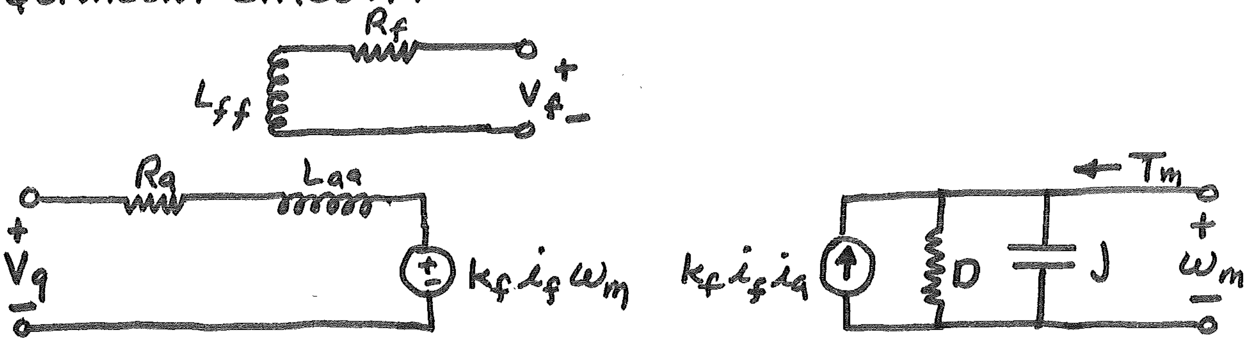
BOB MARKS

5

PROBLEM: A SHUNT MOTOR WITH THE FOLLOWING CONSTANTS WITH AN A-C VOLTAGE: $340 \sin 377t$. FIND THE AVERAGE TORQUE. REPEAT, ASSUMING D-C. EXCITATION @ 240 V (RMS OF 340 V)

$R_f = 50$, $L_{ff} = 25$, $k_f = 0.9$, $R_a = 0.1$; $L_{qq} = .01$, $J_m = 10$ kg-m^2

EQUIVALENT CIRCUIT:



FOR SHUNT: $V_q = V_f = V$, $\omega_m = 0$

D.C. $I_f = V_f / R_f$; $I_q = \frac{V_q}{R_a} = \frac{240}{.1} = 2.4$ kA (RMS)

$= \frac{240}{50} = 4.8$ A (RMS) $\boxed{I_f = \frac{344}{j9429} = -j\,0.036}$

$\tau = k_f i_f i_a = (.9)(4.8)(2400) = 1.03 \times 10^4$ nt.-m.

AC: $V_q = i_q (R_a + j\omega L_{aa}) \Rightarrow i_q = \frac{V_q}{R_a + j\omega L_{aa}}$

$|i_q| = \frac{|V_q|}{(R_a^2 + \omega^2 L_{aa}^2)^{1/2}} = \frac{240}{(377 \cdot .01)} = \frac{240}{3.77} = 62.4$ A (RMS)

$|i_f| = \frac{|V_f|}{|R_f + j\omega L_f|} = \frac{240}{(250 + (377)^2(25)^2)^{1/2}} = .027$ A (RMS)

$\Rightarrow \tau_{AC} = K_f i_q i_f =$ ~~nt~~ 1.48 N-M

BOB MARKS

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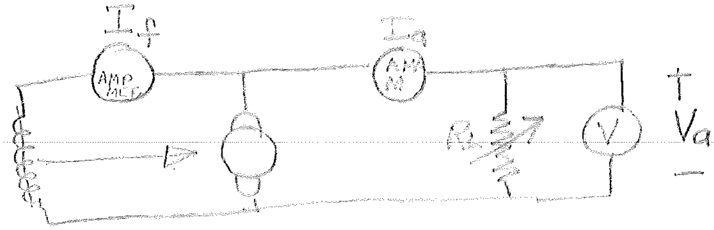
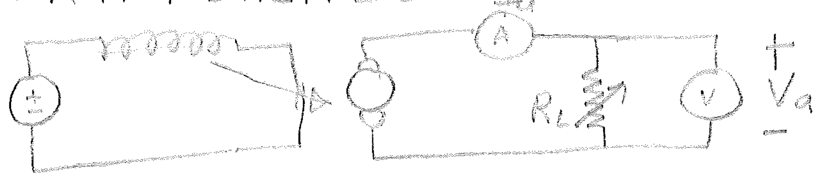
LAB PARTNER: BOB SHAW

1. CIRUITS USED

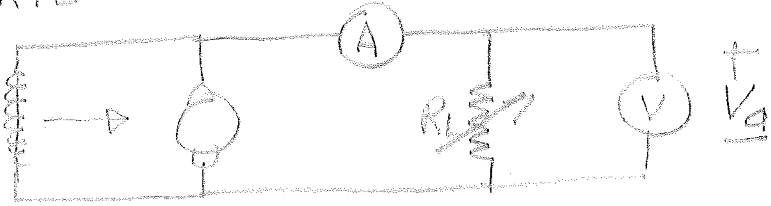
a) SELF EXCITED



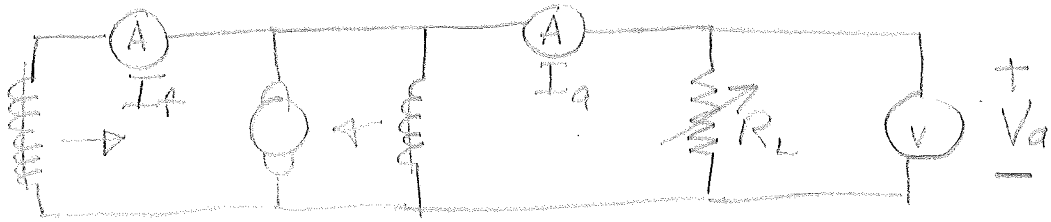
b) SEPERATLY EXCITED



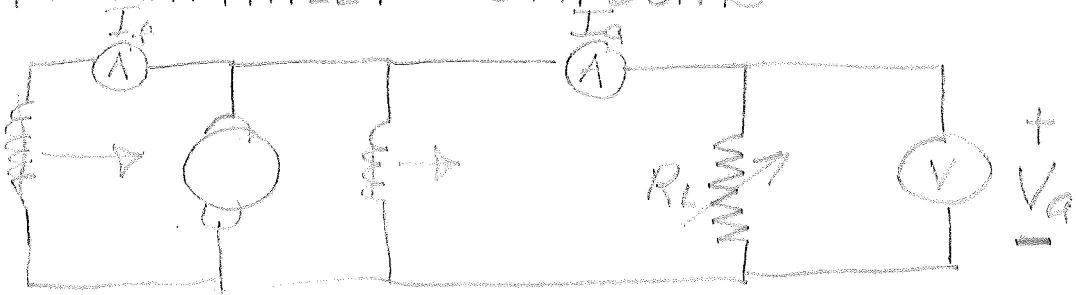
c) SERIES



D) CUMMULATIVE COMPOUND GENERATOR



E) DIFFERENTIALLY COMPOUND



2) FOR A SEPARATELY EXCITED GENERATOR:

$V_q = E_q - R_q I_q \ni E_q$ = VOLTAGE FROM CONSTANT FIELD EXCITATION.

THIS HOWEVER IS AN IDEALIZED EQUATION. THE DATA FROM LAB YIELDS A "CURVED" LINE, AS OPPOSED TO THE PROPOSED IDEAL STRAIGHT LINE. THE CURVE IS DUE TO SATURATION, WHICH REDUCES POLE FLUX, REDUCING THE MUTUAL INDUCTANCE (M_{df}) WITH AN INCREASE ON LOAD.

THE REGULATION, DEFINED AS

$$\text{REG}(\%) = (V_{\text{NO LOAD}} - V_{\text{FULL LOAD}})/V_{FL} \times 100\%$$

CAN BE COMPUTED AS:

$$\text{REG}(\%) = \left(\frac{125-115}{115}\right)100\%$$

$$= 8.8\%$$

SATURATION CAUSES THE VERTICAL PART OF THE SLOPE, ARMATURE REACTION IS CAUSED CAUSES THE LINEAR RESISTANCE DROP, FROM WHEN THE BRUSHES AND FLUX ARE IN LINE. THE "BACK-SWING" IS A RESULT OF BOTH SATURATION AND OVERLOAD.

b) STEADY STATE EQUATION FOR SHUNT GENERATOR:

$$I_f = V_q / R_f$$

$$E_q = K_f I_f \omega_m = V_q + R_q I_q$$

$$V = V_q = V_f$$

$$\Rightarrow V_q \uparrow \rightarrow I_q \downarrow ;\ I_f \downarrow \xrightarrow{\text{DECREASES}} V_q \downarrow ,\ E_q \downarrow \rightarrow I_f \downarrow ,$$

THUS CAUSING DROOOP IN V_q-I_q PLOT, THAN WITH SEPARATLY EXCITED GENERATOR

V_q MAY BE EXPRESSED:

$$V_q = K_f \omega_m \left(\frac{V_q}{R_f}\right) - R_a\left(\frac{V_q}{R_f} + \frac{V_q}{R_L}\right)$$

UPON SOLVING FOR V_q-I_f RELATION, INSTABILITY OCCURS, AS CAUSED BY R_L. V_q-I_f IS THUS NON-LINEAR, FOR ω_m = CONSTANT. RESIDUAL MAGNITISM IS NEEDED FOR STARTING, THUS THE CURVE DOES NOT BEGIN AT THE ORIGIN (5.5 V. RESIDUAL MAGNETISM). BECAUSE OF THIS, THERE IS A BEND IN THE V-I_f CURVE NEAR THE ORIGIN.

c) COMPOUND GENERATORS HAVE BOTH SHUNT AND SERIES WINDINGS. WHEN BOTH FLUXES ARE IN THE SAME DIRECTION, IT IS COMMULATIVELY COMPOUND. THE LATTER SYSTEM INCREASES EXCITATION, COUNTERACTING THE KNEE OF THE SELF EXITED GEN. FOR THIS SYSTEM

$$E = k_1 \omega_m I_f + k_2 \omega_m I_q$$

$$V = E - I_q R_q = k_1 \omega_m I_f + (k_2 \omega_m - R_a) I_q$$

IN THE DIFFERENTIALLY COMPOUND GEN., FLUX FIELDS OF THE SHUNT AND SERIES COILS BUCK. THIS AGAIN CAUSES A DROOP IN THE PLOT, LARGER THAN BEFORE.

$$V = k_1 \omega_m I_f - (k_2 \omega_m + R_a) I_q$$

FOR SHUNT DROOP: $E = k_1 \omega_m I_f$

DIFF. COMP. DROOP: $E = k_1 \omega_m I_f - k_2 \omega_m I$

3) COMMUTATING WINDINGS REDUCE ARCING AND BRUSH, COMMUTATOR, AND COIL HEATING, AS WELL AS REDUCING COMMUTATOR PITTING, CAUSED BY HIGH VOLTAGES AND CURRENTS IN THE CIRCUIT.

INDUCED VOLTAGE: $e = L \frac{\partial i_c}{\Delta t}$

ADDING A SET OF COMMUTING POLES TO THE QUAD AXIS INTRODUCES A NEW SET OF COILS, SUCH THAT ARMATURE CURRENTS FLOW IN SUCH A MANNER AS TO INTRODUCE AN EQUAL AND OPPOSITE VOLTAGE CREATING A NET VOLTAGE OF 0!

4) A GOOD VOLTAGE SOURCE GENERATOR WOULD BE THE SEPARATLY EXCITED SHUNT, FOR IT HAS THE BEST VOLTAGE REGULATION. CARE, HOWEVER, MUST BE TAKEN, NOT TO OVERLOAD. CONVERSLY, THE DIFFERENTIALLY COMPOUNDED MACHINE ACTS AS A FAIR CURRENT SOURCE OVER A CERTAIN LOAD RANGE. THE CUMMULATIVE COMPOUND MOTOR IS A GOOD VOLTAGE SOURCE. THE SEPARATLY EXCITED GEN.'S TORQUE MAY REQUIRE A HAZARDOUS SPEED. IT WOULD BE GOOD, HOWEVER

FOR STARTING HIGH INERTIAL LOADS, DO TO ITS HIGH TORQUE CAPABILITIES. THE SHUNT MOTOR HAS SIMILAR PROBLEMS, (HIGH SPEEDS @ LOW FIELD CURRENTS). THE SERIES MOTOR HAS A HIGH STARTING TORQUE THAT LEVELS OUT WITH SPEED. THIS IS NICE FOR VARIABLE SPEEDS, BUT DANGEROUS UNDER LIGHT LOADS BECAUSE OF HI TORQUE. THE DIFFERENTIAL COMPOUNDED MOTOR'S FIELD BECOMES WEAK WHEN LOADED, AND MAY THUS RUN AWAY (A BUMMER)

5) WHEN THE BRUSHES MOVED, THE GENERATOR STRAINED UNDER LOAD AS THE MACHINE WENT INTO QUADRATURE, I_d WENT UP AND I_q WENT DOWN, SINCE THE FLUXES OPPOSED, THUS, THE OUTPUT DROPPED.

```
              PROGRAM PLURT(INPUT,OUTPUT,TAPE2=INPUT,TAPE5=OUTPUT)
000002        DIMENSION A(11),A1(11),B(11),B1(11),C(11),C1(11),D(21), D1(21),
             1E(32),E1(32),F(7),F1(7),G(7),G1(7)
000002        WRITE(5,18)
000006        DO 90 J=1,11
000010        READ(2,15)A(J),A1(J)
000017        WRITE(5,25)A(J),A1(J)
000027   90   CONTINUE
000031        WRITE(5,19)
000035        DO 91 J=1,11
000037        READ(2,15)B(J),B1(J)
000046        WRITE(5,25)B(J),B1(J)
000056   91   CONTINUE
000060        WRITE(5,20)
000064        DO 92 J=1,11
000066        READ(2,15)C(J),C1(J)
000075        WRITE(5,25)C(J),C1(J)
000105   92   CONTINUE
000107        WRITE(5,21)
000113        DO 93 J=1,21
000115        READ(2,15)D(J),D1(J)
000124        WRITE(5,25)D(J),D1(J)
000134   93   CONTINUE
000136        WRITE(5,22)
000142        DO 94 J=1,32
000144        READ(2,15)E(J),E1(J)
000153        WRITE(5,25)E(J),E1(J)
000163   94   CONTINUE
000165        WRITE(5,23)
000171        DO 95 J=1,7
000173        READ(2,15)F(J),F1(J)
000202        WRITE(5,25)F(J),F1(J)
000212   95   CONTINUE
000214        WRITE(5,24)
000220        DO 96 J=1,7
000222        READ(2,15)G(J),G1(J)
000231        WRITE(5,25)G(J),G1(J)
000241     96 CONTINUE
000243        WRITE(5,16)
000247        CALL SETPLT(1,A,A1,11,1HA,1,1HA)
000256        WRITE(5,16)
000262        CALL SETPLT(1,B,B1,11,1HB,1,1HB)
000271        WRITE(5,16)
000275        CALL SETPLT(1,C,C1,11,1HC,1,1HC)
000304        WRITE(5,16)
000310        CALL SETPLT(1,D,D1,21,1HD,1,1HD)
000317        WRITE(5,16)
000323        CALL SETPLT(1,E,E1,32,1HE,1,1HE)
000332        WRITE(5,16)
000336        CALL SETPLT(1,F,F1,7,1HF,1,1HF)
000345        WRITE(5,16)
000351        CALL SETPLT(1,G,G1,7,1HG,1,1HG)
000360        WRITE(5,16)
000364        STOP
000366     15 FORMAT(2F5.3)
000366     16 FORMAT('7')
000366     18 FORMAT('7DATA A',/,5X,'I',19X,'V')
000366     19 FORMAT('7DATA B',/,5X,'I',19X,'V')
```

DATA AND PLOTS

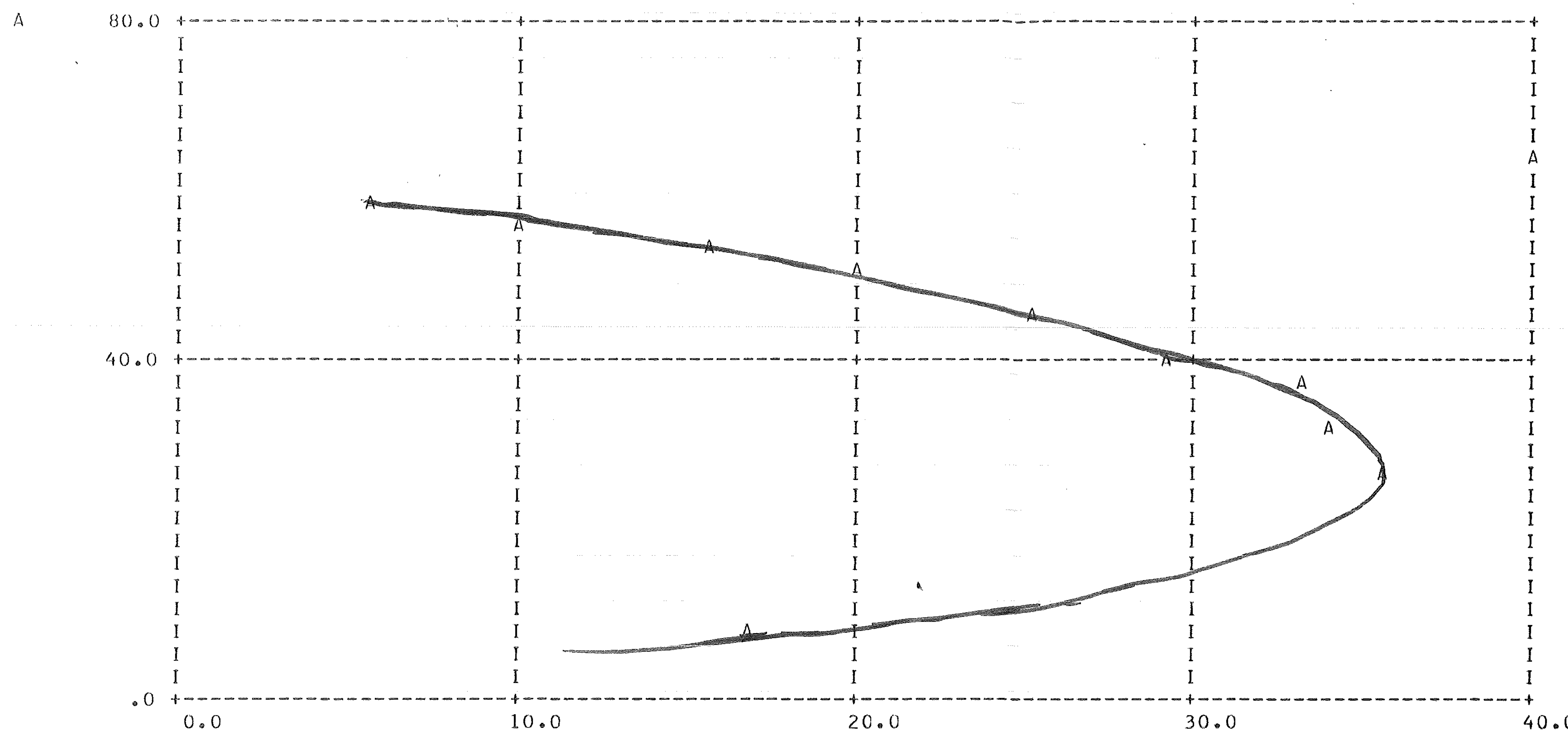
EXTRANEOUS POINTS WERE ADDED HITHER AND YON FOR SCALING PURPOSES OR BY ERROR, AND ARE ACCORDINGLY SNUFFED.

```
000366        20 FORMAT('7DATA C',/,5X,'I',19X,'V')
000366        21 FORMAT('7DATA D',/,5X,'I',19X,'V')
000366        22 FORMAT('7DATA E',/,5X,'I',19X,'V')
000366        23 FORMAT('7DATA F',/,5X,'I',19X,'V')
000366        24 FORMAT('7DATA G',/,5X,'I',19X,'V')
000366        25 FORMAT(6X,F10.6,5X,F10.6)
000366           END

PROGRAM LENGTH INCLUDING I/C BUFFERS
003072

UNUSED COMPILER SPACE
002500
```

DATA A

I	V
5.500000	60.000000
10.000000	57.000000
15.500000	53.000000
20.000000	50.000000
25.000000	45.000000
29.000000	40.000000
33.000000	37.000000
34.000000	33.000000
35.500000	27.000000
16.600000	7.000000
~~40.000000~~	~~63.000000~~

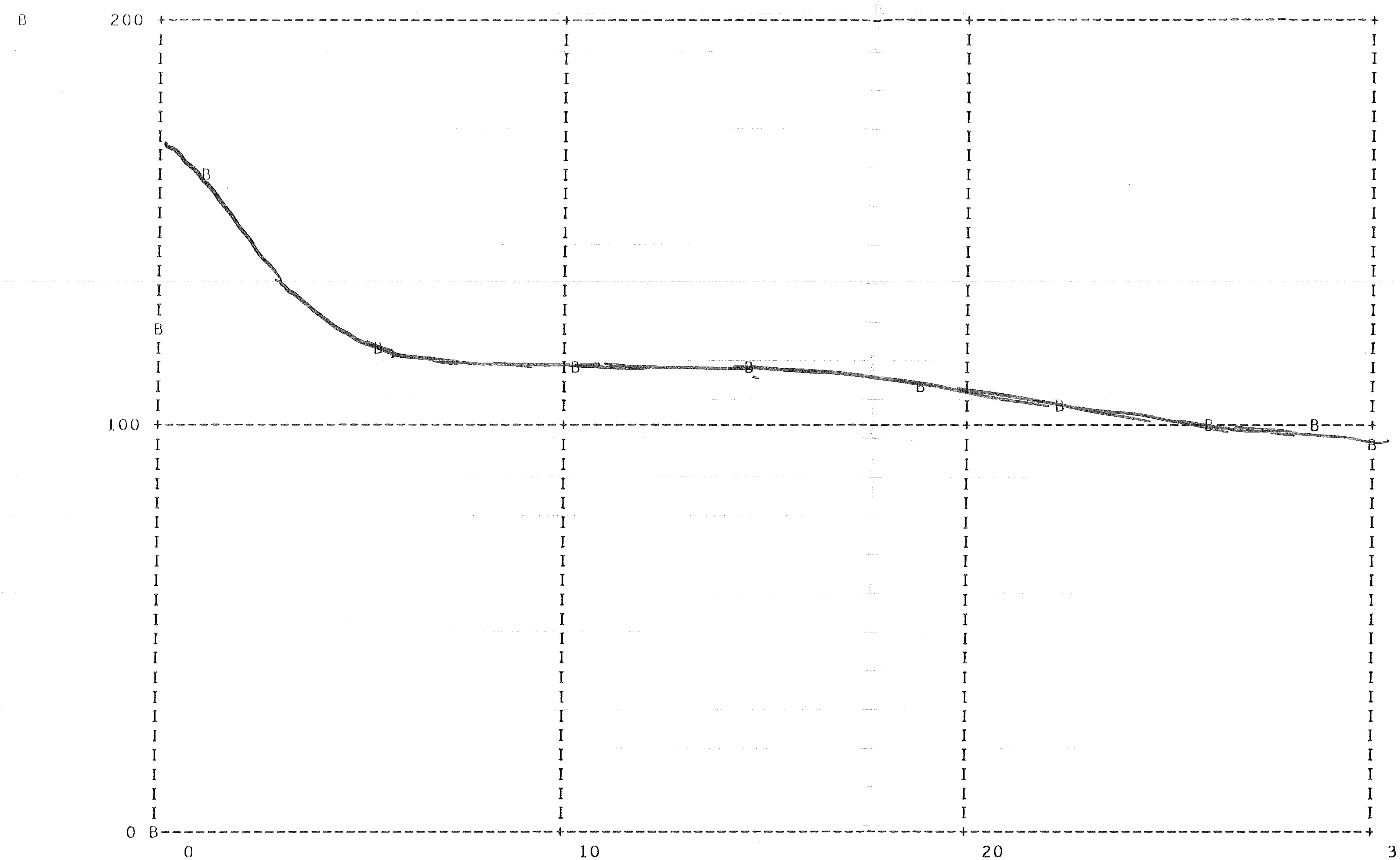
SHUNT
(SERIES DISCONECTED)


A
80.0
40.0
.0
0.0
10.0
20.0
30.0
40.0

DATA B

I	V
~~0.000000~~	~~0.000000~~
0.000000	125.000000
5.400000	120.000000
10.200000	115.000000
14.700000	112.000000
18.800000	108.000000
22.400000	104.500000
25.900000	101.000000
28.700000	98.000000
30.000000	95.000000
~~1.150000~~	~~160.000000~~

SHUNT
(LOADED)

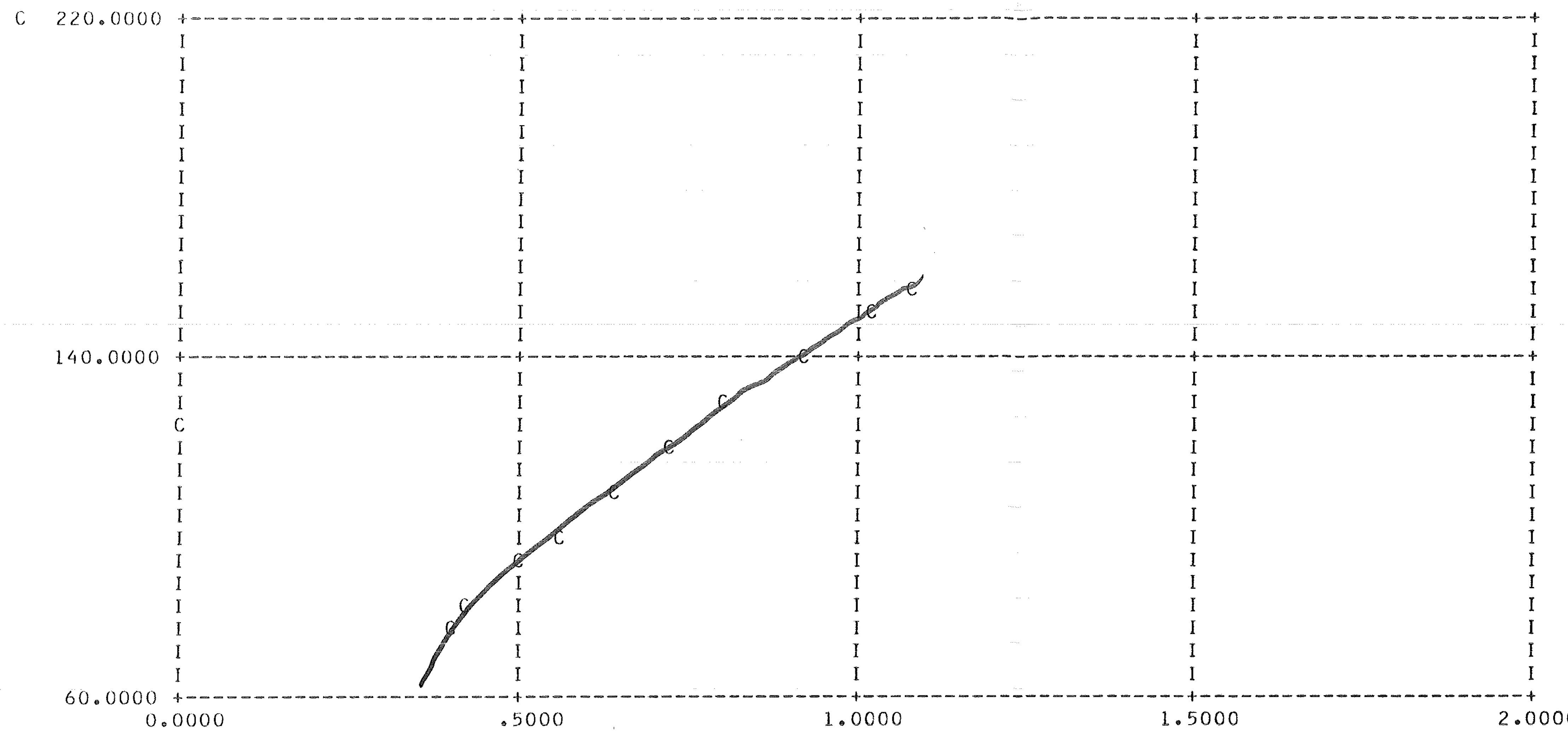


B
200
100
0
0
10
20
3

DATA C

I	V
1.080000	155.000000
1.020000	150.000000
.910000	140.000000
.810000	130.000000
.720000	120.000000
.650000	110.000000
.560000	100.000000
.500000	90.000000
.420000	80.000000
.400000	78.000000
~~0.000000~~	125.000000
1.15	160.

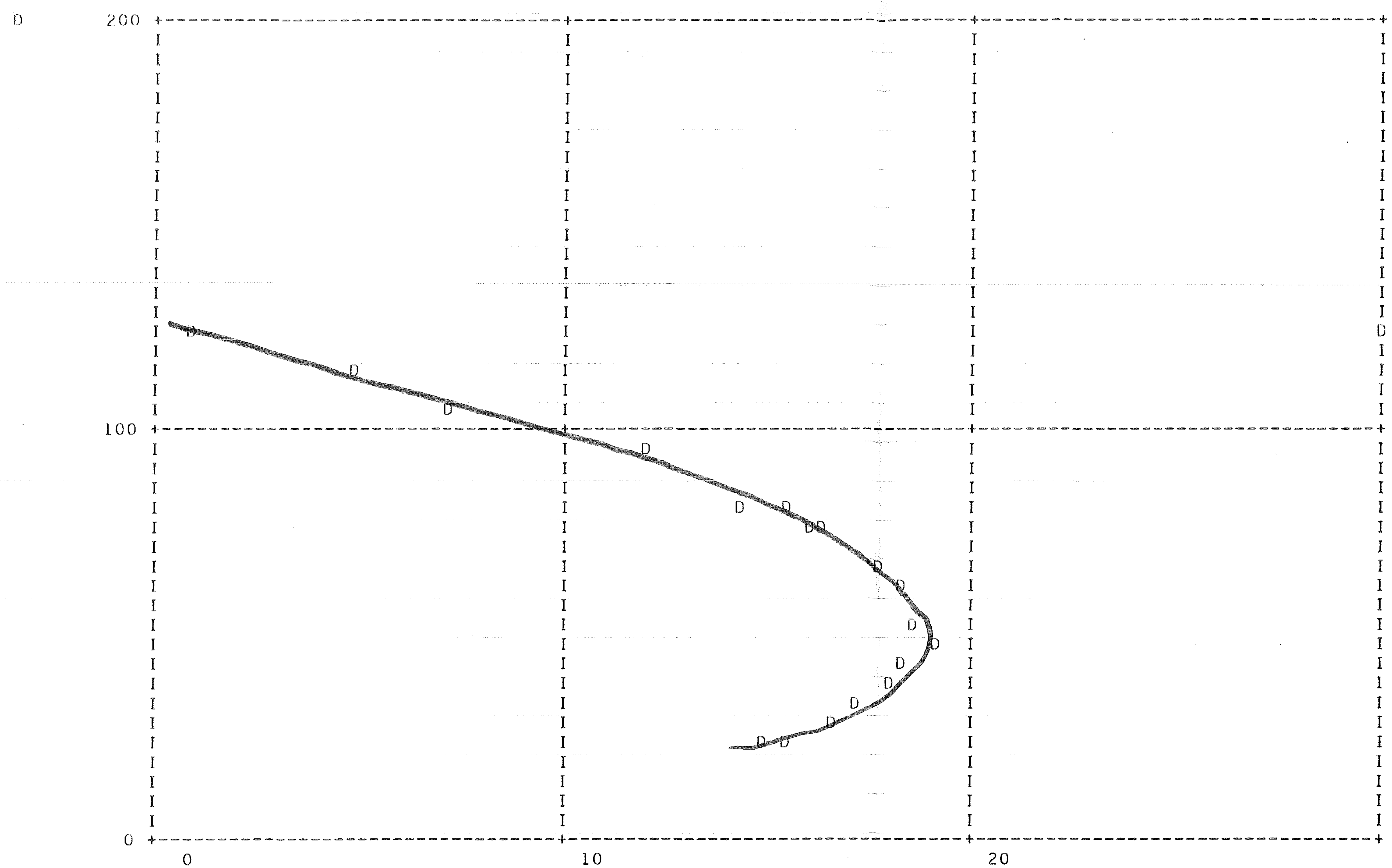
SEPERATLY EXCITED
(NOT LOADED)


C
220.0000
140.0000
60.0000
0.0000
.5000
1.0000
1.5000
2.0000

DATA D

I	V
4.900000	115.000000
7.000000	103.000000
12.100000	93.000000
14.400000	82.500000
15.300000	80.500000
15.900000	77.000000
16.200000	75.000000
16.400000	74.500000
17.700000	67.500000
18.400000	60.000000
18.600000	54.000000
19.000000	48.000000
18.200000	45.000000
18.200000	42.000000
18.000000	38.000000
17.100000	34.000000
16.500000	30.000000
15.400000	26.000000
14.800000	24.000000
~~30.000000~~	~~125.000000~~
~~.800000~~	~~125.000000~~

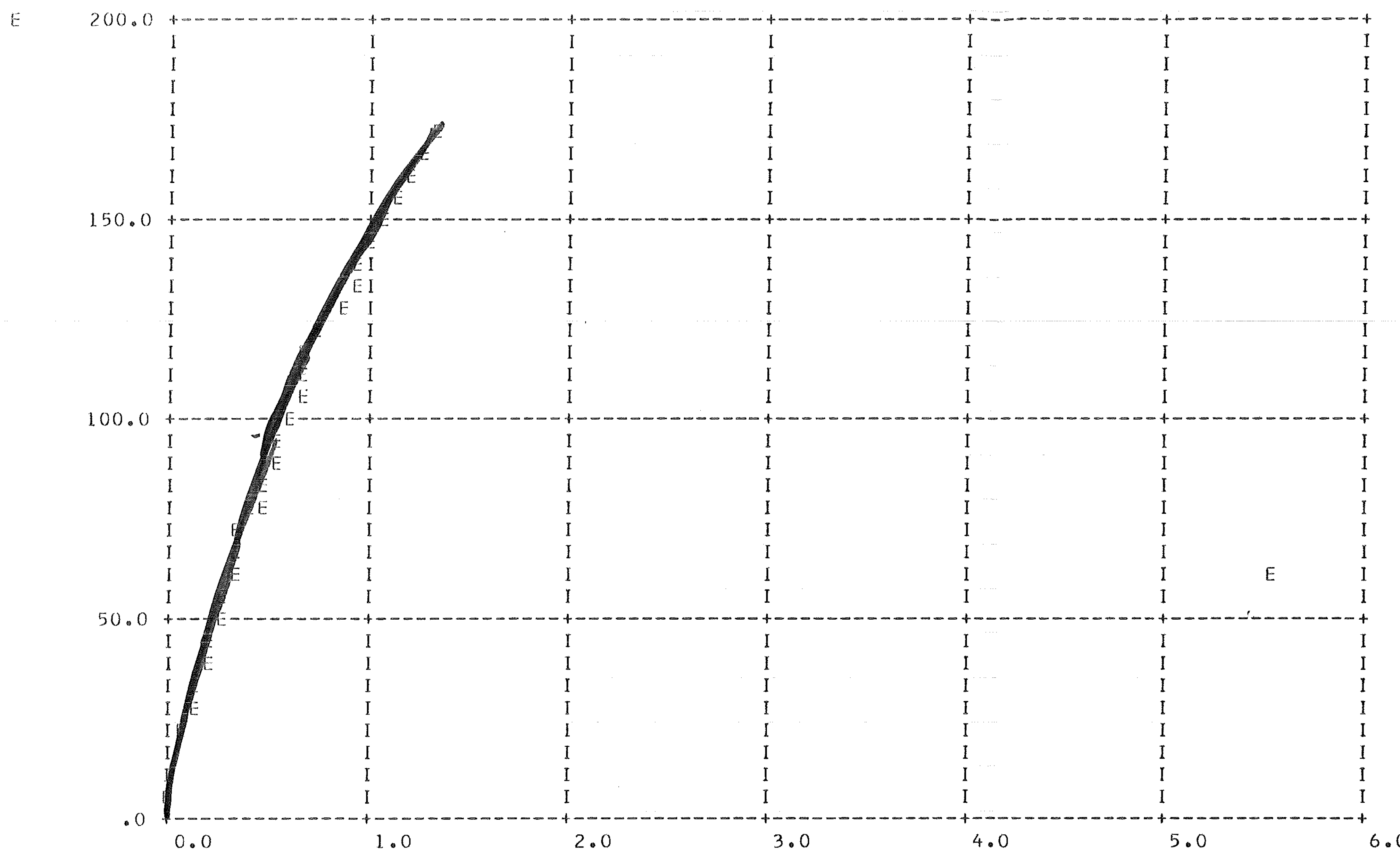
SELF EXCITED
(WITH LOAD)


D
200
100
0
0
10
20
3

DATA E

I	V
~~0.000000~~	~~5.500000~~
.760000	120.000000
.700000	115.000000
.670000	110.000000
.640000	105.000000
.580000	100.000000
.550000	95.000000
.510000	90.000000
.475000	85.000000
.440000	80.000000
.400000	75.000000
.360000	70.000000
.330000	65.000000
.300000	60.000000
.270000	55.000000
.240000	50.000000
.210000	45.000000
.190000	40.000000
.160000	35.000000
.130000	30.000000
.100000	25.000000
.080000	20.000000
.860000	130.000000
.920000	135.000000
.960000	140.000000
1.010000	145.000000
1.060000	150.000000
1.130000	155.000000
1.190000	160.000000
1.250000	165.000000
1.320000	170.000000
~~5.500000~~	~~60.000000~~

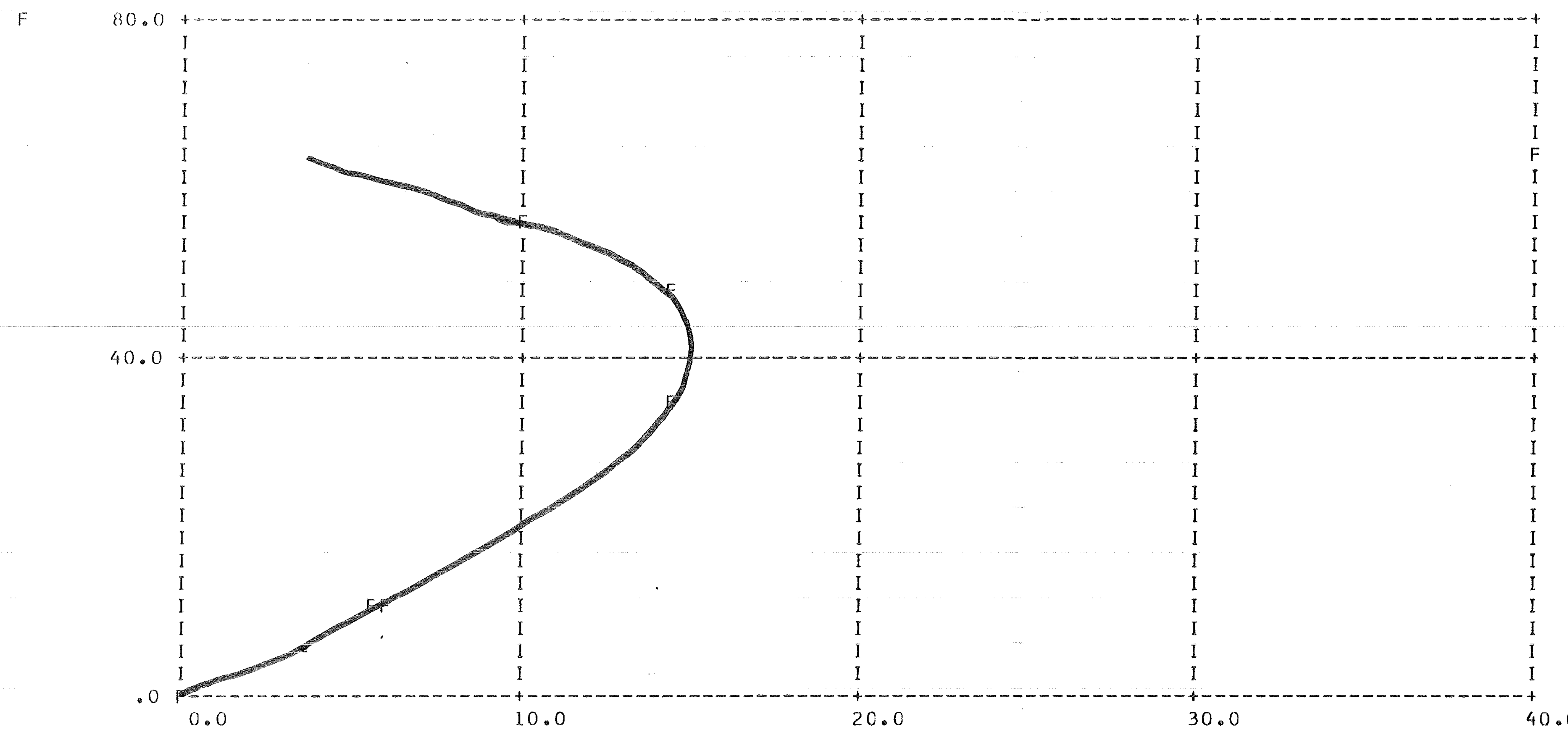
SELF EXCITED
(NO LOAD)


E
200.0
150.0
100.0
50.0
.0
0.0
1.0
2.0
3.0
4.0
5.0
6.0

DATA F

I	V
~~0.000000~~	~~0.000000~~
10.000000	56.000000
14.500000	48.000000
14.500000	34.000000
6.000000	10.000000
~~40.000000~~	~~63.000000~~
5.500000	10.000000

DIFFERENTIALLY COMPOUNDED

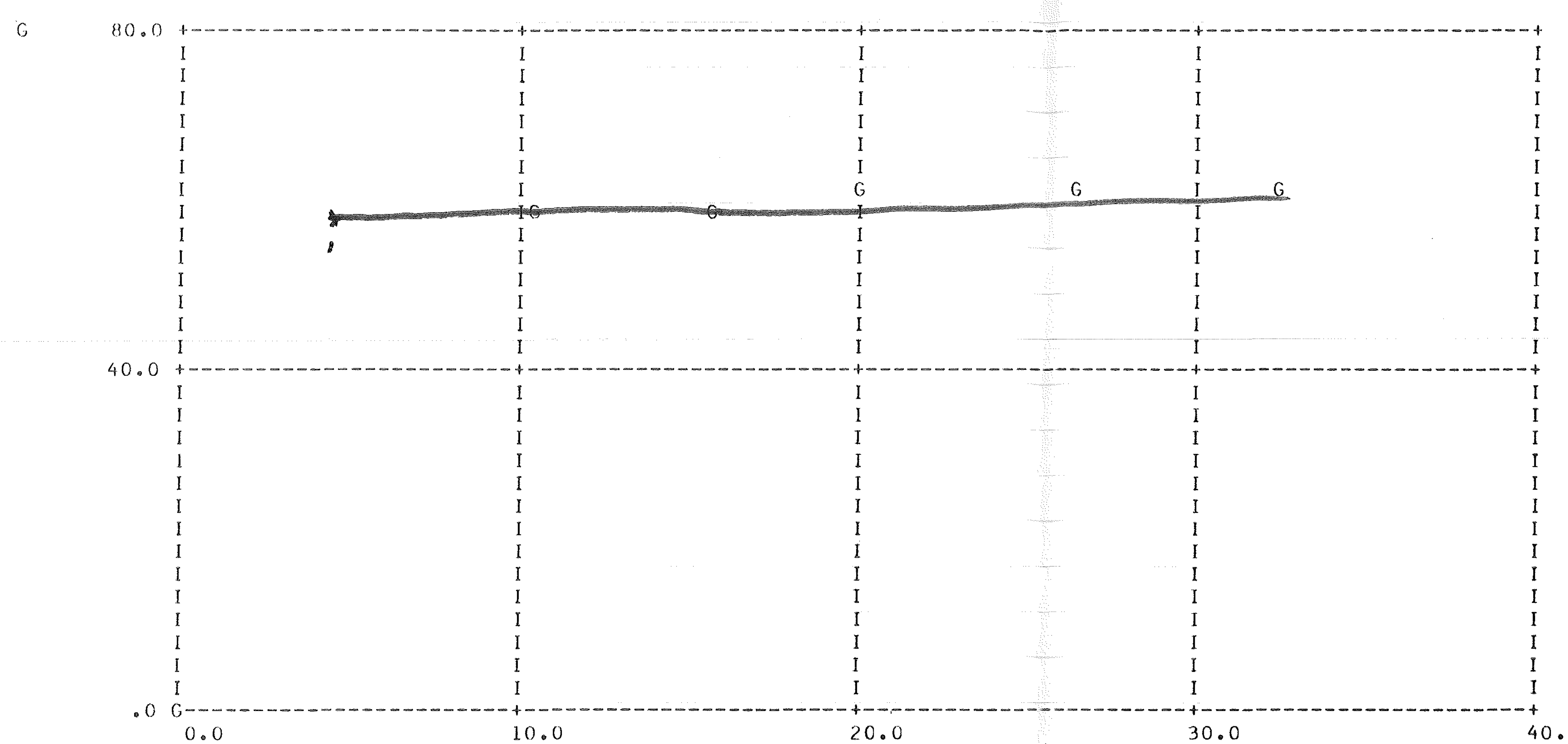


F
80.0
40.0
.0
0.0
10.0
20.0
30.0
40.0

DATA G

I	V
~~0.000000~~	~~0.000000~~
10.500000	60.000000
15.500000	60.000000
20.000000	61.500000
26.500000	62.000000
32.500000	62.500000
~~0.000000~~	~~-0.000000~~

CUMMULATIVELY COMPOUNDED


G
80.0
40.0
.0
0.0
10.0
20.0
30.0
40.0

BOB MARKS

16 Dec 1971

EE 353. ENERGY CONVERSION

LAB PROJECT NO. 1

1. Sketch the circuit used in the demonstration for the open circuit test.

2. Repeat (1) for the short circuit test. Be sure to label voltage and current values.

3. Calculate all parameters of the transformer using the data obtained in the demonstration.

4. Draw the equivalent circuit of the transformer as viewed from the high tension side.

5. Repeat (4) for the low tension side.

6. Draw the approximate equivalent circuit.

7. If the transformer is supplying full load at 120 volts and 0.8 power factor lagging, what is the applied voltage?

8. What is the % regulation?

9. What is the efficiency?

10. Assume the load to vary but that its power factor remains constant; at what load does maximum efficiency occur and what is its value?

15

BOB MARKS

FOR S.C. TEST

$Z = \frac{3V}{17A} = .17647\Omega$ $P.F. = \frac{40}{(3)(17)} = .784313$

$R_{Eq} = Z(P.F.) = .138407$ $(\theta = .669207 \text{ RADIANS} = 38.342740)$

$X_{Eq} = Z \sin(\cos^{-1} P.F.) = .109476$

FROM O.C. TEST

$P.F._1 = \frac{30}{(119)(1.19)} = .21185$ $\theta_1 = 1.357 \text{ RAD} = 77.77^\circ$

$P.F._2 = \frac{38}{(133)(1.91)} = .14959$ $\theta_2 = 1.421 \text{ RAD} = 81.39^\circ$

$P.F._3 = \frac{19}{(83.5)(.54)} = .42138$ $\theta_3 = 1.136 \text{ RAD} = 65.10^\circ$

$P.F._4 = \frac{21}{(93.5)(.66)} = .340301$ $\theta_4 = 1.22356 \text{ RAD} = 70.10^\circ$

$I_{h+e_1} = i_1 P.F._1 = (1.19) P.F._1 = .2521$ $I_{\phi_1} = i_1 \sin(\cos^{-1} P.F._1) = 1.163 \text{ A}$

$I_{h+e_2} = i_2 P.F._2 = (1.91) P.F._2 = .2857$ $I_{\phi_2} = i_2 \sin(\cos^{-1} P.F._2) = .9776 \text{ A}$

$I_{h+e_3} = i_3 P.F._3 = (.54) P.F._3 = .2275$ $I_{\phi_3} = i_3 \sin(\cos^{-1} P.F._3) = .4898 \text{ A}$

$I_{h+e_4} = i_4 P.F._4 = (.66) P.F._4 = .2246$ $I_{\phi_4} = i_4 \sin(\cos^{-1} P.F._4) = .6206 \text{ A}$

$X_{m_1} = \frac{V_1}{\sin(\cos^{-1} P.F._1) i_1} = j102.322\Omega$ $X_{1m_1} = X_{m_1}\left(\frac{450}{120}\right)^2 = 1438.90\Omega$

$X_{m_2} = \frac{V_2}{\sin(\cos^{-1} P.F._2) i_2} = j70.427\Omega$ $X_{1m_2} = X_{m_2}(a)^2 = 990.38\Omega$

$X_{m_3} = \frac{V_3}{\sin(\cos^{-1} P.F._3) i_3} = j170.476\Omega$ $X_{1m_3} = X_{m_3}(a)^2 = 2397.32\Omega$

$X_{m_4} = \frac{V_4}{\sin(\cos^{-1} P.F._4) i_4} = j150.663\Omega$ $X_{1m_4} = X_{m_4}(a)^2 = 2118.70\Omega$

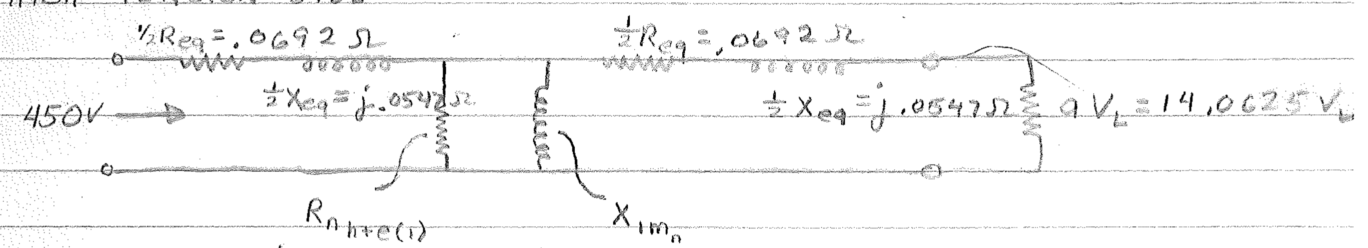
$R_{h+e_1} = \frac{V_1}{P.F._1 i_1} = 472.032\Omega$ $R_{1h+e(1)} = R_{h+e_1} a^2 = 6637.95\Omega$

$R_{h+e_2} = \frac{V_2}{PF_2 i_2} = 747.140\Omega$ $R_{2h+e(1)} = R_{h+e_2} a^2 = 10506.66\Omega$

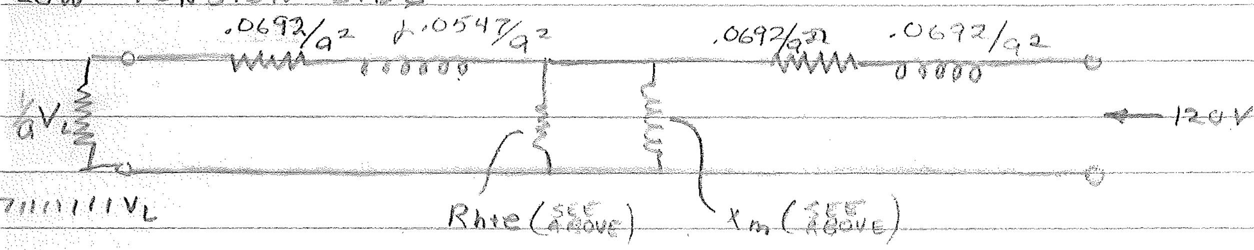
$R_{h+e_3} = \frac{V_3}{PF_3 i_3} = 366.960\Omega$ $R_{3h+e(1)} = R_{h+e_3} a^2 = 5160.38\Omega$

$R_{h+e_4} = \frac{V_4}{PF_4 i_4} = 416.298\Omega$ $R_{4h+e(1)} = R_{h+e_4} a^2 = 5854.19\Omega$

HIGH TENSION SIDE

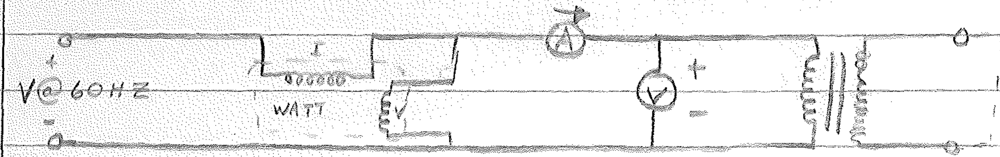


LOW TENSION SIDE



$= .0711111 V_L$

BOB MARKS



XFMR 450/120 ; 2KVA

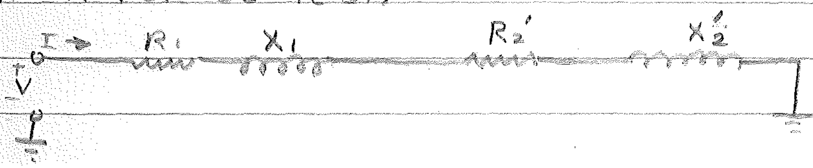
O.C. TEST

	V (V)	I (A)	P (W)
1)	119	1.19	30
2)	133	1.91	38
3)	83.5	.054	19
4)	93.5	.66 A	21

S.C. TEST

V	I	P
3	17	40

CIRCUIT FOR SC TEST:



BOB MARKS

$$\frac{170}{200}$$

EE 353. ENERGY CONVERSION

FINAL EXAM. WINTER, 19[illegible]

Did you know a 100-watt bu[illegible] gives 50 percent more light [illegible] four 25-watt bulbs.

EDUCATION

You should have education [illegible] so that you won't have to lo[illegible] to people; and then more educ[illegible] so that you will be wise eno[illegible] to look down on people.

—[illegible]. [illegible]. Gr[illegible]

The Art of Living Success[illegible]

PART [illegible]

(1) The optimum d.c. machine control problem done in [illegible] the minimum loss problem. Formulate and sol[illegible] the [illegible] problem: i.e., find $i_a(t)$ and $\omega(t)$ which mi[illegible] the [illegible] rotation of the motor (or displacement of the ex[illegible]),
$T = \int_0^T dt$, subject to a given (fixed) angle of rotation [illegible] ~~heat dissipation, Q, boundary conditions~~ $\omega(0)$ [illegible] differential equation, $K_f I_f \, i_a(t) = J\frac{d\omega}{dt} + T_c$ [illegible]

(2) Repeat (1) for the maximum efficiency problem, which [illegible] $i_a(t)$, $\omega(t)$ yielding, under the same conditions as [illegible] displacement, α , for given armature losses, Q.

(3) In both (1) and (2) determine the armature terminal [illegible], [illegible] (a) an armature inductance, $L_a \neq 0$ and (b) $L_a = 0$

Hint: The function, H, corresponding to (2[illegible])(b), to be used [illegible] is

$$H_1 = 1 + \lambda_1 \frac{J^2 R_a}{K_f^2 I_f^2}\left(\frac{d\omega}{dt} + T_c/J\right)^2 + \lambda_2 \omega$$ ([illegible])

$$H_2 = \lambda_3 \frac{J^2 R_a}{K_f^2 I_f^2}\left(\frac{d\omega}{dt} + \frac{T_c}{J}\right)^2 + \omega$$ ([illegible])

$\lambda_1, \lambda_2, \lambda_3$ are Lagrange multipliers.

EE 353. ENERGY CONVERSION FINAL EXAM (Part 2 - Continued).

WINTER QUARTER 1971-72

(4) (a) Problem 15-2, text.

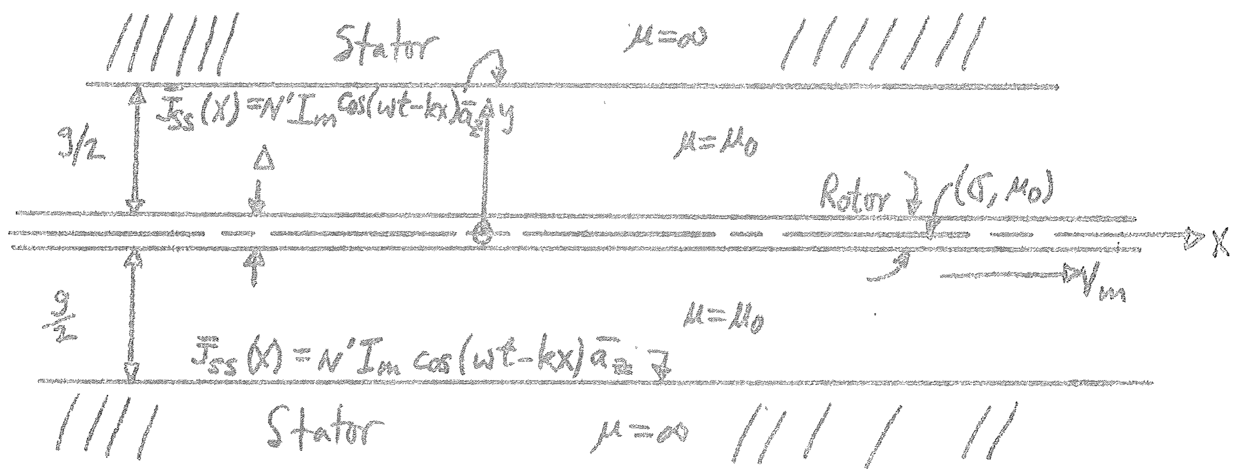
(b) " 15-3, text.

(5) Starting with (1-226) for F_y (p. 74) of the notes on linear induction machines, develop a mechanical "small-signal" model for vertical motion of the linear induction machine. Include gravity, levitation force and inertial reaction (in the vertical direction). Discuss the small displacement motion about a quiescent gap length, G, for two values of slip:

(a) $S < 1/R_m'$ and (b) $S > 1/R_m'$.

Hint: Let $g = G + \delta$, where δ is the small a.c. signal, and expand F_y (eqn. 1-226) in a Taylor series keeping the d.c. and linear (in δ) terms. Solve for the d.c. operating point. Use a spring-mass-gravity model. What is the spring constant? Also assume electrical voltagex to be fixed (do not worry about the electrical transients).

6.



The figure shows the model of an infinitely long linear induction motor with a double sided stator configuration. The rotor is symmetrically placed in the air-gap.

(a) Set up vector-potential equations in the air gap and rotor and solve for the vector potential (assume an electrically thin rotor, i.e., $|\alpha\Delta| \ll 1$).

Hint: Use the symmetry of the model to simplify your calculations.

6. (b) Determine the air-gap flux densities.

(c) Calculate the <u>net</u> time-average surface traction on <u>both sides</u> of the rotor using the Maxwell stress tensor. Use an area of $\ell K \lambda$ in z- and x-directions.

Answer: $$F_x = \frac{4}{\ell \lambda \mu_0}\left[\frac{|V|}{N' V_{syn}}\right]^2 \frac{(S R_m'/2)}{\left(\cosh \frac{kg}{2}\right)^2 + \left(\frac{S R_m'}{2} \sinh \frac{kg}{2}\right)^2}$$

$$F_y = 0.$$

(d) Show that there is a normal force acting on each stator iron, tending to push the stator apart, of amount

$$F_y = \frac{1}{\ell \lambda \mu_0}\left[\frac{|V|}{N' V_{syn}}\right]^2 \frac{1 - \left(S R_m'/2\right)^2}{\left(\cosh kg/2\right)^2 + \left(\frac{S R_m'}{2} \sinh kg/2\right)^2}$$

(e) Compare (c) with the single-sided stator discussed in class. Compare the results of (c) and (d), relative to the y-directed forces, and explain the differences.

7. Lab demonstration problem.

SUPPLEMENT FOR PROB. 6 (Part 2 of Take Home Final).

(6) (f) Plot

$$\frac{F_x}{\frac{1}{2\mu_0}\left[\frac{|K|}{\sigma V_{syn}}\right]^2} = \frac{\left(S\,k_m'/2\right)\ \times 10^{-2}}{\left(\cosh k g/2\right)^2 + \left(\frac{S k_m'}{2}\sinh k g/2\right)^2}$$

$\times 10^{-2}$ $\quad \times 10^{-2}$ $\quad \times 10^{-2}$

versus per-unit synchronous speed $\left(\frac{V_m}{V_{syn}}\right)$.

Consider the two cases of an aluminum rotor and a copper rotor. The necessary constants are:

f = 220 Hz.

V_{syn} = 300 mi/hr = 134.1 m/sec.

Δ = 5/8 inch = 1.59 cm.

Conductivity of copper = σ = 5.9 x 10^7 mho/meter.

" " aluminum = σ = 3.72 x 10^7 mho/meter.

μ_0 = 4π x 10^{-7} H/m.

gap = g = 11/8 inch = 3.5 cm. (The total stator-stator gap = g + Δ = 2").

λ = wavelength = 24 inches = 0.61 meter.

Make any comments regarding peak force, starting force, etc., th
you think will impress me. (The usual rules about profanity rem
in force).

103
18

12.56.

1)* AN ANALOG FOR OBTAINING THE H FUNCTION FOR USE IN EULER'S EQUATION FOR OPTIMALITY IS OBTAINING THE PARAMATER TO BE OPTIMISED (f) IN THE FORM $f = \int_0^T F\,dt$. THE RESTRICTIONS ON f ARE ALSO PUT INTO SIMILAR FORM: $g_n = \int_0^T G_n(t)\,dt$. THE H EQUATION BECOMES:

$$H = F(t) + \sum_{n=1}^{m} \lambda_n G_n,$$

WHERE F AND G_n ARE FUNCTIONS OF TIME AND THE OPTIMUM PARAMETER TO BE FOUND, m IS THE NUMBER OF RESTRICTIVE EQUATIONS, AND λ_n IS A LAGRANGE MULTIPLIER.

IN THIS PROBLEM, T IS TO BE OPTIMIZED,

$$T = \int_0^T dt \Rightarrow F = 1$$

THE RESTRICTIONS ARE α AND Q

$$\alpha = \int_0^T \omega(t)\,dt \Rightarrow G_2 = \omega(t)$$

$$Q = R_a \int_0^T i^2(t)\,dt \Rightarrow G_1 = i_a^2(t)$$

$$G_1 = i_a^2(t) = \frac{J^2}{(K_f I_f)^2}\left(\frac{d\omega}{dt} + \frac{T_o}{J}\right)^2; \text{ FROM } K_f I_f i_a(t) = J\frac{d\omega}{dt} + T_o$$

$$\Rightarrow H = F + \lambda_1 G_1 + \lambda_2 G_2$$

$$= 1 + \lambda_1 \frac{J^2 R_a}{(K_f I_f)^2}\left(\frac{d\omega}{dt} + \frac{T_o}{J}\right)^2 + \lambda_2 \omega$$

✓

* FOR PROBLEMS SUCH AS THOSE WORKED IN NOTES FOR OPTIMUM CONTROL AND PROBLEMS 1 AND 2 IN THIS TEST

$$H = 1 + \lambda_1 \frac{J^2 R_a}{K_f^2 I_f^2}\left(\frac{d\omega}{dt} + \frac{T_o}{J}\right)^2 + \lambda_2 \omega$$

EULER'S EQUATION:

$$\frac{\delta H}{\delta Y} - \frac{\delta}{\delta t}\left(\frac{\delta H}{\delta Y'}\right) = 0$$

$$\frac{\delta H}{\delta Y} = \frac{\delta H}{\delta \omega} = \frac{\delta}{\delta \omega}\left[1 + \lambda_1 \frac{J^2 R_a}{K_f^2 I_f^2}\left(\frac{d\omega}{dt} + \frac{T_o}{J}\right)^2 + \lambda_2 \omega\right]$$

$$= \lambda_2 + \frac{\delta}{\delta \omega}\left[\lambda_1 \frac{J^2 R_a}{K_f^2 I_f^2}\left(\frac{d\omega}{dt} + \frac{T_o}{J}\right)^2\right] \quad ;\left(\frac{\delta \omega}{\delta \omega} = 1\right)$$

$$\frac{\delta}{\delta \omega}\left[\lambda_1 \frac{J^2 R_a}{K_f^2 I_f^2}\left(\frac{d\omega}{dt} + \frac{T_o}{J}\right)^2\right] = 0$$; PG. 6 OF NOTES ON OPTIMUM CONTROL

$$\Rightarrow \frac{\delta H}{\delta \omega} = \lambda_2$$

$$\frac{\delta H}{\delta Y'} = \frac{\delta H}{\delta \omega'} = \frac{\delta}{\delta \omega'}\left[1 + \lambda_1 \frac{J^2 R_a}{K_f^2 I_f^2}\left(\omega' + \frac{T_o}{J}\right)^2 + \lambda_2 \omega\right]$$

$$= \frac{2 J^2 R_a}{K_f^2 I_f^2} \lambda_1 \left(\frac{d\omega}{dt} + \frac{T_o}{J}\right)$$; PG. 6 OF NOTES ON OPTIMUM CONTROL

$$\frac{\delta}{\delta t}\left(\frac{\delta H}{\delta \omega'}\right) = \frac{2 J^2 R_a}{K_f^2 I_f^2} \lambda_1 \frac{\delta^2 \omega}{\delta t^2}$$

FROM EULER'S EQUATION:

$$\frac{2 J^2 R_a}{K_f^2 I_f^2} \lambda_1 \frac{d^2 \omega}{dt^2} = \lambda_2$$

OR $$\frac{d^2 \omega}{dt^2} = \frac{K_f^2 I_f^2}{2 J^2 R_a}\left(\frac{\lambda_2}{\lambda_1}\right)$$; A CONSTANT

PARTIAL INTEGRATION YIELDS:

$$\omega(t) = \frac{K_f^2 I_f^2}{4 J^2 R_a}\left(\frac{\lambda_2}{\lambda_1}\right) t^2 + C_1 t + C_2 \quad \ni C_1 \text{ AND } C_2 \text{ ARE CONSTANT}$$

EMPLOYING BOUNDARY CONDITIONS:

$$\omega(0) = 0 = C_2$$

$$\omega(T) = 0 = \frac{K_f^2 I_f^2}{4 J^2 R_a}\left(\frac{\lambda_2}{\lambda_1}\right) T^2 + C_1 T$$

$$\Rightarrow C_2 = \frac{-K_f^2 I_f^2}{4 J^2 R_a}\left(\frac{\lambda_2}{\lambda_1}\right) T$$

HENCE $$\omega(t) = \frac{K_f^2 I_f^2}{4 J^2 R_a}\left(\frac{\lambda_2}{\lambda_1}\right)\left(t^2 - Tt\right)$$

NOW $$\alpha = \int_0^T \omega(t)\, dt$$ (α IS KNOWN, AND CONTAINS EXPRESSION λ_2/λ_1, WHICH CAN THUS BE EXPRESSED IN TERMS OF α)

$$= \int_0^T \frac{K_f^2 I_f^2}{4 J^2 R_a}\left(\frac{\lambda_2}{\lambda_1}\right)\left(t^2 - Tt\right)$$

$$= \frac{K_f^2 I_f^2}{4 J^2 R_a}\left(\frac{\lambda_2}{\lambda_1}\right)\left(\frac{t^3}{3} - \frac{Tt^2}{2}\right)\Big|_0^T$$

$$= \frac{-K_f^2 I_f^2}{24 J^2 R_a}\left(\frac{\lambda_2}{\lambda_1}\right) T^3$$

$$\Rightarrow \left(\frac{\lambda_2}{\lambda_1}\right) = \frac{-24\alpha J^2 R_a}{K_f^2 I_f^2 T^3}$$

$$\therefore \omega(t) = \frac{-K_f^2 I_f^2}{4 J^2 R_a}\left(\frac{24\alpha J^2 R_a}{K_f^2 I_f^2 T^3}\right)(t^2 - Tt)$$

$$= \frac{+6\alpha}{T^3}(Tt - t^2)$$

NOW $K_t I_f i_a(t) = J\frac{d\omega}{dt} + T_o$

$$\Rightarrow i_a(t) = \frac{1}{K_t I_f}\left(J\frac{d\omega}{dt} + T_o\right)$$

$$= \frac{1}{K_t I_f}\left[\frac{6\alpha J}{T^3}(T - 2t) + T_o\right]$$

THE EXPRESSIONS ABOVE FOR $i_a(t)$ AND $\omega(t)$ FOR MINIMUM T ARE THE SAME EQUATIONS DERIVED FOR THE MINIMUM LOSS PROBLEM, THUS THE SYSTEM MAY DESIGNED FOR CONCURRENT SPEED AND LOSS OPTIMALITY!

AN EXPRESSION FOR THE OPTIMUM VALUE OF T CAN BE DERIVED AS FOLLOWS:

$$Q = R_a \int_0^T i_a^2(t)\,dt$$

$$= \frac{R_a}{(K_t I_f)^2}\int_0^T \left[\frac{6\alpha J}{T^3}(T - 2t) + T_o\right]^2 dt$$

$$= \frac{R_a}{(K_t I_f)^2}\int_0^T \left[\left(\frac{6\alpha J}{T^3}\right)^2 (T - 2t)^2 + 2 T_o \frac{6\alpha J}{T^3}(T - 2t) + T_o^2\right] dt$$

$$= \frac{R_a}{(K_t I_f)^2}\int_0^T \left(\frac{6\alpha J}{T^3}\right)^2 (T^2 - 4tT + 4T^2) + \frac{12 T_o \alpha J}{T^3}(T - 2t) + T_o^2 \Big] dt$$

$$= \frac{R_a}{(K_t I_f)^2}\left[\left(\frac{6\alpha J}{T^3}\right)^2 (T^2 t - 2t^2 T + 4tT^2) + \frac{12 T_o \alpha J}{T^3}(Tt - t^2) + T_o^2 t\right]_0^T$$

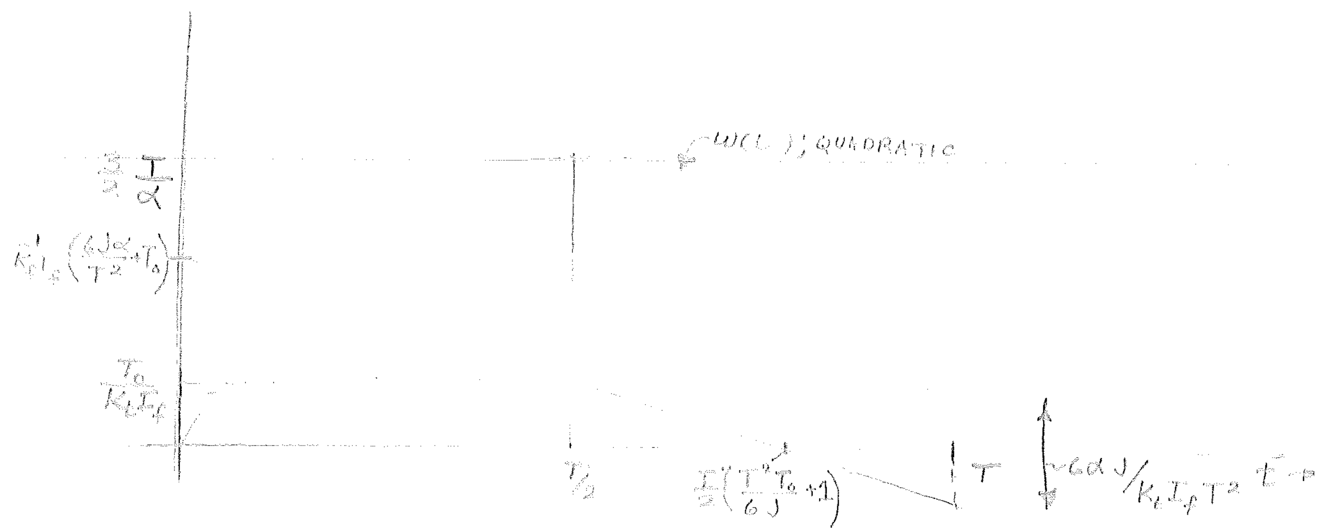
$$= \frac{R_a}{(K_t I_f)^2}\left[\left(\frac{6\alpha J}{T^3}\right)^2 (3T^3) + T_o^2 T\right]$$

$$= \frac{R_a}{(K_t I_f)^2}\,\frac{108\alpha^2 J^2}{T^3} + \frac{T_o^2 T R_a}{(K_t I_f)^2}$$

$$\Rightarrow T_o^2 R_a T^4 - Q (K_t I_f)^2 T^3 + 108 R_a \alpha^2 J^2 = 0$$

THE ABOVE FOURTH ORDER POLYNOMIAL CAN BE SOLVED, YIELDING FOUR VALUES OF T. THE SMALLEST POSITIVE REAL OF T IS THE OPTIMUM.

GENERALLY:



2) USING ANALOG FROM PROBLEM 1

$$\alpha = \int_0^t \omega(t)\,dt \Rightarrow f = \omega(t)$$

$$Q = R_a \int_0^T i_a^2(t)\,dt = \int_0^T \frac{J^2 R_a}{K_t^2 I_f^2}\left(\frac{d\omega}{dt} + \frac{T_o}{J}\right)^2 dt$$

$$\Rightarrow G_3 = \frac{J^2 R_a}{K_t^2 I_f^2}\left(\frac{d\omega}{dt} + \frac{T_o}{J}\right)^2$$

HENCE: $H = \omega + \lambda_3 \frac{J^2 R_a}{K_t^2 I_f^2}\left(\frac{d\omega}{dt} + \frac{T_o}{J}\right)^2$

EULER'S EQUATION

$$\frac{\delta H}{\delta \omega} - \frac{\delta}{\delta t}\left(\frac{\delta H}{\delta \omega'}\right) = 0$$

$$\frac{\delta H}{\delta \omega} = \frac{\delta}{\delta \omega}\left[\omega + \lambda_3 \frac{J^2 R_a}{K_t^2 I_f^2}\left(\frac{d\omega}{dt} + \frac{T_o}{J}\right)^2\right]$$

$= 1$; AS IN NOTES ON OP. CONT., PG 6

$$\frac{\delta H}{\delta \omega'} = \frac{\delta}{\delta \omega'}\left[\omega + \lambda_3 \frac{J^2 R_a}{K_t^2 I_f^2}\left(\frac{d\omega}{dt} + \frac{T_o}{J}\right)^2\right]$$

$= \lambda_3 \frac{2 J^2 R_a}{K_t^2 I_f^2}\left(\frac{d\omega}{dt} + \frac{T_o}{J}\right)$; AS IN NOTES

$$\frac{d}{\delta t}\left(\frac{\delta H}{\delta \omega'}\right) = \lambda_3 \frac{2 J^2 R_a}{K_t^2 I_f^2}\frac{\delta^2 \omega}{\delta t^2}$$

FROM EULER $\Rightarrow \frac{2 J^2 R_a}{K_t^2 I_f^2}\frac{\delta^2 \omega}{\delta t^2} = 1$

OR $\frac{d^2 \omega}{dt^2} = \frac{K_t^2 I_f^2}{2 J^2 R_a \lambda_3}$; A CONSTANT

PARTIAL INTEGRATION YIELDS

$$\omega(t) = \frac{K_t^2 I_f^2}{4 J^2 R_a \lambda_3} t^2 + C_1 t + C_2 = 0$$

EMPLOYING BOUNDRY CONDITIONS:

$$\omega(0) = 0 = C_2$$

$$\omega(T) = 0 = \frac{K_t^2 I_f^2}{4 J^2 R_a \lambda_3} T^2 + C_1 T$$

$$\Rightarrow C_1 = \frac{-K_t^2 I_f^2}{4 J^2 R_a \lambda_3} T$$

$$\therefore \omega(t) = \frac{K_t^2 I_f^2}{4 J^2 R_a \lambda_3}\left(t^2 - Tt\right)$$

SOLVING FOR λ_3 IS NEXT IN ORDER, FROM RESTRICTION $Q = R_a \int_0^T i_a^2(t)\,dt$. $i_a(t)$ CAN BE FOUND FROM EXPRESSION FOR $\omega(t)$: $I_f K_t i_a(t) = J\frac{d\omega}{dt} + T_o$

$$\frac{d\omega}{dt} = \frac{K_t^2 I_f^2}{4J^2 R_a \lambda_3}(2t - T)$$

$$\Rightarrow i_a(t) = \frac{1}{I_f K_t}\left[\frac{K_t^2 I_f^2}{4 J R_a \lambda_3}(2t - T) + T_o\right]$$

$$= \frac{K_t I_f}{4 J R_a \lambda_3}(2t - T) + \frac{T_o}{I_f K_t}$$

$$Q = \int_0^T R_a \left[\frac{K_t I_f}{4 J R_a \lambda_3}(2t - T) + \frac{T_o}{I_f K_t}\right]^2 dt$$

$$= \int_0^T R_a \left[\left(\frac{K_t I_f}{4 J R_a \lambda_3}\right)^2 (4t^2 - 4tT + T^2) + \frac{T_o}{2 J R_a \lambda_3}(2t - T) + \left(\frac{T_o}{I_f K_t}\right)^2\right] dt$$

$$= \left[\frac{1}{R_a}\left(\frac{K_t I_f}{4 J \lambda_3}\right)^2 \left(\frac{4}{3}t^3 - 2t^2 T + T^2 t\right) + \frac{T_o}{2 J \lambda_3}(t^2 - Tt) + R_a \left(\frac{T_o}{I_f K_t}\right)^2 t\right]_0^T$$

$$= \frac{T^3}{3 R_a}\left(\frac{K_t I_f}{4 J \lambda_3}\right)^2 + R_a \left(\frac{T_o}{I_f K_t}\right)^2 T$$

$$\Rightarrow \frac{T^3}{3 R_a}\left(\frac{K_t I_f}{4 J \lambda_3}\right)^2 = Q - R_a \left(\frac{T_o}{I_f K_t}\right)^2 T$$

$$\therefore^{*} \frac{1}{\lambda_3} = \frac{-4J}{K_t I_f}\left[\frac{3 R_a}{T^3}\left\{Q - R_a\left(\frac{T_o}{I_f K_t}\right)^2 T\right\}\right]^{\frac{1}{2}}$$

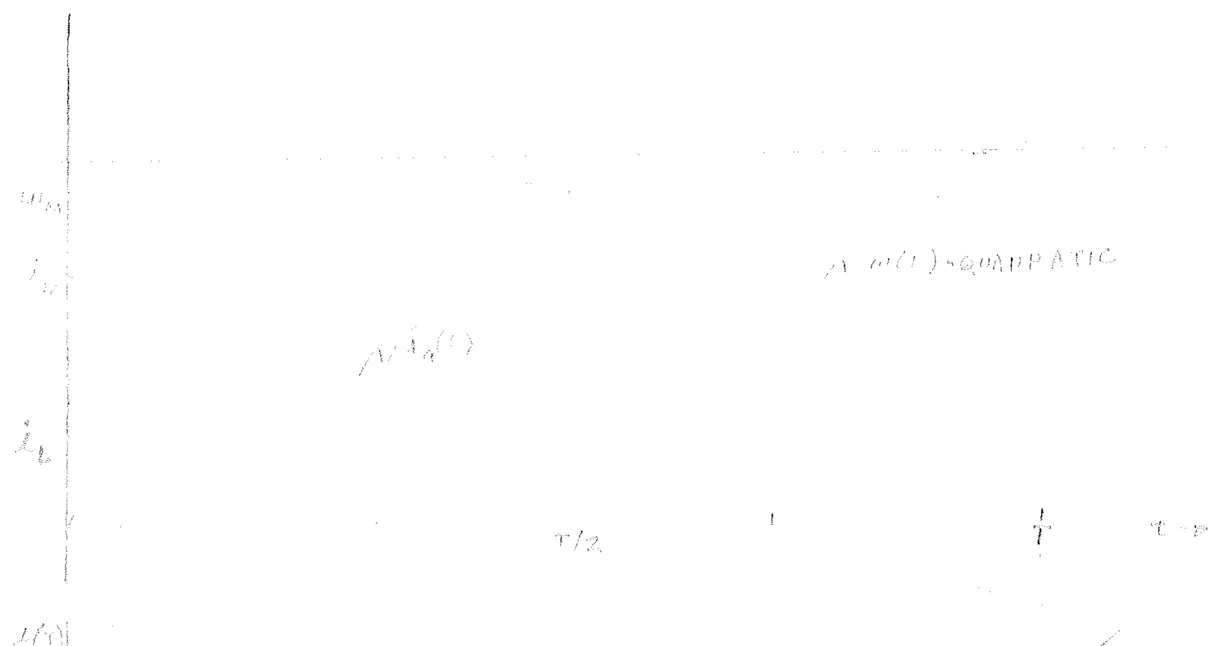
$$\Rightarrow \omega(t) = \left(\frac{K_t^2 I_f^2}{4 J^2 R_a}\right)\left(\frac{4J}{K_t I_f}\right)\left[\frac{3 R_a}{T^3}\left\{Q - R_a\left(\frac{T_o}{I_f K_t}\right)^2 T\right\}\right]^{\frac{1}{2}}(t^2 - Tt)$$

$$= \frac{-K_t I_f}{J\,T}\left[\frac{3}{R_a T}\left\{Q - R_a\left(\frac{T_o}{I_f K_t}\right)^2 T\right\}\right]^{\frac{1}{2}}(t^2 - Tt)$$

AND $$i_a(t) = \left(\frac{K_t I_f}{4 J R_a}\right)\left(\frac{4J}{K_t I_f}\right)\left[\frac{3 R_a}{T^3}\left\{Q - R_a\left(\frac{T_o}{I_f K_t}\right)^2 T\right\}\right]^{\frac{1}{2}}(2t - T) + \frac{T_o}{I_f K_t}$$

$$= \frac{-1}{T}\left[\frac{3}{R_a T}\left\{Q - R_a\left(\frac{T_o}{I_f K_t}\right)^2 T\right\}\right]^{\frac{1}{2}}(2t - T) + \frac{T_o}{I_f K_t}$$

*NEGATIVE VALUE CHOSEN FOR POSITIVE ROTATION



POINTS OF INTEREST

$$\omega_m = \frac{K_t I_f T}{4J}\sqrt{\frac{3}{R_a T}\left[Q - R_a\left(\frac{T_o}{I_f K_t}\right)^2 T\right]}$$

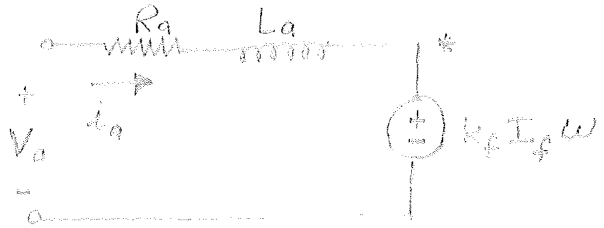
$$i_m = \frac{T_o}{I_f K} + \sqrt{\frac{3}{R_a T}\left[Q - R_a\left(\frac{T_o}{I_f K_t}\right)^2 T\right]}$$

$$i(T) = \frac{T_o}{I_f K} - \sqrt{\frac{3}{R_a T}\left[Q - R_a\left(\frac{T_o}{I_f K_t}\right)^2 T\right]}$$

$$i_L = T_o / I_f K_t$$

$$\omega(T) = \omega(0) = 0$$

4)



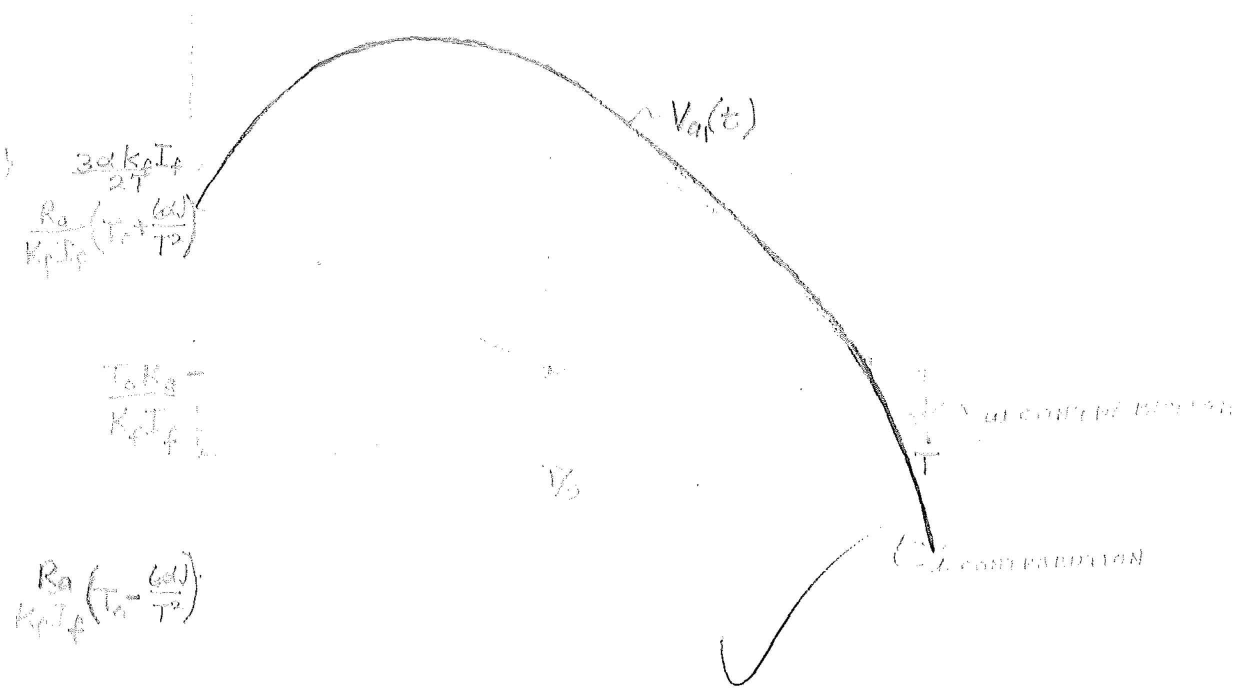
a) FOR OPTIMUM T (NOTE $K_i = K_f$)

CASE 1a; $L_a = 0$

$$\Rightarrow V_a = R_a i_a(t) + K_f I_f \omega(t)$$

$$= \left[\frac{R_a}{K_f I_f} \left[\frac{6\alpha J}{T^3} (T - 2t) + T_0 \right] + \frac{6\alpha K_f I_f}{T^3} (Tt - t^2) \right] \left[\mu(t) - \mu(t-T) \right]$$

$$= \left(-\frac{6\alpha K_f I_f}{T^3} t^2 + \frac{6\alpha}{T^2} \left(K_f I_f - \frac{2 R_a}{K_f I_f} \frac{J}{T} \right) t + \frac{R_a}{K_f I_f} \left[\frac{6\alpha J}{T^2} + T_0 \right] \right) \cdot \left[\mu(t) - \mu(t-T) \right]$$



*CHAPTER 6, "ROTOR WINDINGS AND COMMUTATOR ACTION", Pg 244
NOTE: BRUSHES ON Q AXIS

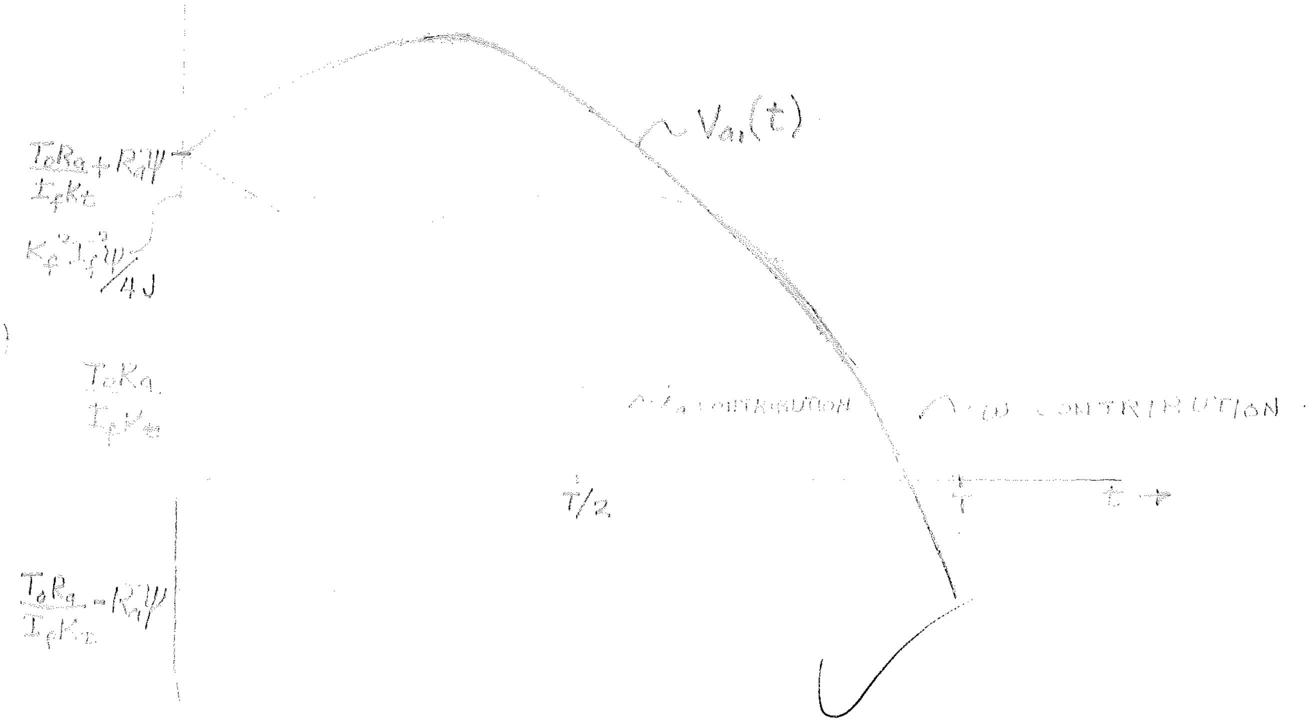
CASE 1b – OPTIMUM α WITH $L_a = 0$

AGAIN: $V(t) = R_a\, i_a(t) + K_f I_f\, \omega(t)$

LET $\psi = \left(Q - R_a\left(\frac{T_o}{I_f K_t}\right)^2 T\right)^{\frac{1}{2}} \left(\frac{3}{R_a T}\right)^{\frac{1}{2}}$

$$\Rightarrow V(t) = \left[R_a \frac{\psi}{T}(T - 2t) + \frac{T_o R_a}{I_f K_t} + \frac{K_f^2 I_f^2 \psi}{JT}(Tt - t^2)\right]\left[\mu(t) - \mu(t-T)\right]$$

$$= \left[\frac{-K_f^2 I_f^2 \psi}{JT} t^2 + \psi\left(\frac{K_f^2 I_f^2}{J} - \frac{2R_a}{T}\right)t + \left(R_a\psi + \frac{T_o R_a}{I_f K_t}\right)\right]\left[\mu(t) - \mu(t-T)\right]$$



CASE 2a

WITH $L_a \neq 0$

$$V_{a2} = R_a i_a + L_a \frac{d i_a}{dt} + K_f I_f \omega$$

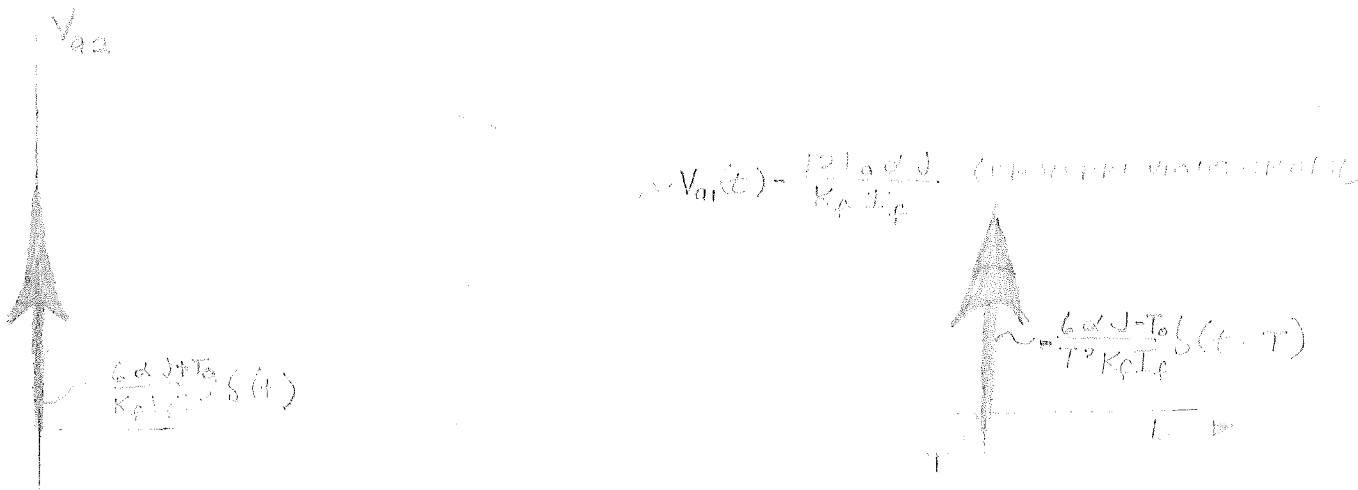
$$= \left[R_a \frac{1}{K_f I_f} \left[\frac{6\alpha J}{T^3}(T-2t) + T_o \right] + \frac{6\alpha K_f I_f}{T^3}(Tt - t^2) \right] \left[\mu(t) - \mu(t-T) \right]$$

$$+ L_a \frac{d}{dt} \left[\frac{1}{K_f I_f} \left\{ \frac{6\alpha J}{T^3}(T-2t) + T_o \right\} \left\{ \mu(t) - \mu(t-T) \right\} \right]$$

$$= V_{a1} + L_a \left[\frac{-12\alpha J}{K_f I_f} \left\{ \mu(t) - \mu(t-T) \right\} + \frac{1}{K_f I_f} \left\{ \frac{6\alpha J}{T^2}(T-2t) + T_o \right\} \left\{ \delta(t) - \delta(t-T) \right\} \right]$$

$$= V_{a1} + L_a \left[\frac{-12\alpha J}{K_f I_f} \left\{ \mu(t) - \mu(t-T) \right\} + \frac{6\alpha J + T_o}{T^2 K_f I_f} \delta(t) - \frac{6\alpha J - T_o}{T^2 K_f I_f} \delta(t-T) \right]$$

THE VOLTAGE WITH L_a IS V_{a1} WITH A D.C. VALUE AND IMPULSES AT 0 AND T, WHICH ALLOW FOR THE INSTANTANEOUS CHANGE OF i_a THRU L_a



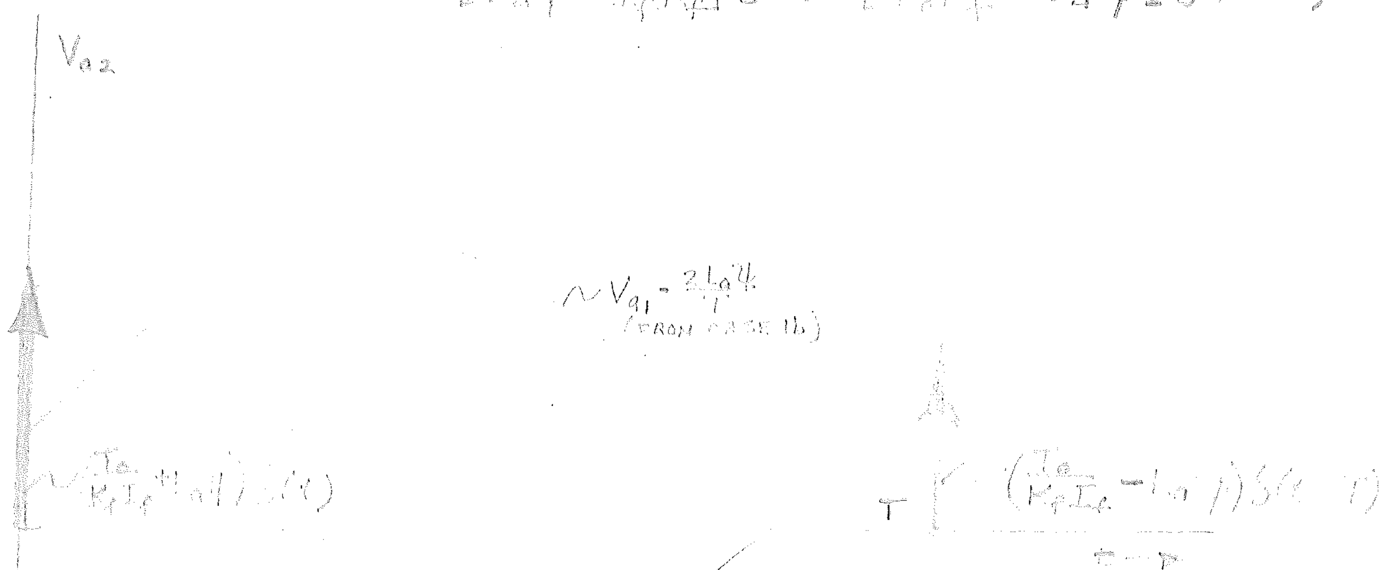
CASE 2b: OPTIMUM α, $L_a \neq 0$

AGAIN: $$V_{a2}(t) = \left[R_a i_a(t) + K_f I_f \omega(t) + L_a \frac{d}{dt}(i_a(t))\right]\left[u(t) - u(t-T)\right]$$

$$= V_{a1}(t) + L_a \frac{d}{dt}\left[\left(\frac{\psi}{T}(T-2t) + \frac{T_o}{I_f K_f}\right)\left(u(t) - u(t-T)\right)\right]$$

$$= V_{a1}(t) - \frac{2L_a\psi}{T}\left[u(t) - u(t-T)\right] + \left[\frac{L_a\psi}{T}(T-2t) + \frac{T_o}{I_f K_f}\right]\left[\delta(t) - \delta(t-T)\right]$$

$$= V_{a1}(t) - \frac{2L_a\psi}{T}\left[u(t) - u(t-T)\right] + \left[L_a\psi + \frac{T_o}{I_f K_f}\right]\delta(t) + \left[\frac{T_o}{I_f K_f} - L_a\psi\right]\delta(t-T)$$



AGAIN, THE VOLTAGE V_{a2} (FOR CASE b) IS V_{a1}, WITH AN ADDED D.C. COMPONENT AND IMPULSES @ $t=0$ AND $t=T$ TO ALLOW INSTANTANEOUS CHANGE OF i_a THRU L_a.

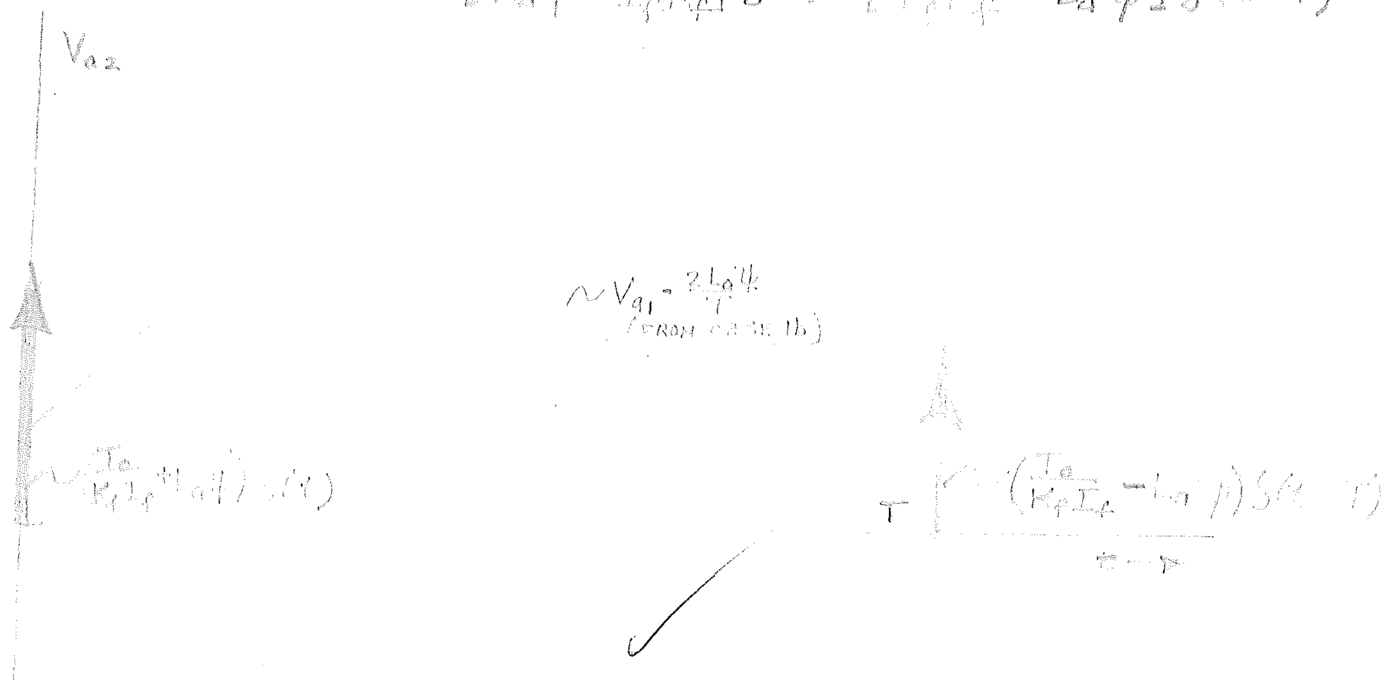
CASE 2b: OPTIMUM α, $L_a \neq 0$

AGAIN: $V_{a2}(t) = \left[R_a i_a(t) + K_f I_f \omega(t) + L_a \frac{d}{dt}(i_a(t))\right]\left[\mu(t) - \mu(t-T)\right]$

$$= V_{a1}(t) + L_a \frac{d}{dt}\left[\left(\frac{\psi}{T}(T-2t) + \frac{T_o}{I_f K_f}\right)\left(\mu(t) - \mu(t-T)\right)\right]$$

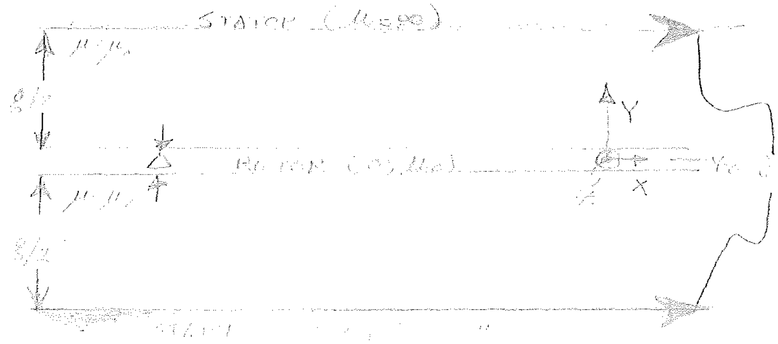
$$= V_{a1}(t) - \frac{2L_a\psi}{T}\left[\mu(t) - \mu(t-T)\right] + \left[\frac{L_a\psi}{T}(T-2t) + \frac{T_o}{I_f K_f}\right]\left[\delta(t) - \delta(t-T)\right]$$

$$= V_{a1}(t) - \frac{2L_a\psi}{T}\left[\mu(t) - \mu(t-T)\right] + \left[L_a\psi + \frac{T_o}{I_f K_f}\right]\delta(t) + \left[\frac{T_o}{I_f K_f} - L_a\psi\right]\delta(t-T)$$



AGAIN, THE VOLTAGE V_{a2} (FOR CASE b) IS V_{a1}, WITH AN ADDED D.C. COMPONENT AND IMPULSES @ $t = 0$ AND $t = T$ TO ALLOW INSTANTANEOUS CHANGE OF i_a THRU L_a.

6)



$\frac{80}{100}$

9) USING ...

$\nabla^2 \hat{A} = \ldots (\ldots \hat{v} \times \hat{B})$

IF (HIGH) ... THE TERM ... MAY BE ...

low

ALSO, ...

SOLUTION, ...

WHERE

... OF A_1, A_2, A_3 ... (1) CAP

... $\Delta/2$, BOTH HAVE THE ...

AND ... $K y_\Delta + A_4 \cos$ ...

(1) ... $A_1 \sinh \frac{K\Delta}{2} + A_2 \cosh \frac{K\Delta}{2} = A_3 \sinh$ ...

ALSO, THE TANGENTIAL ... H_x, NEED HAVE A ...

AT ... GAP ...

(2) $\Rightarrow$ ... $\frac{K\Delta}{2}$ ...

ALSO, @ ... H_x ...

(3) ... $\Rightarrow$ ... A_4 ...

FURTHERMORE, ...

(4) ...

$$\begin{bmatrix} \cos \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \cosh \frac{K(g+\Delta)}{2} & \sinh \frac{K(g+\Delta)}{2} & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \ldots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \ldots \end{bmatrix}$$

EMPLOYING ...

$\det = -\psi \sinh$ ... $\cosh \frac{K\Delta}{2} \cosh$ ...

$+ \cosh$ ...

SOLUTION OF A_1:

$$A_1 = \frac{1}{\det} \begin{vmatrix} 0 & \ldots & \ldots \\ 0 & \ldots & \ldots \\ \ldots & \ldots & \ldots \end{vmatrix}$$

$= \frac{1}{\det} \Big[$ ... $\cosh \frac{K\Delta}{2} \Big]$

SOLUTION OF A_2

$$A_2 = \frac{1}{det}\begin{vmatrix} \cosh \frac{K\Delta}{2} & 0 & -\psi \sinh \frac{\psi K\Delta}{2} \\ \sinh \frac{K\Delta}{2} & 0 & -\cosh \frac{\psi K\Delta}{2} \\ \text{[illegible]} & \frac{N' I_m \mu_r}{\mu} & 0 \end{vmatrix}$$

$$= \frac{1}{det}\left[-\frac{N' I_m \mu_r}{\mu}\left(\psi \sinh \frac{K\Delta}{2} \sinh \frac{\psi K\Delta}{2} - \cosh \frac{K\Delta}{2} \cosh \frac{\psi K\Delta}{2}\right)\right]$$

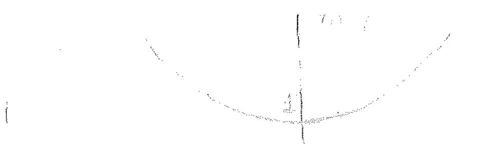
SOLUTION OF A_3

$$A_3 = \frac{1}{det}\begin{vmatrix} \cosh \frac{K\Delta}{2} & \text{[illegible]} & \text{[illegible]} \\ \text{[illegible]} & \text{[illegible]} & \text{[illegible]} \\ \text{[illegible]} & \text{[illegible]} & \text{[illegible]} \end{vmatrix}$$

$$= \frac{1}{det}\left[\text{[illegible]} - \sinh^2 \frac{K\Delta}{2}\right]$$

$$= \frac{1}{det}\left[N' \text{[illegible]}\right]$$

NOW, α_1 [illegible] $|\alpha_1 \Delta|$ [illegible] 1



FOR VA[illegible] [illegible]

[illegible]

THUS TH[illegible]

A_1 [illegible]

= [illegible]

[illegible] $\sinh \frac{K\Delta}{2}$ [illegible]

20/20

10/10

b) $\vec{B} = \nabla \times \vec{A}$

$\vec{A} = A_m e^{j(\omega t - kx)}$ [illegible] WHERE A_m IS A FUNCTION OF Y

$\therefore \vec{B} = \left(\frac{\partial}{\partial Y} A_m \bar{a}_x + j [illegible] \bar{a}_Y\right) e^{j(\omega t - kx)}$

FOR $Y > 0$; $\vec{B} = [[illegible] KY + A_2 [illegible] KY) \bar{a}_x + jk (A_1 \sinh KY + [illegible] KY) \bar{a}_Y] e^{[illegible]}$

$= k(A_1 \cosh KY + A_2 \sinh KY) \bar{a}_x + jk(A_1 \sinh KY + A_2 \cosh KY) \bar{a}_Y] e^{[illegible]}$

IN GAP FOR [illegible]

FOR $Y < 0$; $\vec{B} = [(A_1 \cosh [illegible] - A_2 \sinh KY) \bar{a}_x + j(A_1 \sinh KY - A_2 \cosh KY) \bar{a}_Y] k e^{[illegible]}$

c) THE MAGNETIC STRESS TENSOR

$$\bar{\bar{\sigma}}_m = \begin{bmatrix} \frac{B_x^2 - B_Y^2 - B_z^2}{2\mu_0} & \frac{B_x B_Y}{\mu_0} & \frac{B_x B_z}{\mu_0} \\ \frac{B_Y B_x}{\mu_0} & [illegible] & [illegible] \\ [illegible] & [illegible] & [illegible] \end{bmatrix}$$

$$= \begin{bmatrix} \frac{B_x^2 - B_Y^2}{2\mu_0} & \frac{B_x B_Y}{\mu_0} & \\ [illegible] & [illegible] & \\ 0 & 0 & [illegible] \end{bmatrix}$$

[illegible] $B_z = 0$

[illegible]

$B_x B_Y^* = [illegible]$

$= [illegible] [5R_m [illegible] \frac{KA}{2} [illegible]]$

$= [illegible] [5R_m [illegible] (\sinh \frac{KA}{2} [illegible] \frac{KA}{2} [illegible])]$

$Re\left[\frac{B_x B_Y^*}{2\mu_0}\right] = \frac{5R_m N_1^2 I_m^2 \mu_0}{\phi} [illegible] \sinh [illegible] \cosh \frac{KA}{2}$

$$A_3 = \frac{2N' I_m \ldots /K}{jSR_m \cosh\frac{Kg}{2} + 2\sinh\frac{Kg}{2}}$$

AND

$$A_4 = \frac{2N' I_m \ldots}{jSR_m' \cosh\frac{Kg}{2} + 2\sinh\frac{Kg}{2}}$$

GAP $A_m = \ldots$; $A_m = A_4 \cosh K_1 Y$ (ROTOR)

$Y > 0$

NOW ...

FOR $Y < 0$

("$\rightarrow$" IS ...)

$A_1 \rightarrow \frac{-jSR_m N' \ldots}{jSR_m \ldots \frac{Kg}{2}} = {}^*A_1$

$A_2 \rightarrow -\frac{2N' I_m \ldots /K}{jSR_m \cosh \ldots}$

$A_3 \rightarrow \frac{2N \ldots}{jSR_m \ldots} \ldots A_3$

THEN FOR $Y < 0$

GAP

$A_m = A_1 \ldots KY \ldots$

ROTOR

$A_m \ldots$

Bx is an odd function

By is an even function.

Just the opposite.

$A_m = -A_1 \sinh kY + A_2 \cosh kY$ $Y<0$ (in gap)

in order to get an even function.

don't change this. This remains the same throughout the rotor.

[illegible]

$$\frac{B_x B_x' - B_y B_y'}{4\mu_0} = \frac{N'^2 [illegible]}{[illegible]}\left[[illegible] \frac{KA}{2} \cdot S^2 R_m'^2 \cosh^2 \frac{KA}{2} + \cosh^2 \frac{KA}{2} - \sinh^2 \frac{KA}{2}\right]$$

$$= \frac{N'^2 I_m \mu_0}{2\phi}\left[S^2 R_m'^2 + 1\right]$$

$$\Rightarrow \langle p \rangle = \frac{N'^2 I_m^2 \mu_0}{\phi}\left[S R_m' \hat{a}_y + \tfrac{1}{2}(1 - S^2 R_m'^2)\hat{a}_y\right] \quad \text{FOR } Y > 0$$

FOR $Y < 0$ $(Y \to -Y;\ \hat{a}_y \to -\hat{a}_y)$ [illegible] CHANGE

$$\therefore \langle P \rangle_{Y<0} = \frac{N'^2 I_m^2 \mu_0}{\phi}\left[S R_m \hat{a}_x - \tfrac{1}{2}(1 - S^2 R_m'^2)\hat{a}_y\right]$$

NOW FORCE = $\langle p \rangle \cdot$ AREA $\ni$ AREA $= \lambda \ell$

$$\Rightarrow F_{UP} = \frac{N'^2 I_m^2 \mu_0 \lambda \ell}{\phi}\left[S R_m' \hat{a}_x + \tfrac{1}{2}(1 - S^2 R_m'^2)\hat{a}_y\right]$$

$$F_{DOWN} = \frac{N'^2 I_m^2 \mu_0 \lambda \ell}{\phi}\left[S R_m' \hat{a}_x - \tfrac{1}{2}(1 - S^2 R_m'^2)\hat{a}_y\right]$$

$$F_{TOT} = F_{UP} + F_{DOWN}$$

$$= \frac{2 N_1'^2 I_m^2 \mu_0 \lambda \ell}{\phi} S R_m' \hat{a}_x$$

$$= \frac{N_1^2 I_m^2 \mu_0 [illegible] \lambda S R_m \hat{a}_x}{4 \cosh^2 [illegible] + S^2 R_m'^2 \sinh^2 \frac{KA}{2}} \Rightarrow F_x [illegible]$$

NOW, $N I \mu_0 \lambda \ell$ [illegible] $V_x / N' V_{SYN}$

$$\Rightarrow F_x = \frac{4}{[illegible]}\left[\frac{[illegible]}{N' V_{SYN}}\right]^2 [illegible]$$

10/20

d) [illegible]
AND [illegible]

$$[illegible] \left[[illegible] + \tfrac{1}{2}\left(1 - [illegible]\right)\bar{a}_y \right]$$

OR

$$F = \frac{\mu_0 N^2 I_{01}^2 \ell \lambda}{\phi} \left[[illegible]\, a_x + \tfrac{1}{2}\left(1 - [illegible]\right) \bar{a}_y \right]$$

$$\Rightarrow F_y = \frac{\mu_0 N^2 I_{01}^2 \ell \lambda}{2\phi} \left([illegible] - 1 \right)$$

AGAIN, NI [illegible] $|M| / N' V_{[illegible]}$

$$\Rightarrow F_y = \frac{1}{2\lambda\mu_0} \left| \frac{M}{N' V_{[illegible]}} \right|^2 \frac{(1 - [illegible])}{[illegible]} [illegible]$$

e) THE COMBINED YIELD [illegible] BY AN
[illegible] ROTOR.
[illegible] FROM THE BOTH STATORS ACT
EQUAL FORCE [illegible] AS
IN A DOUBLE [illegible] IN SUCH A MANNER [illegible] STATOR,
ON THE ROTOR [illegible] THE
TO CANCEL OUT [illegible]
SYSTEM YIELDS EQUAL AND [illegible] ROTOR,
FORCES BETWEEN [illegible]
PRODUCING THE Y COMPONENT [illegible]
AND THE X COMPONENT [illegible] THE
A NOTE SHOULD BE MADE [illegible]
ROTOR IN THE [illegible] IN STABLE EQUILIBRIUM,
CONFIGURATION [illegible]
FOR A SMALL DISPLACEMENT [illegible]
CENTER [illegible]
COMPONENT [illegible]

Are you saying that you evaluated the stress-tensor using $B(g+\frac{d}{2})$? Show me a step or two

10/20

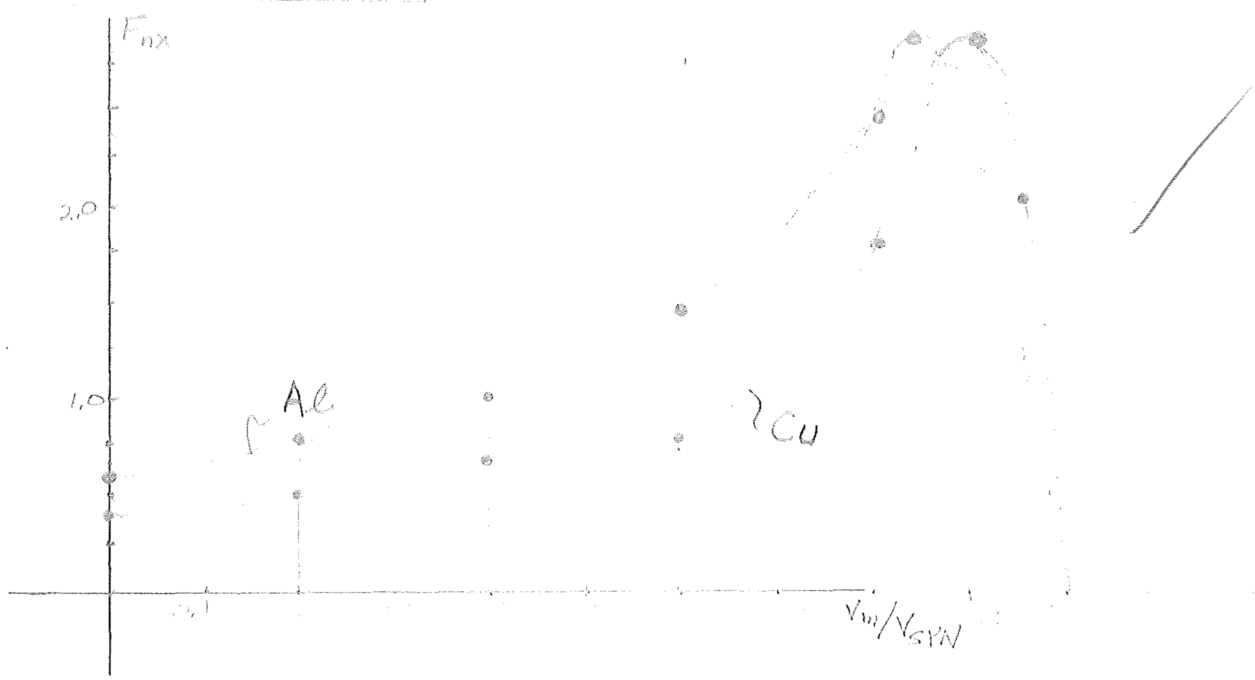
✓

10/10

4) $\left| \frac{|v|}{N' V_{SYN}} \right|$ = [illegible]

DATA

V_m / V_{SYN}	F_{nx} COPPER	F_{nx} ALUMINUM
0	[illegible]	0.61[illegible]
0.2	0.48	[illegible]
0.4	0.6[illegible]	1.01
0.6	0.76	1.44
0.8	1.74	2.[illegible]
0.9	2.5	2.[illegible]
0.95	2.6	2.0
1.0	0	0



THE PEAK FORCE IS, AS EXPECTED, INDEPENDENT OF THE CONDUCTIVITY, SUCH IS OBVIOSLY NOT TRUE CONCERNING THE SPEED AT WHICH THE PEAK FORCE OCCURS; THE LARGER THE CONDUCTIVITY THE GREATER THE SPEED AT WHICH MAXIMUM FORCE IS GENERATED

O.K.

20/20

*5.2) FROM [illegible] "[illegible]"

$$[V] = [R][i] + [L]\frac{d}{dt}[i] + \frac{d[L]}{dt} \cdot \frac{d\alpha_m}{dt} \cdot [i]$$

LET $i_a = I_m \sin(p\alpha_m - \theta) = I_m \sin(p\omega_m t - \phi)$

$$\Rightarrow V_f = R_{ff} i_f + L_{fa}\frac{di_a}{dt} + L_{fa}\frac{di_a}{dt} + L_{fb}\frac{di_b}{dt} + L_{fc}\frac{di_c}{dt} - p\omega_m M_{afm} i_a \sin p\alpha_m$$

$$= R_f i_f - p\alpha_m L_{fa} I_m \cos(p\alpha_m - \phi) + \frac{p\omega_m M_{afm} I_m}{2}(\cos\phi - \cos(2p\alpha_m - \phi))$$

$$V_a = -(i_a R_{aa} + L_{aa}\frac{di_a}{dt} - p\omega_m M_{afm} I_f \sin p\alpha_m)$$

$$= -(R_{aa} I_m \sin(p\alpha_m - \phi) + p\omega_m(M_{afm} I_f \sin p\alpha_m - L_a I_m)$$

$$V_b = -(L_{ba}\frac{di_a}{dt} - p\omega_m M_{afm} I_f \sin(p\alpha_m - \frac{2\pi}{3}))$$

$$= -p\omega_m [L_{ba} I_m \cos(\omega t - \phi) - M_{afm} I_f \sin(\omega t - \frac{2\pi}{3})]$$

$$V_c = -(L_{ca}\frac{di_a}{dt} - p\omega_m M_{afm} I_f \sin(p\alpha_m - \frac{4\pi}{3}))$$

$$= p\omega_m [M_{afm} I_f \sin(p\alpha_m - \frac{4\pi}{3}) - L_{ca} I_m \cos(\omega t - \phi)]$$

[illegible] PRIOR REVISIONS TO DELETION

[illegible]

[illegible] minimized [illegible] $\int_0^T i(t)\,dt$.

[illegible]

$$\alpha = \int_0^T \omega(t)\,dt.$$

$$Q = R_a \int_0^T i_a^2(t)\,dt$$

$$= \frac{R_a J^2}{K_f^2 I_f^2} \int_0^T \left(\frac{d\omega}{dt} + T_0/J \right)^2 dt.$$

[illegible] used $K_f I_f \, i_a(t) = J \frac{d\omega}{dt} + T_0$.

[illegible] $\lambda_1 \dfrac{R_a J^2}{K_f^2 I_f^2} \left(\dfrac{d\omega}{dt} + T_0/J \right)^2 + \lambda_2 \omega.$

"Q" integrand "α" integrand

Euler equation: $\dfrac{\partial H}{\partial \omega} - \dfrac{d}{dt}\left(\dfrac{\partial H}{\partial \omega'} \right) = \lambda_2 - \dfrac{d}{dt}\left(2\lambda_1 \dfrac{R_a J^2}{K_f^2 I_f^2} \left[\dfrac{d\omega}{dt} + \dfrac{T_0}{J} \right] \right) = 0$

or $\boxed{\dfrac{\lambda_2 K_f^2 I_f^2}{2 \lambda_1 R_a J^2} = \dfrac{d^2\omega}{dt^2}}$

[illegible] $\dfrac{\partial^2 H}{\partial \omega'^2} = 2\lambda_1 \dfrac{R_a J^2}{K_f^2 I_f^2}$. Hence, if $\lambda_1 > 0$, then [illegible] we have a minimum solution, according to the [illegible] of (2.0). If $\lambda_1 < 0$, we have maximized the time [illegible]

$$\omega(t) = \left(\frac{n_2}{n_1}\right)\left(\frac{K_t I_f}{J}\right)^2 \frac{t^2}{4R_a} + at + b, \quad b = 0, \text{ because } \omega(0) = 0.$$

$$\omega(T) = \left(\frac{n_2}{n_1}\right)\left(\frac{K_t I_f}{J}\right)^2 \frac{T^2}{4R_a} + aT = 0, \quad \text{because } \omega(T) = 0.$$

$$\therefore \quad a = -\left(\frac{n_2}{n_1}\right)\left(\frac{K_t I_f}{J}\right)^2 \frac{T}{4R_a}$$

$$\therefore \quad \boxed{\omega(t) = \left(\frac{n_2}{n_1}\right)\left(\frac{K_t I_f}{J}\right)^2 \frac{t}{4R_a}(t - T).}$$

$$\text{Now} \quad \alpha = \int_0^T \omega(t)\,dt = \left(\frac{n_2}{n_1}\right)\left(\frac{K_t I_f}{J}\right)^2 \frac{1}{4R_a}\int_0^T (t^2 - tT)\,dt$$

$$= -\left(\frac{n_2}{n_1}\right)\left(\frac{K_t I_f}{J}\right)^2 \frac{T^3}{24R_a}.$$

$$\therefore \quad \boxed{\left(\frac{n_2}{n_1}\right) = -\frac{24R_a}{T^3}\left(\frac{J}{K_t I_f}\right)^2 \alpha,}$$

Exactly as on p. 7 of my notes

$$\text{and} \quad \boxed{\begin{aligned} \omega(t) &= \frac{6\alpha t}{T^2}\left(1 - t/T\right), \\ i(t) &= \frac{1}{K_t I_f}\left[J\frac{d\omega}{dt} + T_0\right] = \left(\frac{6\alpha J/T^2 + T_0}{K_t I_f}\right) - \frac{12\alpha J}{K_t I_f T^3}\,t. \end{aligned}}$$

[illegible] must find T, however, the [illegible] minimizing subject to the constraints α and Q.

[illegible]

$$[illegible] \frac{[illegible]}{T^3} + \frac{R_a T_0^2 T}{K_f^2 T_f^2}$$

Then, [illegible]

$$\boxed{[illegible] \; T^3 Q + \frac{12 \alpha^2 T^2 R_a}{K_f^2 T_f^2} = 0.}$$

This [illegible]) can be solved in order to determine [illegible] of α, β and other known constants.

[illegible] (for simplicity), then

$$\boxed{[illegible] \left(\frac{12 \alpha^2 T^2 R_a}{Q K_f^2 T_f^2} \right)^{1/3}}$$

[illegible] required to turn the motor α degrees, while [illegible] Q joules. The result makes sense; if you wish to [illegible] the platform must turn slower (less [illegible]) i.e., T increases. Also, if α increases, it will [illegible] Q, to turn the platform α degrees.

(4). [illegible] to be maximized: $\alpha = \int_0^T \omega(t)\,dt$.

[illegible]: $\Phi = R_a \int_0^T i_a^2(t)\,dt$

$$= \frac{R_a J^2}{K_f^2 I_f^2} \int_0^T \left(\frac{d\omega}{dt} + T_0/J \right)^2 dt.$$

$$\therefore \boxed{[illegible] + \lambda_3 \frac{R_a J^2}{K_f^2 I_f^2} \left(\frac{d\omega}{dt} + T_0/J \right)^2}$$

[illegible] $\frac{\partial H_3}{\partial \omega} - \frac{d}{dt}\left(\frac{\partial H}{\partial \omega'} \right) = 1 - 2\lambda_3 \frac{R_a J^2}{K_f^2 I_f^2} \cdot \frac{d^2\omega}{dt^2} = 0$.

Then [illegible] $\frac{K_f^2 I_f^2}{2\lambda_3 R_a J^2}$. This is the usual equation, with the usual solution (this time T is known)

$$\boxed{[illegible] \frac{K_f^2 I_f^2}{[illegible] R_a J^2} T \left(\frac{t^2}{T} - t \right)}$$

[illegible] $\frac{1}{\lambda_3} = \frac{-24 J^2 R_a \alpha}{K_f^2 I_f^2 T^3}$

$$\therefore \boxed{[illegible] \left(t - \frac{t^2}{T} \right), \; i_a(t) = \left(\frac{6\alpha J / T^2 + T_0}{K_f I_f} \right) - \frac{12\alpha J}{K_f I_f T^3} t}$$

(5)

[illegible] form of Q. This is readily done (see [illegible] (eq. [illegible]))

$$Q = \frac{12\alpha J^2 R_a}{K_f^2 I_f^2 T^3} + \frac{R_a I_0^2 T}{K_f^2 I_f^2}$$

$$\alpha = \left[\frac{(Q K_f^2 I_f^2 - R_a I_0^2 T)(T^3)}{12 J^2 R_a}\right]^{1/2}$$

Let us check to see if we have maximized the original functional (the distance to be covered)

$$\frac{\partial^2 H_3}{\partial w^2} = \frac{2\lambda_3 R_a J^2}{K_f^2 I_f^2} = \frac{R_a J^2}{(K_f I_f)^2}\left(-\frac{(K_f I_f)^2 T^3}{12 J^2 R_a \alpha}\right)$$

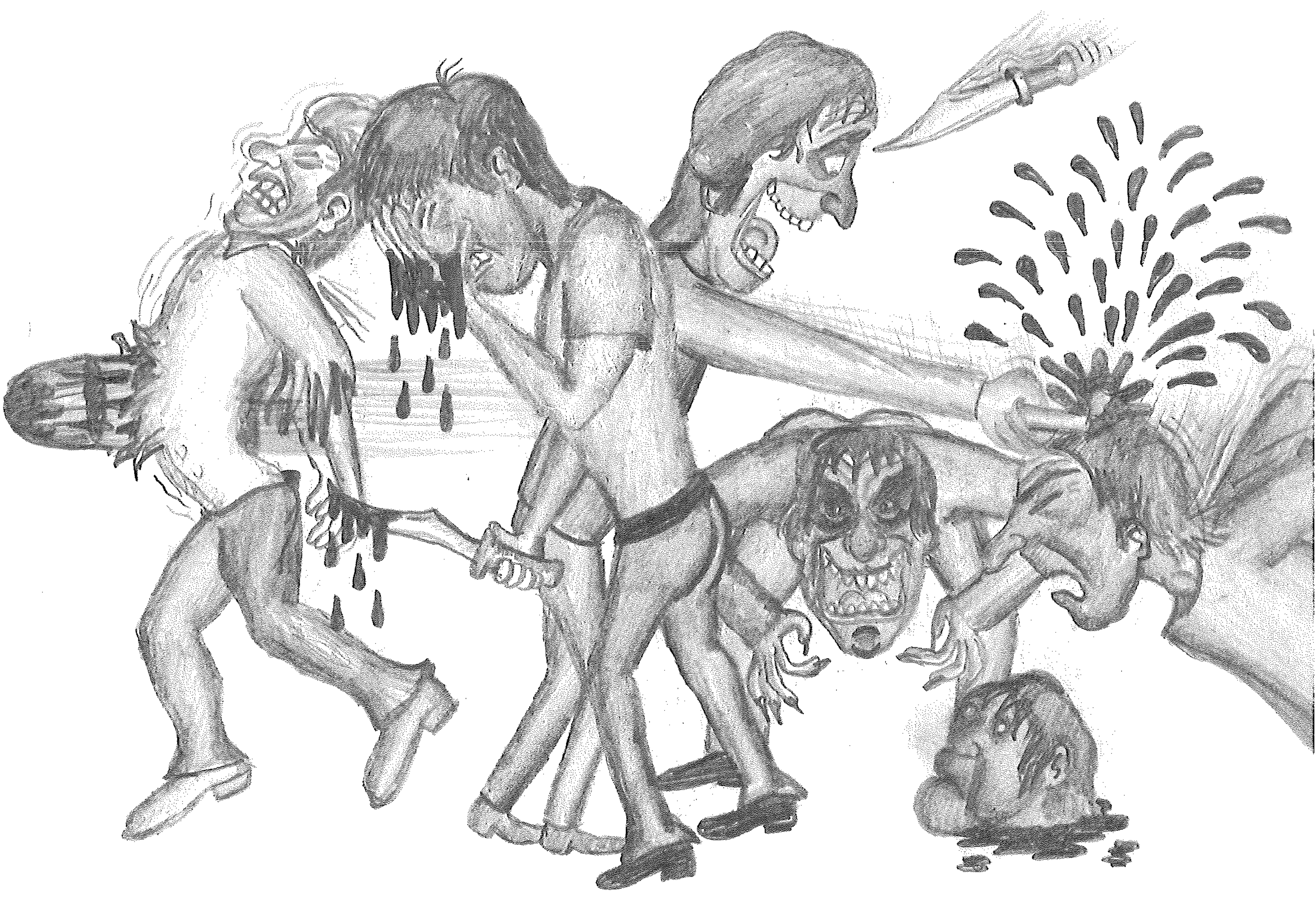
$$= -\frac{T^3}{12\alpha} < 0$$

Because the partial derivative is negative, we have, indeed, maximized the functional of interest. The point of these two problems (other than to learn about armature control of d.c. machines) is that the same $w(t)$, $i_a(t)$ waveforms furnish optimal solutions for three distinct motor control problems involving both maximization or minimization.

1) (a) $v_a(t) = i_a(t) R_a + L_a \frac{di_a}{dt} + K_f I_f w(t)$

(b) $v_a(t) = i_a(t) R_a + K_f I_f w(t)$.

Any discontinuities in armature current, $i_a(t)$, will cause $\frac{di_a}{dt} \to \infty$, i.e., there will be large voltage "spikes" or impulses. If the motor controller is [illegible] the armature [illegible], the current waveform [illegible]



Electrical Engineering Department
Rose-Hulman Institute of Technology
Terre Haute, Indiana
March 16, 1972

EE498 - Senior Projects

Hand in your project proposals no later than March 23, 1972. You may consult with me prior to that time about possibilities for your project.

The proposal should be brief and to the point. It should tell exactly what you propose to do in detail. It should be typed on 8-1/2 x 11 plain white paper. This proposal when approved will constitute a contract between student and instructor. The possibilty of getting an A in the course depends entirely upon your achieving everything you set out to do as described in the proposal.

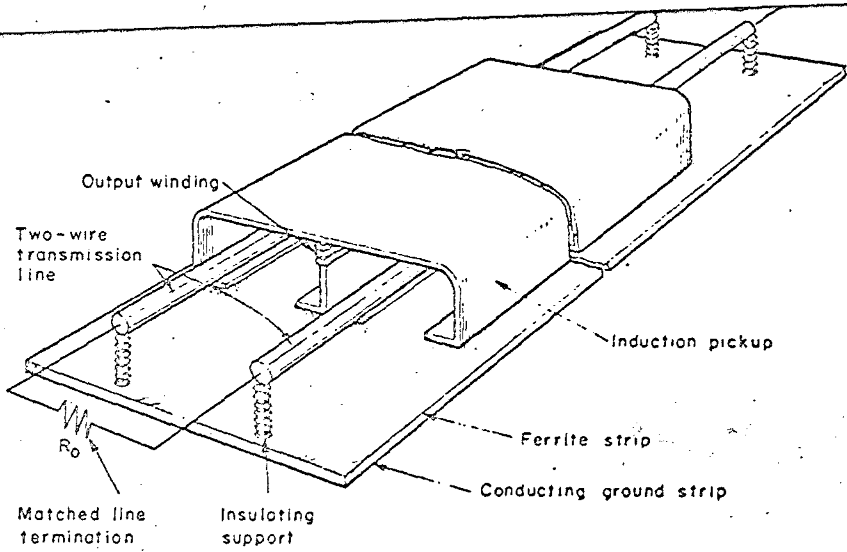
You may renegotiate the contract during the first four weeks of the course in order to decrease the scope of the prohect with a accompanying decrease in possible grade. Anyone who does not finish his project and turn in the formal report will receive an incomplete in the course.

We will have one formal class meeting each week which all students enrolled in EE498 are expected to attend. It is not required that you be in the lab working on your project during the scheduled lab period Friday morning.

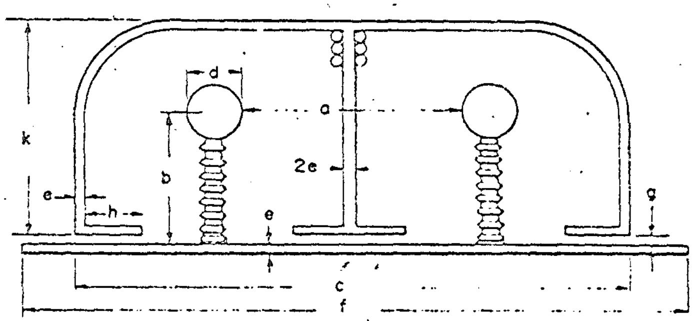
The proposal should give some motivation from your choice of project. It should give a bibliography if possible and a short resume of work which you have already done prior to this quarter. It should list the name of an EE staff member that the course instructor may consult with on your project (if not JHD).

We will have oral presentations of your proposals given on March 31 during the lab period (Attendance required). An interim report (hand written) is to be truned in no later than April 20, 1972. This report will be a quick summary of your progress on the project.

JHDerry



(a) Schematic view of transmission line with induction pickup



(b) Cross section view of (a) Typical dimensions (cm) a = 40, b = 25, c = 120, d = 10, e = 1, f = 100, g = 1, h = 12, ℓ = length = 2000, k = 40

FIGURE 6-1 - Typical Structure for Induction Pickup

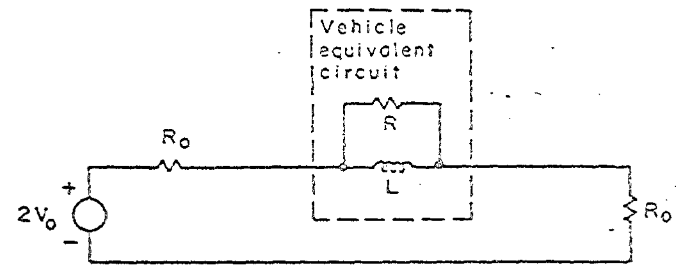


FIGURE 6-2 - Circuit model for transmission line supplying power to a vehicle by means of an induction pickup. For a given pickup magnetizing inductance, L, with $\omega L >> R_o$, maximum power in R occurs for $R = \omega L$. $P(max) = (V_o/R_o)^2 \omega L/2$.

6.0 A LINEAR INDUCTION POWER-TRANSFER SYSTEM FOR VEHICLES

6.1 Introduction

The usual method of transferring electric power to a moving vehicle is by means of a sliding contact. For a high speed vehicle this technique appears to have problems, both because of the speed, and because of the large power required. Moreover, the normal use of dc or 60 cps power poses problems because the high voltage which is required for efficient power transmission is not convenient for propulsion, so heavy on-board voltage reducing equipment may be required.

This report considers the possibility of increasing the power frequency and transferring power by inductive coupling instead of by sliding contact. Some of the important design parameters are considered and rough calculations for a typical system are presented.

6.2 Pickup Design Considerations

There are several ways in which an induction pickup could extract power from a power line. Perhaps the simplest would be a large ferrite rod antenna inserted between the wires in a two wire line. Clearly, this type of structure will not be very efficient at 60 hertz but at a frequency of several hundred kilo-hertz, a substantial power could be transformed. Unfortunately, the use of too high a frequency creates problems because of the relatively short wavelength and because of the small skin depth for currents in the transmission line wires. Moreover, it is easier to generate, control, and utilize electric power if the frequency is in a range suitable for high power semiconductor devices.

Hence, it is desirable that steps be taken to increase the magnetic coupling between the power line and the pickup coil in order that the transmission frequency can be kept as low as possible.

Figure 6-1 shows a tentative design for an induction pickup in which an "E" shaped ferrite core is used with the pickup coil wound around the center leg. A thin ferrite strip is placed under the transmission line, so that the flux encircling the transmission line wires sees a relatively small air gap between the pickup and this strip. The transmission line and pickup coil form a transformer with the "E" core and ferrite strip analogous to the E-I laminations used in conventional transformers.

The pickup could, in principle, be operated with a constant voltage or a constant current in the primary. However, if we assume that the transmission line is many wavelengths long, it appears that the only practical mode of operation is with a constant amplitude travelling wave on the line. For example, the line could be excited at one end and terminated in a matched load at the other end. An equivalent circuit for this mode is shown in Fig. 6-2 where R_o is the characteristic impedance of the line, L is the primary magnetizing inductance of the transformer, and R is the load referred to a one turn pickup coil.

Clearly, the larger the magnetizing inductance, the larger the power which can be delivered to the load. As a practical matter, the inductive reactance of the pickup, L, will be small compared with R_o so the transformer will effectively operate with a constant current in the primary. If several vehicles are coupled to the line at the same time, there will be some reduction in current, but this does not appear to be significant for a multiply excited system of the type to be described.

6.5 Conclusions

On the basis of relatively crude calculations, it appears to be feasible to transmit power from a transmission line to a moving vehicle by means of an induction pickup. For a typical example, it is possible to transfer a power of 2.5 megawatts to a pickup 20 meters long, assuming a transmission frequency of 18 khz, a current of 400 amperes, and a voltage of 100 kv. The weight of the pickup is on the order of 2 kilograms per kilowatt of power output capability, but this could be reduced by using a smaller clearance between the pickup and the ferrite ground plane, or by using a higher line frequency. The assumed clearance was 1 cm with an allowance of ± 10 cm for side to side motion of the pickup with respect to the transmission line. The system efficiency is between 69% and 98%, depending on the economics of transmission line construction, and assuming 100 vehicles consuming 2 megawatts each on a 1000 km two-way track.

It is anticipated that two major problems are generating high powers at 18 khz, and maintaining a constant voltage on a transmission line which is many wavelengths long. Another problem is the rather large weight and size of the pickup, but since the pickup serves as a transformer, on-board transformers can be eliminated. The possible feasibility of an induction pickup suggests that further study is warranted in order to arrive at more accurate design criteria and performance expectations. Experiments on simple scale models could be used to verify the design relations.

REFERENCES

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5. Massachusetts Institute of Technology, Project Transport Progress Report, Part A, June 14-15, 1966.

6. Massachusetts Institute of Technology, Project Transport Progress Report, Part B, June 14-15, 1966.

7. Williams, Laithwaite, and Piggot, "Brushless Variable-Speed Induction Motors," Proc. IEE, June, 1966.

8. Ooi, Boon-Teck, "Fringe End Effects of Short Stator," Massachusetts Institute of Technology, S.M. Thesis, January, 1966.

9. Thornton, R.D., "Semiconductor Switched DC Motors," Massachusetts Institute of Technology, Part A, Project Transport Progress Report, June 14-15, 1966.

would then be 50 kv which does not appear to be too high considering the minimum spacing between pickup and line is 8 cm.

The excitation frequency for the transmission line should be as low as possible consistent with transfer of the desired amount of power. As a typical example, if the frequency is selected to be 18 khz, if the pickup has the dimensions shown in Fig. 6-1, and if the line current is 400 amperes, then the maximum power which can be delivered to the vehicle is about 2.5 megawatts. The choice of 18 khz appears to be reasonable because it can be converted to dc or lower frequency ac by means of high power solid state devices. Also, 18 khz is above the normal audio range so that objectionable noise is minimized.

The choice of line frequency is also governed by wavelength considerations. Typically power might be fed into the line at intervals on the order of 30 to 100 km, and it is desirable to minimize the number of wavelengths between any two feed points, if a reasonably low standing wave ratio is to be maintained in spite of the presence of a number of vehicles on the line. An excitation frequency of 18 khz will produce a wavelength of about 14 kilometers, which is probably on the lower edge of practicality. It would thus be desirable to avoid using a frequency much above this value.

6.4 Efficiency of the Induction Power Transmission System

The power loss on the transmission line will be primarily I^2R loss caused by currents in the line. If the wires were aluminum tubes of diameter d and with wall thickness greater than the skin depth δ, then the resistance per unit length of a two wire line is about $(2\rho/\pi d\delta)$

where ρ is the resistivity. For aluminum $\rho = 2.8 \times 10^{-8}$ ohm-m and, at 18 khz, $\delta = 0.63$ mm. Thus for dimensions shown in Fig. 6-1, $\bar{R}$ is about 0.28 ohms per kilometer. With I = 400 amperes, the power loss would be 45 kw/km. This amount of loss is high and probably not practical. There are, however, a number of ways of reducing this loss. For example, a number of insulated wires could be twisted in such a way that the current is shared between the various wires. As a lower limit on loss we could calculate the resistance assuming that the entire cross section of the line is carrying current. The resistance is then less than 0.01 ohms per kilometer so the power loss is reduced to less than 2 kw/km or 2 megawatts for the entire 1000 km line. Other losses in the line appear to be negligible, with, for example, ferrite losses amounting to less than 10 watts per kilometer.

For purpose of example, assume that there are 100 evenly spaced vehicles on a 1000 km two-way track. Assume further that the vehicle speed is 480 km/hr (300 mph), and each vehicle consumes 2 megawatts of power. These assumptions imply a vehicle spacing of 20 km (12 miles) or 40 seconds. If the power-feeding stations are located every 50 km (31 miles), then each station will supply an average of 2 1/2 vehicles on each track, for a total power of 10 megawatts per station or 200 megawatts overall. The I^2R loss for the simple aluminum tube wires would be 90 kilowatts, giving a transmission efficiency of 69%. The lower limit on loss for stranded wires would be 4 megawatts, giving a transmission efficiency of 98%. The economically optimum solution probably lies somewhere between these extremes.

transmission lines can be tied together at quarter wavelength intervals so that any quarter wavelength section can be de-energized without interrupting operation on the remaining sections.

Figure 6-3 shows a typical two-way system with quarter wavelength tie points and a constant voltage generator connected across the lines every 3/4 wavelength. The generators are so phased that power flows in opposite directions in the two lines and thus the rms voltage and current on the lines are nearly independent of position. It is assumed that only one or two vehicles will be between any two generators, so the vehicles will not cause more than about a 5% variation in line voltage.

In order for the transmission line to operate in the travelling wave mode, it is necessary that the ratio of line-to-line voltage to line current be equal to the characteristic impedance of the line. There is relatively little design freedom in the choice of the line impedance because it depends on the logarithm of the ratio of various cross sectional dimensions. For any choice that appears at all reasonable for use with an induction pickup, this impedance is in the range of 200 to 300 ohms. It appears that the impedance of the magnetizing inductance of the pickup is about an order of magnitude less than this, so it is desirable to use the lowest possible line impedance. In other words, the power transferred to the pickup depends almost entirely on the line current, and we would like to minimize the voltage required to produce this current. As a typical example, the transmission line shown in Fig. 6-1 is estimated to have a characteristic impedance on the order of 250 ohms and typical operating conditions might be 100 kv line-to-line voltage and 400 amperes line current. The line to ground voltage

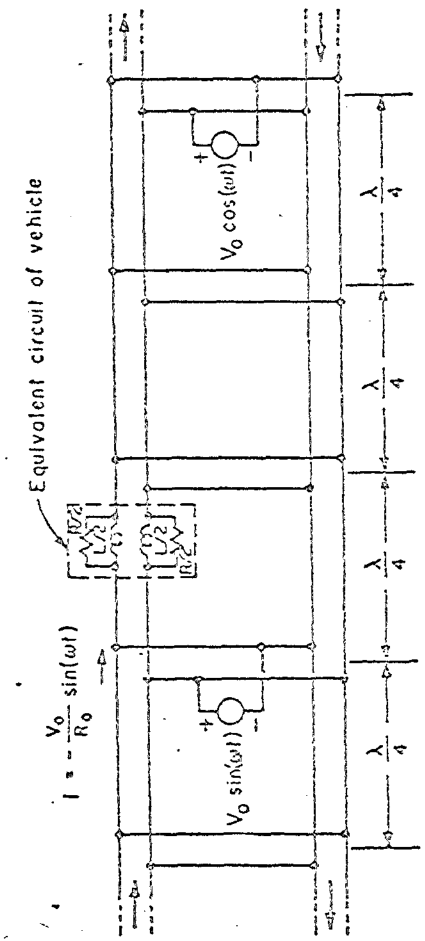


FIGURE 6-3 - Circuit for exciting two transmission lines with travelling waves. Arrows indicate direction of power flow. Quarter-wave-length interconnections are used to minimize standing waves caused by discontinuities (vehicles) on the line. Typically $R_o << R, \omega L$. Excitation sources are located an odd number of quarter-wave-lengths apart.

The pickup design shown in Fig. 6-1 has a magnetic field air gap with a relatively large area and short length. The flux density in the air gap is quite low so the ferrite magnetic circuit has less than 10% of the area of the air gap. A wide spacing is allowed between the transmission line and the pickup in order that a high voltage line can be used. Also, a considerable latitude is allowed for sideways motion of the pickup with respect to the transmission line. The height of the transmission line above ground is made large enough to minimize power losses in the ferrite strip when the pickup is not present, and to minimize the characteristic impedance of the line. A conductor under the ferrite provides a low loss ground plane and minimizes the external field of the line. An aluminum frame over the pickup provides both structural strength for the pickup and shielding for the vehicle. It is envisioned that the pickup would be an integral part of the vehicle and would extend almost the full length of the vehicle.

Typical dimensions are shown in Fig. 6-1. Perhaps the most important dimension is the clearance between the pickup and the ferrite ground strip (i.e., "g" in Fig. 6-2. This distance is assumed to be 1 cm but the power output could be doubled by reducing this distance to 1/2 cm. A minimum clearance of about 8 cm is assumed between the pickup and the transmission line, with an allowable side-to-side motion for the pickup of about ± 10 cm with respect to the transmission line. The shoes on the bottom of the pickup are made as long as practical without interfering with the transmission line supports. The length of the pickup is assumed to be 20 meters based on a vehicle length of about 30 meters. The magnetic material in the pickup is assumed to be ferrite, but laminated

steel could equally well be used. The mass of the pickup is estimated at about 4,000 kg.

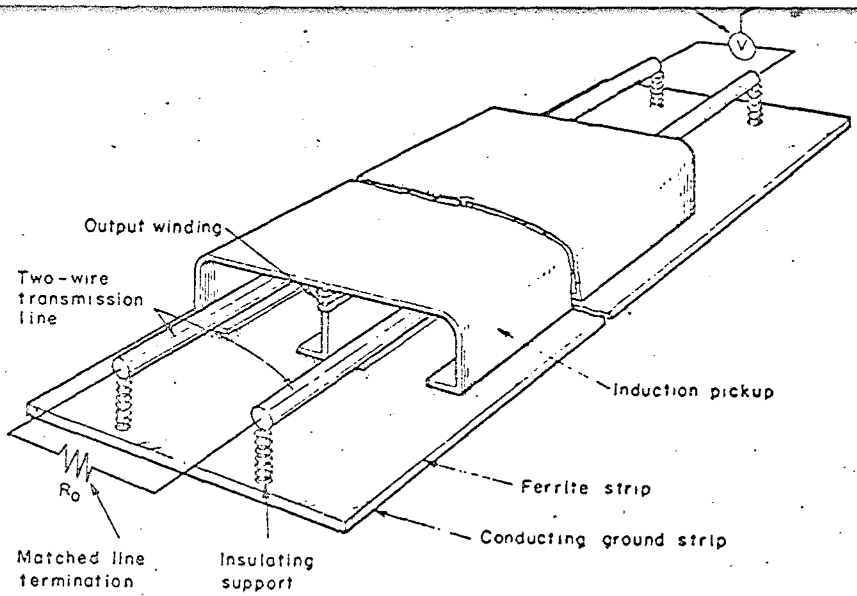
Since the pickup behaves like a transformer, almost any output voltage can be developed by using an appropriate turns ratio. If desired, the pickup could be divided into two or more sections with each section behaving almost as an independent pickup. This mode might be desirable if there were two or more separate motors used for propulsion.

6.3 Transmission Line Design Considerations

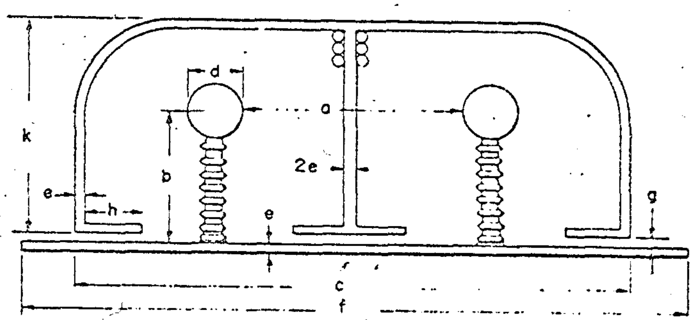
The transmission line can be operated in several possible modes, and the best choice can only be made after consideration of a number of economic factors. For example, one possibility is to divide the line into a number of short sections with each section excited only when a vehicle is present. Such a system would minimize line losses, but at an increase in the cost of control equipment.

For purpose of example, it is assumed that it is desirable to excite the entire line, and that the vehicles are nearly evenly spaced. Power is assumed to be fed into the line at periodic intervals in order to maintain a nearly constant voltage on the line. The entire line can be thought of as a resonant structure, except that power is flowing from one end of the line to the other.

Instead of actually transmitting power from one end of the line to a dummy load at the other end, a two-way track system can be used with power flowing in opposite directions in the two parallel transmission lines. The two lines are then chosen to be an integer number of quarter wavelengths long and are joined at each end. In addition, the two



(a) Schematic view of transmission line with induction pickup



(b) Cross section view of (a) Typical dimensions (cm) a = 40, b = 25, c = 120, d = 10, e = 1, f = 100, g = 1, h = 12, ℓ = length = 2000, k = 40

FIGURE 6-1 - Typical Structure for Induction Pickup

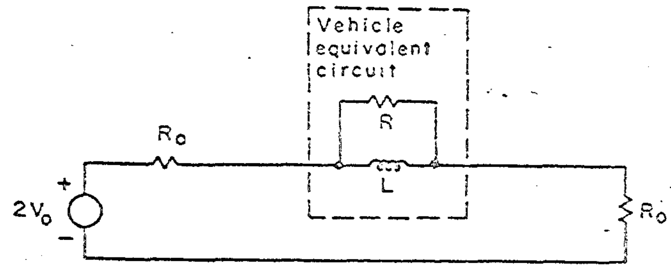


FIGURE 6-2 - Circuit model for transmission line supplying power to a vehicle by means of an induction pickup. For a given pickup magnetizing inductance, L, with $\omega L >> R_o$, maximum power in R occurs for $R = \omega L$. $P(max) = (V_o/R_o)^2 \omega L/2$.